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Chapter 9

REALIZED VOLATILITY AND CORRELATION ESTIMATORS UNDER NON-GAUSSIAN MICROSTRUCTURE NOISE

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Abstract

Realized volatility and correlation estimators suffer from microstructure noise, resulting in biased and imprecise estimators. This suggests that estimators do not converge for high-frequency levels, where noise especially exists. To solve the problem of noise, some approaches have been suggested in literature. In particular, the subsampling and averaging approach works well. Moreover, the realized volatility literature usually assumes Gaussian microstructure noise despite the fact that noise in real world financial markets does not follow a Gaussian process. In this article, we suggest what we believe to be more realistic microstructure noise processes. The fractional stable noise that we suggest is the most realistic process compared to other noise processes investigated in our simulations. Therefore, realized volatility and correlation estimators should be unbiased and converge faster under this type of noise process. Empirically, some of the estimators exhibit heavy tails and some exhibit dynamic behaviors. This is especially so in the case of absolute based realized correlations which exhibit negative asymmetry in the dependence structure between minute by minute frequency data of CAC and FTSE stock indices.

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1. Introduction

Asset return volatilities are central in finance, in particular in asset pricing, portfolio allocation, and market risk measurement. Therefore the field of financial econometrics devotes considerable attention to time-varying volatility and associated tools for its measurement, modeling and forecasting. On the other hand, correlations are critical inputs for many of the common tasks of financial management. Hedges require estimates of the correlation between the assets in the hedge. If the correlations and volatilities are changing, then the hedge ratios should be adjusted to account for the most recent information. Asset allocation and risk assessment also rely on correlations.

The most popular approach to obtain volatility estimates is using statistical models that have been proposed in literature on autoregressive conditional heteroscedasticity (ARCH) and stochastic volatility. Another method of extracting information about volatility is to formulate and apply economic models that link the information contained in options to the volatility of the underlying asset. This approach is based on the market's assessment of future volatility. All these approaches have the following in common: (1) the resulting volatility measures are only valid under the specific assumptions of the models used and (2) they are generally uncertain which or whether any of these specifications provide a good description of actual volatility. Meanwhile, interest in the field of high-frequency finance is growing. Exploiting high-frequency data, volatility and correlation can be measured more accurately in a model-free approach. The realized volatility approach is claimed to be consistent under general nonparametric conditions. In other words, these types of measures provide more precise ex-post observations of the actual volatility compared to the traditional sample variances based on daily or coarser frequency data. In fact, sampling as often as possible would, in theory, produce exact estimates of the true variance in the limit.

However, this is not the practical case. The presence of market microstructure noise in high-frequency financial data complicates the estimation of financial volatility and correlation making the approach unreliable. There is a considerable bias in estimation at the higher frequency due to intervention of noise. While the realized volatility approach suggests sampling at the highest possible frequency to attain the highest precision, market microstructure frictions exist at the highest levels of frequency. For this reason, sparse sampling or lower frequencies have been recommended to reduce market microstructure contamination. For example, optimal sampling schemes have been investigated by Bandi and Russell [Ban05b] and Aït-Sahalia et al. [Ait05]. Meanwhile, researchers have investigated some methods to cope with the problem at the highest frequency in the presence of noise. These methods include a kernel-based correction by Zhou [Zho96], a moving average filter by Maheu and McCurdy [Mah02], an autoregressive filter by Bollen and Inder [Bol02], and a subsampling and averaging approach by Zhang et al. [Zha05] However, in the literature on realized volatility, market microstructure noise is usually assumed to follow an i.i.d process. For example, Barndorff-Nielsen, Hansen, Lunde and Shephard [Bar04a], Zhang et al. [Zha05], Bandi and Russell [2005b], and Hansen and Lunde [2006] assume that noise follows a Gaussian process.

As Rachev and Mittnik [Rac00] explain, although earlier theories build on Bachelier's original theory on speculation assumed the normal distribution, the excess kurtosis found in the studies by Mandelbrot and Fama in the 1960s led them to reject the normal assumption and propose the stable Paretian distribution as a statistical model for asset returns. In subsequent years, a number of empirical investigations supported this conjecture. See, for example, Mittnik, Rachev, and Paolella [Mit98]. The implication of rejecting the random walk hypothesis is that researchers must accept that returns in financial markets are not independent but instead exhibit trends. Sun, Rachev, and Fabozzi [Sun07] argue that in addition to the empirical evidence, there are theoretical arguments that have been put forth for rejecting both the Gaussian assumption and the random walk assumption. One of the most compelling arguments against the Gaussian random walk assumption is that markets exhibit a fractal structure. Samorodnitsky and Taqqu [Sam94] demonstrate that the properties of some self-similar processes can be used to model financial markets that are characterized as being non-Gaussian and non-random walk. Such financial markets have been stylized by long-range dependence, volatility clustering, and heavy tailedness. A distribution that is rich enough to encompass those stylized facts exhibited in return data is the stable distribution. Long-range dependence, self-similar processes, and stable distribution are very closely related [Sun07].

In an empirical study comparing 27 German stocks included in the DAX, Sun, Rachev, and Fabozzi [Sun07] conclude that an ARMA-GARCH model assuming a fractional stable noise outperforms other ARMA-GARCH models assuming independent and identically distributed (i.i.d.), stable, generalized Pareto, generalized extreme value, and fractional Gaussian noises. These results suggest that a non-Gaussian assumption about microstructure noise seems more realistic. Indeed, it seems that the best way of dealing with market microstructure noise may depend on the real world types and properties of noise.

Inspired by Sun, Rachev, and Fabozzi [Sun07], this article examines some volatility and correlation estimators under different assumptions about microstructure noise. In particular, the impacts of different assumptions about microstructure noise including i.i.d. or white noise, stable noise, fractional Gaussian noise, and fractional stable noise on accuracy and especially on the bias in estimation of volatility and correlation are investigated and compared.

The rest of the article is structured as follows: In section 2, assumptions about price, return, and microstructure noise processes in financial markets are made and then different volatility and correlation estimators are explained. In section 3, self-similar processes of interest are described and some simulation algorithms for generating noise are identified. Behavior of realized volatility and correlation estimators under different real noise processes are investigated via Monte Carlo simulations in section 4. Using 1 minute frequency data for the FTSE and CAC index series, in section 5 we empirically consider the behavior of the estimators. Section 6 concludes and discusses some issues.

2. Realized Power Volatility and Correlation

To fix the notations, assume that p denotes a price process. The logarithmic price $y = \log p$ can be directly observed so that

$$y = y^* + u, \tag{1}$$

where y^* denotes the logarithmic efficient price of an asset, i.e., the price that would exist in the absence of market microstructure noise, and u denotes a microstructure noise in the observed logarithmic price as induced by, for instance, asymmetries in information, transaction costs, price discreteness and bid-ask bounce effects. A certain time period t (i.e. one day) is fixed and availability of n high-frequency compounded prices over t is assumed. Given Equation (1), we can readily define continuously returns over any intra-period interval of length $\delta = \frac{t}{n}$ and write

$$y_{t_{i+1}} - y_{t_i} = y^*_{t_{i+1}} - y^*_{t_i} + u_{t_{i+1}} - u_{t_i},$$

or

$$r_{t_i} = r^*_{t_i} + \epsilon_{t_i}, \tag{2}$$

where r_{t_i} is a return on day t at time i, and where $t = 1, ..., T$ and $i = 1, ..., n$. Here n is the number of intraday observations. The following two assumptions are imposed on the price process and market microstructure effects.

Price process assumptions: The logarithmic efficient price process, y^*, is assumed to be a continuous stochastic volatility semimartingale. More precisely,

1: the logarithmic efficient price process

$$y^*_t = \alpha_t + m_t, \tag{3}$$

where α_t (with $\alpha_0 = 0$) is a continuous drift process of finite variation defined as $\int_t^{t+1} \phi(s)ds$ and m_t is a continuous local martingale defined as $\int_t^{t+1} \sigma(s)dWs$, with $\{W_t : t \geq 0\}$ denoting a standard Brownian motion.

2: the spot volatility process, σ_t, is the cádlág process and bounded away from zero.

3: the integrated variance process $\int_t^{t+1} \sigma^p(s)ds$ is bounded, almost surely, for all $t < \infty$.

The price process assumed in (3) is consistent with the asset pricing theory and suggests that the efficient return process evolves over time as a stochastic volatility martingale difference plus an adapted process of finite variation. The stochastic spot volatility can display jumps, diurnal effects, high-persistence (possibly of the long memory type), and nonstationarities. In addition, leverage effects (i.e., dependence between σ and the Brownian motion W) are allowed.

Microstructure noise assumptions: For the microstructure frictions, it is supposed that:

1: microstructure noise in the price process, u'_{t_i}, follows the fractional stable noise.

2: u'_{t_i} are independent of the $y^{*'}_{t_i}$ for all i and all n.

In general, the characteristics of the noise returns, ϵ's, may depend on the sampling frequency.

2.1. Volatility Estimators

For the decomposition of continuous logarithmic process (3), the integral of the instantaneous variances over the day, that is,

$$IPV = \int_{t}^{t+1} \sigma^{p}(s)ds, \tag{4}$$

where the power or order p is a positive value, provides an ex post measure of the true, latent or Integrated Power Variance (IPV) process associated with day t. In the literature, the special case where $p = 2$, such that

$$IV = \int_{t}^{t+1} \sigma^{2}(s)ds, \tag{5}$$

is usually called Integrated Volatility (IV). By cumulating the intraday squared returns, as shown in Merton [Mer80], Andersen et al. [And98], Andersen et al. [And01a] and Andersen et al. [And01b], one can consistently approximate the integrated volatility in equation (5) to a higher precision as the number of intraday observations increases ($n \to \infty$). In particular, one can obtain an estimate of the Realized Volatility, $\widehat{RV}$, denoted as

$$\widehat{RV} = \sum_{t_i}^{T_n} r_{t_i}^{2}, \tag{6}$$

over a period t, with $0 = t_0 \leq t_1 \leq ...t_n = T$ and where $i = 1, ..., n$ is ith intraday observation with an integer n. The basic concept of accumulative intraday squared returns has been extended by Barndorff-Nielsen and Shephard [Bar03a] to a wider class called realized power variation of order p, that is, the sums of absolute powers of increments of a process, $\widehat{RP}$,

$$\widehat{RP} = \sum_{t_i}^{T_n} |r_{t_i}|^{p}, \tag{7}$$

where $i = 1, ..., n$ is ith intraday observation with an integer n and p, the power or order, is a positive value. The quantity of Realized Power variation $\widehat{RP}$, as a proxy, is supposed to approximate the daily increments of the power variation of the semimartingale that drives the underlying logarithmic price process, i.e., Integrated Power Volatility (IPV) in (4). It is supposed to be a consistent and unbiased estimator for IPV, as the frequency increases or $n \to \infty$. While (6) is simply a special case of (7), another well-known special case of (7) is called Realized Absolute volatility estimator (RA), as follows

$$\widehat{RA} = \sum_{t_i}^{T_n} |r_{t_i}|, \tag{8}$$

where $p = 1$ with a fixed t. This estimator is also considered as a proxy for IPV in (4). Barndorff-Nielsen and Shephard [Bar03a] provided a limiting distribution theory for realized power variation.

The theoretical validity of the convergence of the estimators depends considerably on the observability of the efficient or equilibrium price process. However, it is widely accepted that the equilibrium price process and, as a consequence, the equilibrium return data are contaminated by market microstructure effects. However, market microstructure noise, especially at very high-frequencies, puts the convergence and in particular the biasedness of the realized variation estimators into challenge [Zho96], [And02], [Zha05]. In practice, microstructure frictions may include for example bid-ask spreads, discreteness of the price grid, asymmetries in information, transaction costs, and lunch-time effects. For solving the problem of microstructure noise, several approaches have been proposed including for example a kernel-based correction introduced by Zhou [Zho96], an optimal sampling introduced by Bandi and Russell [Ban05a], a moving average filter introduced by Maheu and McCurdy [Mah02], an autoregressive filter introduced by Bollen and Inder [Bol02], and a subsampling and averaging approach introduced by Zhang et al. [Zha05]. It has been experimentally found by Ghysels and Sinko [Ghy06b] that the subsampling and averaging class of the estimators is the best volatility predictor of all microstructure noise correctors.

Zhang et al. [Zha05] proposed the Two-Scale Realized Volatility estimator (TSRV), which encompasses the realized squared volatility estimators from two time scales. From the returns on a slow time scale, an estimator $\widehat{RV}_{avg}$ is obtained which is constructed based on subsampling and averaging procedure, and $\widehat{RV}_{all}$ is computed from the returns on a fast time scale using the latter as a means for bias-corrector of the estimator. Then the estimator $\widehat{TSRV}$ is approximated by

$$\widehat{TSRV} = \left(1 - \frac{\bar{n}}{n}\right)^{-1} \left(\widehat{RV}_{avg} - \frac{\bar{n}}{n}\widehat{RV}_{all}\right), \tag{9}$$

where $\widehat{RV}_{all}$ is equivalent to (6) and $\widehat{RV}_{avg} = \frac{1}{K}\sum_{k=1}^{K}\sum_{t_i,t_{i+1}\in g^{(k)}} r_{t_i}^2$ where the K number of samples are allocated to g subgrids. The estimator $\widehat{TSRV}$ is then consistent and unbiased for integrated volatility (5). Advocating subsampling and averaging approach, Safari and Seese [Saf07] extended the realized power variation (7) to Two-Scale realized Power Volatility (TSPV) estimator which is a consistent and unbiased estimator for IPV in (4) under microstructure noise and at the same time is somewhat immune against large values or jumps compared to TSRV estimator particularly when $p = 1$. The estimator $\widehat{TSPV}$, which is a generalized case of $\widehat{TSRV}$, is approximated by

$$\widehat{TSPV} = \left(1 - \frac{\bar{n}}{n}\right)^{-1} \left(\widehat{RP}_{avg} - \frac{\bar{n}}{n}\widehat{RP}_{all}\right), \tag{10}$$

where $\widehat{RP}_{all}$ is equivalent to (7) and $\widehat{RP}_{avg}$ is computed by

$$\widehat{RP}_{avg} = \frac{1}{K}\sum_{k=1}^{K}\sum_{t_i,t_{i+1}\in g^{(k)}} |r_{t_i}|^p . \tag{11}$$

In (9) and (10) we have $\bar{n} = \frac{n-K+1}{K}$. To understand subsampling procedure used here, the full grid of G arrival times, $G = \{t_0, ..., t_n\}$, is first defined and then it is partitioned into K nonoverlapping subgrids $g^{(k)}$ with $k = 1, ..., K$. The first subgrid starts from t_0 and takes every Kth arrival time, i.e., $g^{(1)} = (t_0, t_{0+K}, t_{0+2K}, ...,)$, the second subgrid starts from t_1 and takes every Kth arrival time, i.e., $g^{(2)} = (t_1, t_{1+K}, t_{1+2K}, ...,)$ and so on. Given the kth subgrid of arrival times, the corresponding realized variation estimator can be defined as $\widehat{RP}^{(k)} = \sum_{t_i, t_{i+1} \in g^{(k)}} |r_{t_i}|^r$, where t_i and t_{i+1} denote consecutive elements in $g^{(k)}$. For the estimator, the all scale plays a bias-correcting role while the averaging scale reduces the variance of the estimator. The optimal number of subgrids K provided by Zhang et al. [Zha05] is prescribed as

$$K = cn^{2/3}, \tag{12}$$

where c can be estimated by $c = (\frac{16\sigma_\epsilon^4}{TE\eta^2})^{1/3}$ where $\eta^2 = \frac{3}{4}\int_t^{t+1} \sigma^4(s)ds$. The term σ_ϵ^4 is square of the variance of noise, while $\int_t^{t+1} \sigma^4(s)ds$ is the integrated quarticity. The σ_ϵ^2 is estimated by $\widehat{\sigma_\epsilon^2} = \frac{1}{2n}\widehat{RP}$ and $\eta^2 = \frac{4}{3}(\widehat{RP})^2$ at some reasonable lower frequency. The estimator $\widehat{TSPV}$ is an unbiased and consistent estimator for IPV in (4) under the microstructure noise as the frequency increases. Both the two-scale estimators are consistent and unbiased under microstructure noise kind of i.i.d process.

2.2. Correlation Estimators

Realized correlation estimator is conditionally constructed based on realized volatility. According to the theory of realized variation, Andersen et al. [And01a] and Andersen et al. [And01b] developed the concept of realized correlation constructed on realized standard deviation, $\widehat{RV}_{std} = \widehat{RV}^{1/2}$, and covariance, $\widehat{RCOV}_{xy} = \sum_{t_i}^{T_n} r_{t_{ix}}.r_{t_{iy}}$. The Realized Correlation, $\widehat{RCOR}_{xy}$, is represented as follows

$$\widehat{RCOR}_{xy} = \widehat{RCOV}_{xy}/(\widehat{RV}_{std,x}.\widehat{RV}_{std,y}), \tag{13}$$

where x and y are high frequency time series of two asset returns. Barndorff-Nielsen and Shephard [Bar04b] provided an asymptotic distribution theory for realized covariance and correlation estimators. They claim that the estimators converge in probability to the corresponding true covariance and correlation. This is quite true under some general conditions, but the microstructure noise remains still as an influential problem. Hence, Safari and Seese [Saf07] continue to extend realized power volatility and in particular two-scale estimator into correlation estimators in order to deal with microstructure noise which may cause a remarkable bias in estimation. Motivated by robustness of absolute transformation in realized power variation against large values specially when p is set around or equal to 1, realized power standard deviation is derived as $\widehat{RP}_{std} = \widehat{RP}^{1/2}$ and thus Realized Power Correlation as follows

$$\widehat{RPCOR}_{xy} = \widehat{RCOV}_{xy}/(\widehat{RP}_{std,x}.\widehat{RP}_{std,y}). \tag{14}$$

For two-scale correlation, first a two-scale covariance (TSCOV) has to be defined. The $\widehat{TSCOV}_{xy}$ is defined as

$$\widehat{TSCOV}_{xy} = \left(1 - \frac{\bar{n}}{n}\right)^{-1} \left(\widehat{RCOV}_{xy,avg} - \frac{\bar{n}}{n}\widehat{RCOV}_{xy,all}\right) \tag{15}$$

where $\widehat{RCOV}_{xy,all}$ is the same as $\widehat{RCOV}_{xy}$, built on the full grid and the $\widehat{RCOV}_{xy,avg}$ estimator can be estimated by

$$\widehat{RCOV}_{xy,avg} = \frac{1}{K}\sum_{k=1}^{K} \sum_{t_i,t_{i+1}\in g^{(k)}} r_{t_{ix}}.r_{t_{iy}}. \tag{16}$$

Given the two-scale power standard deviation as $\widehat{TSPV}_{std} = \widehat{TSPV}^{1/2}$, then the Two-Scale Power based Correlation $\widehat{TSPCOR}_{xy}$ estimator is prescribed by

$$\widehat{TSPCOR}_{xy} = \widehat{TSCOV}_{xy}/(\widehat{TSPV}_{std,x}.\widehat{TSPV}_{std,y}), \tag{17}$$

where all estimators are based on a fixed interval of time and where x and y are two assets or high-frequency time series[1]. The power or absolute based correlation estimators are expected to estimate the true or Integrated Power Correlation, $IPCOR_{xy}$, consistently and to converge as the frequency increases. We have

$$IPCOR_{xy} = \frac{\int_t^{t+1} \Sigma_{xy}(s)ds}{\sqrt{\int_t^{t+1} \sigma_x^r(s)ds \int_t^{t+1} \sigma_y^r(s)ds}}, \tag{18}$$

where $\int_t^{t+1} \Sigma_{xy}(s)ds$ is the true or Integrated Covariance. For more information see [Saf07]. While the $\widehat{RPCOR}_{xy}$ estimator may face to the bias problem due to market microstructure noise, the two-scale estimator, $\widehat{TSPCOR}_{xy}$, corrects the bias by its subsampling procedure under microstructure noise of type i.i.d. However, this type of noise maybe not to be the real world case.

One of the most important and influential issues may occur when utilizing high-frequency data for studying dependence and comovement by covariance and correlation estimators is non-synchronous observations caused by for example missing observations, different working hours per day, and different holidays between countries. Non-synchronous observations mainly lead to unreliable estimations. To cope with such the problem, the best way found insofar is to synchronize observations. Let t_i and τ_j be the instants at which the returns x and y are being observed. Hayashi and Yoshida [Hay05] and Corsi [Cor07] proposed a covariance estimator

[1] For the sake of simplicity, throughout of the paper we consider only the covariance and correlation estimators between two assets. However, the matrices of assets can be simply applied on the estimators.

$$\widehat{RCOV}_{xy} = \sum_{t_i}^{T_n} \sum_{\tau_j}^{T_m} r_{t_{i_x}}.r_{\tau_{j_y}}.I[min(t_i, \tau_j) > max(t_{i-1}, \tau_{j-1})], \quad (19)$$

where $I[.]$ is the indicator function which takes the value of one only when the observations of two returns instantaneously overlap. This estimator consistently estimates the covariance of non-synchronous processes. Throughout the paper this synchronization is applied for estimation of covariances. In case of two-scale estimator, the indicator function is applied to the model, i.e., this model can be extended to the two-scale model.

3. Self-similar Processes

The self-similar processes are the processes that are invariant under suitable translations of time and scale. They are important in probability theory because of their connection to limit theorems and they are of great interest in modeling heavy-tailed and long-memory phenomena. In fact, Lamperti [Lam62] used the term semi-stable in order to underline that the role of self-similar processes among stochastic processes is analogous to the role of stable distributions among all distributions. A process $\{X(t)\}_{t\geq 0}$ is called self-similar [Lam62] if for some $H > 0$,

$$X(at) \stackrel{d}{=} a^H X(t),$$

for every $a > 0$, where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions of the processes on the left and right. The process $X(t)$ is also called an H-self-similar process and the parameter H is called the self-similarity index or Hurst exponent. Weron et al. [Wer05] argue that if we interpret t as time and $X(t)$ as space then above equation tells us that every change of time scale $a > 0$ corresponds to a change of space scale a^H. The bigger H, the more dramatic is the change of the space coordinate. The equation, indeed, means a scale-invariance of the finite-dimensional distributions of $X(t)$. This property of a self-similar process does not imply the same for the sample paths. Therefore, pictures trying to explain self-similarity by some zooming in or out on one sample path, are, by definition, misleading. In contrast to the deterministic self-similarity, the self-similarity of stochastic processes does not mean that the same picture repeats itself exactly as we go closer. It is rather the general impression that remains the same.

3.1. Fractional Gaussian Processes

Many of the interesting self-similar processes have stationary increments. A process $\{X(t)\}_{t\geq 0}$ is said to have stationary increments if for any $b > 0$,

$$[X(t+b) - X(b)] \stackrel{d}{=} [X(t) - X(0)].$$

The fractional Brownian motion $\{B_H(t)\}_{t\geq 0}$ has the integral representation

$$B_H(t) = \int_{-\infty}^{\infty} [(t-u)_+^{H-1/2} - (-u)_+^{H-1/2}] dB(u), \tag{20}$$

where x_+=max(x,0) and $B(u)$ is a Brownian motion. It is H-self-similar stationary increments (H-sssi) and it is the only Gaussian process with such properties for $0 < H < 1$ [Sam94]. The classic Brownian motion $B(t)$, used by Einstein and Smoluchowski, is simply a special case of the fractional Brownian motion when $H = 1/2$.

In modeling of long-memory phenomena, the stationary increments of H-self-similar processes are of special interest since any H-self-similar process with stationary increments $\{X(t)\}_{t \in R}$ induces a stationary sequence $\{Y_j\}_{j \in Z}$, where $Y_j = X(j+1) - X(j)$ and $j = ..., -1, 0, 1, ...$. The sequence Y_j corresponding to the fractional Brownian motion is called fractional Gaussian noise [Mer03]. It is called a standard fractional Gaussian noise if varY_j=1 for every $j \in Z$. The fractional Gaussian noise has some remarkable properties. If H=1/2, then its autocovariance function $r(k) = R(0,k) = 0$ for $k \neq 0$ and hence it is the sequence of independent identically distributed (i.i.d.) Gaussian random variables. The situation is quite different when $H \neq 1/2$, namely the Y_j's are dependent and the time series has the autocovariance function.

3.2. Fractional Stable Processes

While the fractional Brownian motion can capture the effect of long-range dependence, it has less power to capture heavy tailedness [Sun07]. The existence of abrupt discontinuities in financial data, combined with the empirical observation of sample excess kurtosis and unstable variance, confirms the stable Paretian hypothesis identified by Mandelbrot [Man83]. It is natural to introduce the stable Paretian distribution in self-similar processes in order to capture both long-range dependence and heavy tailedness. There are many different extensions of fractional Brownian motion to the stable distribution. The most commonly used extension of the fractional Brownian motion to the α-stable case is the linear fractional stable motion (also called the fractional Levy stable motion). Samorodnitsky and Taqqu [Sam94] define the process $\left\{Z_\alpha^H(t)\right\}_{t \in R}$ by the following integral representation

$$Z_\alpha^H(t) = \int_{-\infty}^{\infty} [(t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}] dZ_\alpha(u), \tag{21}$$

where $Z_\alpha(u)$ is a symmetric Levy α-stable motion. The integral is well defined for $0 < H < 1$ and $0 < \alpha \leq 2$ as a weighted average of the Levy stable motion $Z_\alpha(u)$ over the infinite past with the weight given by the above integral kernel denoted by $f_t(u)$.

The process $Z_\alpha^H(t)$ is the H-sssi. Assume that H-self-similarity follows from the above integral representation and the fact that the kernel $f_t(u)$ is d-self-similar with $d = H - 1/\alpha$, when the integrator $Z_\alpha(u)$ is 1/α-self-similar. This implies [Wer05] the following important relation

$$H = d + \frac{1}{\alpha}.$$

The process $Z_\alpha^H(t)$ is reduced to the fractional Brownian motion if one sets α=2. When H=1/α, then the Levy α-stable motion is obtained which is an extension of the Brownian motion to the α-stable case. Contrary to the Gaussian case (α=2), the Levy α-stable motion ($0 < \alpha < 2$) is not the only 1/α-self-similar Levy α-stable process with stationary increments (this is true for $0 < \alpha < 1$ only). The increment process corresponding to the fractional Levy stable process is called a Fractional Stable Noise (FSN). By analogy to the case of α=2, fractional stable noise has the long-range dependence when $H > 1/\alpha$ and the negative dependence when $H < 1/\alpha$. If $H = 1/\alpha$, the increments of fractional Levy stable motion are i.i.d. symmetric α-stable variables. We note that there is no long-range dependence when $0 < \alpha \leq 1$ because H is constrained to lie in the interval (0, 1).

3.3. Simulation Algorithms for the Noise Processes

A fast Fourier transform method for synthesizing approximate self-similar sample paths for Fractional Gaussian Noise has been presented by Paxson [Pax97]. The method is fast and appears to generate close approximations to true self-similar sample paths. A simulation procedure based on this method that overcomes some of the practical implementation issues has been prescribed by Bardet et al. [Bar03b]. The procedure follows these steps:

1. Choose an even integer M. Define the vector of the Fourier frequencies $\Omega = (\theta_1, ..., \theta_{M/2})$, where $\theta_t = 2\pi t/M$ and compute the vector $F = f_H(\theta), ..., f_H(\theta_{M/2})$, where

$$f_H(\theta) = \frac{1}{\pi} sin(\pi H)\Gamma(2H+1)(1-cos\theta)\sum_{i\in N} |2\pi t + \theta|^{-2H-1},$$

and $fH(\theta)$ is the spectral density of fractional Gaussian noise.

2. Generate $M/2$ i.i.d. exponential (exp(1)) random variables $E_1, ..., E_{M/2}$ and $M/2$ i.i.d. uniform $(U[0,1])$ random variables $U_1, ..., U_{M/2}$.

3. Compute $Z_t = exp(2i\pi U_t)\sqrt{F_t E_t}$, for $t = 1, ..., M/2$.

4. From the M-vector: $\widetilde{Z} = (0, Z_1, ..., Z_{(M/2)-1}, Z_{M/2}, \bar{Z}_{(M/2)-1}, ..., \bar{Z}_1)$.

5. Compute the inverse FFT of the complex Z to obtain the simulated sample path.

Using the Fast Fourier Transform (FFT) algorithm, Stoev and Taqqu [Sto04] provide an efficient method for simulation of a class of processes with symmetric α-stable (SαS) distributions, namely the linear fractional stable motion (LFSM) processes. The paths of the LFSM process are generated by using Riemann-sum approximations of its SαS stochastic integral representation. They introduce parameters $n, N \in \aleph$ and express the fractional stable noise $Y(t)$ as

$$Y_{n,N}(t) := \sum_{j=1}^{nN}\left(\left(\frac{j}{n}\right)_+^{H-(1/\alpha)} - \left(\frac{j}{n}-1\right)_+^{H-(1/\alpha)}\right) L_{\alpha,n}(nt-j), \tag{22}$$

Where $L_{\alpha,n}(t) := M_\alpha((j+1)/n) - M_\alpha(j/n)$, and $j \in \Re$. The parameter n is mesh size and the parameter M is the cut-off of the kernel function. The authors use the Fast Fourier Transformation (FFT) algorithm for approximating $Y_{n,N}(t)$. Consider the moving average process $Z(m), m \in \aleph$,

$$Z(m) := \sum_{j=1}^{nM} g_{H,n}(j) L\alpha(m-j), \tag{23}$$

where

$$g_{H,n}(j) := \left(\left(\frac{j}{n}\right)_+^{H-(1/\alpha)} - \left(\frac{j}{n} - 1\right)_+^{H-(1/\alpha)} \right) n^{-1/\alpha}, \tag{24}$$

and $L_\alpha(j)$ is the series of i.i.d. standard stable Paretian random variables. Since $L_{\alpha,n}(j) \stackrel{d}{=} n^{-1/\alpha} L_\alpha(j)$, where $j \in \Re$, then the letter equations (23) and (24) imply that $Y_{n,N}(t) \stackrel{d}{=} Z(nt)$, for $t = 1, ..., T$. Let $\tilde{L}_\alpha(j)$ be the $n(N+T)$-periodic with $\tilde{L}_\alpha(j) := L_\alpha(j)$, for $j = 1, ..., n(N+T)$ and let $\tilde{g}_{H,n}(j) := g_{H,n}(j)$, for $j = 1, ..., nN$, $\tilde{g}_{H,n}(j) := 0$, for $j = nN+1, ..., n(N+T)$. Then

$$\{Z(m)\}_{m=1}^{nT} \stackrel{d}{=} \left\{ \sum_{j=1}^{n(N+T)} \tilde{g}_{H,n}(j) \tilde{L}_\alpha(n-j) \right\}_{m=1}^{nT}, \tag{25}$$

because for all $m = 1, ..., nT$, the summation in equation (23) involves only $L_\alpha(j)$ with indices j in the range $-nN \leq j \leq nT - 1$. Using a circular convolution of the two $n(N+T)$-periodic series $\tilde{g}_{H,n}$ and $\tilde{L}_\alpha$ computed by using their Discrete Fourier Transforms (DFT), the variables $Z(n)$, $m = 1, ..., nT$ (i.e., the fractional stable noise) can be generated.

In the next section, assuming that microstructure noise process follows i.i.d. noise, fractional Gaussian noise, stable noise, and fractional stable noise processes, we examine the behavior of the realized volatility estimators nested in GARCH model and realized correlation estimators as model-free and compare performance of the estimators under different noise assumptions.

4. Behavior on Finite Samples: Simulation Experiments

4.1. Volatility Simulation Set-up and Results

Drost and Werker [Dro96] and Nelson [Nel90] propose a continuous-time diffusion model of the GARCH(1,1) as

$$dp_t = \sigma_t dW_{1,t}, \tag{26}$$

$$d\sigma_t^2 = \theta(\omega - \sigma_t^2)dt + (2\lambda\theta)^{1/2} \sigma_t^2 dW_{2,t}, \tag{27}$$

where $W_{1,t}$ and $W_{2,t}$ represent independent standard Brownian motions. For high-frequency observations, such the model seems suitable. But even high-frequency data are observed discretely. However, Drost and Werker [Dro96] argue that the discretely sampled returns from the continuous-time process defined by Eqs. (26) and (27), satisfy the weak GARCH(1,1) model

$$\sigma^2_{(n),t} = \psi_n + \alpha_n r^2_{(n),t-1/n} + \beta_n \sigma^2_{(n),t-1/n}, \tag{28}$$

where n is the number of observations per day t, and $\sigma^2_{(n),t} \equiv P_{(n),t-1/n}(r^2_{(n),t})$ denotes the best linear predictor of $r^2_{(n),t}$. The relations between discrete and continuous time model parameters are highlighted by Drost and Werker [Dro96]. This implies that the models (27) and (28) are compatible. In turn, the h-period linear projection from the weak GARCH(1,1) model of (28) is simply specified by Baillie and Bollerslev [Bai92] as

$$P_{(n),t}(r^2_{(1/h),t+h}) = P_{(n),t}\left(\left[\sum_{j=1,\ldots,nh} r_{(n),t+j/n}\right]^2\right)$$

$$= \sum_{j=1,\ldots,nh} P_{(n),t}(r^2_{(1/h),t+j/n})$$

$$= \sum_{j=1,\ldots,nh} \left[\sigma^2_{(n)} + (\alpha_n + \beta_n)^j (\sigma^2_{(n),t} - \sigma^2_{(n)})\right]$$

$$= nh\sigma^2_{(n)} + (\alpha_n - \beta_n)\left[1 - \alpha_n - \beta_n^{nh}\right] \times [1 - \alpha_n - \beta_n]^{-1} (\sigma^2_{(n),t} - \sigma^2_{(n)}), \tag{29}$$

where $\sigma^2_{(n)} \equiv \psi_n(1 - \alpha_n - \beta_n)^{-1}$. The model (29) is what Andersen et al. [And98] and Andersen et al. [And99] established for simulations of realized volatility.

In realized volatility literature, market microstructure noise is usually assumed to follow an i.i.d process. For example, Barndorff-Nielsen, Hansen, Lunde and Shephard [Bar04a], Zhang et al. [Zha05], Bandi and Russell [2005b], and Hansen and Lunde [2006] assume it follows a Gaussian process. We evaluate the simulations of the model (29) with different microstructure noise assumptions including the i.i.d noise, fractional Gaussian noise, stable noise, and fractional stable noise. Different realized volatility estimators are cast in the model (29) with different noise assumptions and then the bias and variance of estimations are calculated. The Gaussian noise is set equal to 1% of the value of the variable of interest. Random variables for simulations are generated according to minute-by-minute frequency for 4 years assuming 252 working days a year. For generating the non-Gaussian noises, the described procedures in last section is followed based on minute-by-minute frequency observations of CAC 40 and FTSE 100 which will be described in subsection 5.1. The number of sample paths for all simulations is 15,000 realizations. Regarding to two-scale based estimators we allow the sampling observations to be regularly allocated. For all three alternative estimators, we assume equally-distant sampling interval. Three estimators including TSRV, RA, and TSAV (among them the two latter estimators are the RP and TSPV

Table 1. Results of volatility simulations assuming different noise (simulated based on CAC data)

Assumptions	$\widehat{TSRV}$		$\widehat{RA}$		$\widehat{TSAV}$	
	RMSE	Bias	RMSE	Bias	RMSE	Bias
White noise	0.001612	1.686e-005	0.001125	1.749e-005	0.000853	1.383e-005
Fractional Gaussian noise	0.001487	1.675e-005	0.001060	1.688e-005	0.000820	1.347e-005
Stable noise	0.001489	1.676e-005	0.001072	1.692e-005	0.000821	1.349e-005
Fractional Stable noise	0.001306	1.502e-005	0.000885	1.644e-005	0.000819	1.318e-005

Table 2. Results of volatility simulations assuming different noise (simulated based on FTSE data)

Assumptions	$\widehat{TSRV}$		$\widehat{RA}$		$\widehat{TSAV}$	
	RMSE	Bias	RMSE	Bias	RMSE	Bias
White noise	0.002859	1.113e-004	0.002125	1.750e-004	0.001847	1.002e-005
Fractional Gaussian noise	0.002853	1.107e-004	0.002104	1.750e-004	0.001823	9.993e-006
Stable noise	0.002731	9.685e-005	0.001873	1.607e-004	0.001765	9.885e-006
Fractional Stable noise	0.002548	7.071e-005	0.001546	1.418e-004	0.001508	8.453e-006

estimators of power 1) are compared. But what is more important here is the comparison of different microstructure noise assumptions. For evaluation of estimators, we use RMSE and Bias statistics.

In terms of RMSE and Bias of estimation, the results of Monte Carlo simulations are contained in Table 1. A horizontal comparison of different volatility estimators is an indication of different consistency of the estimators. In general, the TSAV estimator yields less variation and bias than others at minute-by-minute simulated frequency. This is in line with the results of [Saf07]. Table 2, which provides the results of simulations using the simulated noise values based on 1 minute frequency real data of FTSE, yields the same results. However, a vertical comparison of the values contained in the tables is more important purpose of the paper. Both the tables report that the GARCH(1,1) model assuming the fractional stable noise outperforms the other models assuming the White noise, Fractional Gaussian noise, and Stable noise. Among the models with different noises, the model assuming White noise has the worst results of fitting. These results are consistent with those of obtained by Sun, Rachev, and Fabozzi [Sun07].

4.2. Correlation Simulation Set-up and Results

Meddahi [Med02] and Barndorff-Nielsen and Shephard [Bar04b] suggest actual correlation estimator used for simulation study as follows

$$\frac{\int_{t-1}^{t} \Sigma_{xy}(s)ds}{\sqrt{\int_{t-1}^{t} \Sigma_{xx}(s)ds \int_{t-1}^{t} \Sigma_{yy}(s)ds}},$$

where Σ_{xy} represents actual covariance and Σ_{xx} shows the variance. The differences between realized correlation estimators (13), (14), and (17) and the actual correlation represent

Table 3. Results of correlation simulation

Assumptions	$\widehat{RSCOR}_{xy}$		$\widehat{RACOR}_{xy}$		$\widehat{TSACOR}_{xy}$	
	RMSE	Bias	RMSE	Bias	RMSE	Bias
White noise	0.003841	5.478e-005	0.003341	4.654e-004	0.002760	2.783e-005
Fractional Gaussian noise	0.003839	5.478e-005	0.003317	4.652e-004	0.002760	2.744e-005
Stable noise	0.003802	5.463e-005	0.003321	4.657e-004	0.002764	2.346e-005
Fractional Stable noise	0.003754	5.418e-005	0.003206	4.574e-004	0.002608	1.837e-005

errors of realized correlation estimators in estimating the true correlation. For correlation simulation we follow Meddahi [Med02] and Barndorff-Nielsen and Shephard [Bar04b] and estimate the errors between realized correlation and actual correlation estimators. Obviously, this set-up for simulation is model free.

As for microstructure noise, the size of the standard deviation of the White noise is set again equal to 1% of the generated data. Other types of noise have been previously simulated and used for volatility estimators based on minute-by-minute real CAC and FTSE data that are used here for correlation simulation. Other conditions for volatility simulations are held.

According to Table 3, the White noise which is usually assumed when modeling realized volatility, has the worst errors in terms of RMSE and Bias of estimation. Instead, models with Fractional Stable noise have the best performance of estimation. This fact is true for the three correlation estimators.

In the following section, distributional and dynamic properties of estimators will be experimentally compared. Since there exists no two-scale realized squared correlation, we compare the results of our measures with realized squared based correlation.

5. Empirical Behaviors of the Estimators

5.1. Data Description

The empirical behaviors of the described realized volatility and correlation estimators are studied here utilizing CAC40 and FTSE100 stock index data at every 1 minute frequency. The sample indices cover a period of shorter than 4 calendar years from Oct. 27, 2003 to July 10, 2007. This period totally includes 929 trading days with 436425 observations.

A few important basic statistics of returns or increments on index series are appeared in Table 4. The mean of the returns are positive at 1 minute frequency, implying an average positive return over this period at minute by minute frequency for investors who have invested in an assumed portfolio of index. The table reports excess kurtosis much higher than 3 (coefficients equal to 524 for CAC and to 434 for FTSE). The distributions of the time series are highly leptokurtic. Simply, this is an indication of heavy tail in distribution of the time series. This conveys that there is a higher probability for extreme events to appear in our data sample than that in normally distributed data sample. Also, departure from symmetry in distribution of the time series can be observed clearly by the skewness in the coefficient reported in the table equal to -0.807 and -0.375 for CAC and FTSE. They are negative, meaning that negative extreme values which are more common than positive extreme values

Table 4. Some descriptive statistics and test of return of indices

Statistic	CAC40	FTSE100
Minimum	-2.74e-02	-2.16e-02
Maximum	3.15e-02	1.87e-02
Mean	1.33e-06	1.03e-06
Median	0.00e+00	0.00e+00
Sum	5.89e-01	4.54e-01
Variance	1.37e-07	8.20e-08
Skewness	-8.07e-01	-3.75e-01
Kurtosis	5.24e+02	4.34e+02
Jarque-Bera test of normality	2.2e-16	2.2e-16
Hurst exponent by Whittle Estimator	5.17e-01	5.24e-01

are generally populated in the left side of the distribution. The popular Jarque-Bera test of normality simply indicates that the investigated time series with p-value equal to 2.2e-16 at 95 percent of confidence do not form a normal distribution. The Chi squared with 2 degrees of freedom in Jarque-Bera test of normality is 5.06e+09 and 3.46e+09 for CAC and FTSE, respectively. Finally, estimated Hurst exponents approximated by the Whittle [Whi63] Estimator[2] equal to 0.517 (std=0.00037) and 0.524 (std=0.00029) imply almost a random walk and no memory in returns series of CAC and FTSE. These facts are in line with common sense. As Rachev et al. [Rac05] state empirical evidence does not support the assumption that many important variables in finance follow a normal distribution.

5.2. Unconditional and Conditional Distributions

We now turn our attention to the distribution of volatility and correlation time series constructed based on different estimators. It would be interesting if we could observe some of those important facts that have been documented in the literature of financial time series over a long history of research here in high-frequency based time-varying volatilities and correlations, too.

Figure 1 depicts the time series of realized volatility estimators. Some periods of market turmoil with higher volatility on the time series are evident. Over these market turmoils, CAC and FTSE almost simultaneously display high and low volatilities. Hence two markets look to be interdependent on volatility. Simply squared based volatility tends to report volatility much smaller than two others on average, as can be seen in Table 5. However, the means of realized volatility approximated by absolute based estimators are close. In comparison, all estimators show a mean of volatility reported in Table 5 much higher than a constant variance of returns reported in Table 4.

[2]Throughout the paper the Whittle Estimator is approximated by the FARIMA model.

Figure 1. Time series of realized volatility measures constructed based on two-scale squared, absolute, and two-scale absolute estimators. Volatility can be observed as time-varying. Two markets display almost simultaneously high and low volatilities. Hence they look to be interdependent on volatility.

Figure 2. Kernel density distributions of different realized volatility series look skewed rightward. The shapes are not exactly the same. The tails of squared based volatility of series are heavier than others.

Table 5. Basic statistics and tests of realized volatility measures

Statistic	CAC			FTSE		
	$\widehat{TSRV}$	$\widehat{RA}$	$\widehat{TSAV}$	$\widehat{TSRV}$	$\widehat{RA}$	$\widehat{TSAV}$
Mean	6.49e-05	9.33e-02	1.30e-01	3.89e-05	7.66e-02	8.94e-02
Median	4.38e-05	8.74e-02	1.22e-01	2.69e-05	7.27e-02	8.49e-02
Variance	5.83e-09	7.63e-04	1.49e-03	1.82e-09	3.81e-04	5.19e-04
Skewness	5.66e+00	2.11e+00	2.11e+00	5.53e+00	2.87e+00	2.92e+00
Kurtosis	4.87e+01	6.85e+00	6.82e+00	4.41e+01	1.39e+01	1.48e+01
Jarque-Bera	2.2e-16(97415)	2.2e-16(2527)	2.2e-16(2508)	2.2e-16(80431)	2.2e-16(8874)	2.2e-16(9941)
Hurst via Whittle	0.541(4.8e-09)	0.572(0.00033)	0.576(0.00064)	0.551(1.4e-09)	0.591(0.00017)	0.596(0.00023)

According to Table 5, the unconditional distributions of volatilities are positively skewed. The kurtosis coefficients are all higher than that of a normal distribution. Meanwhile, this coefficient for squared based volatility is much higher than that of others. Positive extreme values in volatility time series, which are the reason for high leptokurtosis, lead their unconditional distributions to possess a right heavy tail. These heavy right tails can be seen in Figure 2. Exploiting 5 minute exchange rate data, Andersen et al. [And01b] found that the distributions of realized daily variances are skewed to the right and leptokurtic. In line with these findings, using 5 minute stock exchange data, Andersen et al. [And01a] also confirm that the unconditional distributions of realized variances are highly right-skewed. With p-value of 2.2e-16, null hypothesis of normality for all volatility series is significantly rejected with the Jarque-Bera test. The values in parenthesis reported in the table for Jarque-Bera test is Chi squared with 2 degrees of freedom.

Plots in Figure 3 are more informative and compare the empirical distribution with that of the normal. The size of discrepancy from bisector represents deviation from normality. In the right tail, the discrepancy is much higher.

Regarding to different estimators for multivariate conditional correlations, it is easy to study dependence structure between CAC and FTSE. Now with respect to theoretical developments, we are able to empirically investigate how high-frequency correlations between the markets change over the time. The aforementioned estimators of correlation are applied on the returns series described in Table 3. For the purpose of comparison, we proceed with a squared based, absolute based, and two-scale realized correlation estimators.

After estimation of dependence structure between returns of the markets, the descriptive statistics of time-varying realized correlation series are given in Table 6. On average, there exists a positive dependence between two returns of the markets, meanwhile the dependency is reported by squared based correlation so stronger than absolute based ones. According to the constant variance of correlation series, squared based correlation highly fluctuates around its mean. Graphically, the fluctuations of correlations are shown in Figure 4, the top plots. With regard to our 1 minute frequency data, we found that there is a slightly asymmetry to the right side of distribution of squared based correlation whereas absolute based correlations include negative asymmetry in their distributions. This finding is consistent with that of Safari and Seese [Saf07] where a relationship between the returns of NASDAQ and DAX at 5 minute frequency was investigated. While the distribution of squared based correlation looks platykurtic, the distribution of others indicates excess kurtosis and therefore existence of fat tail. Since there are some extreme negative values in absolute based correlation series and they dominate positive extreme values, the heavy tails present in left side of the distributions. Kernel density plots in Figure 4 exhibit the heavy tails. Longer negative tail in multivariate absolute based realized correlations can be documented in such a way that the extreme values are usually populated in the left tail of the distributions as can be observed in QQ plots of Figure 4. Jarque-Bera test for null normality reveals statistically none of distributions pose normality at the 5 percent level of significance.

Presence of negative asymmetry in correlation conveys the meaning that negative shocks in returns have greater impact than positive shocks in CAC and FTSE markets. In the other words, downside comoves are greater than upside comoves between markets.

Negative asymmetry in the correlation between stock markets and even between stock

Figure 3. Quantile Quantile-normal plots compare the empirical volatility series to the theoretical distribution. More matching to the straight line means more approaching to normality. The right tails are easily observable by longer distance from the straight line.

market and other markets (i.e. bond market) has been documented by several researchers in different ways. Kroner and Ng [Kro98] introduce Asymmetric Dynamic Covariance (ADC) model which encompasses several popular multivariate GARCH models within its framework while allowing for asymmetric effects to appear within the conditional variance-covariance process. Cappiello, Engle and Sheppard [Cap06] propose a new generalized autoregressive conditional heteroskedastic process, the asymmetric generalized dynamic conditional correlation (AG-DCC) model, to permit conditional asymmetries in correlation dynamics. Using equity returns, Longin and Solnik [Lon01] derive the distribution of extreme correlation for a wide class of return distributions. Empirically, they reject the null hypothesis of multivariate normality for the negative tail, but not for the positive tail. They suggest that correlation increases in bear markets, but not in bull markets.

According to Rachev [Rac05] actually correlation is one particular measure of depen-

Table 6. Basic statistics and test of realized correlations

Statistic	$\widehat{RSCOR}_{xy}$	$\widehat{RACOR}_{xy}$	$\widehat{TSACOR}_{xy}$
Mean	4.47e-04	3.98e-07	2.11e-07
Median	1.88e-03	6.83e-07	3.16e-07
Variance	1.94e-03	5.27e-10	1.21e-10
Skewness	3.06e-02	-8.86e-01	-3.68e-02
Kurtosis	8.00e-01	9.66e+00	5.07e+00
Jarque-Bera test	2.9e-06(25.5)	2.2e-16(3769.9)	2.2e-16(1003.4)
Hurst via Whittle	0.513(7.00e-05)	0.558(6.64e-05)	0.563(9.01e-05)

dence among many. Another approach is to model dependency using copulas. From copula theory, we find that for continuous multivariate distribution functions, the univariate margins and the multivariate dependence structure can be separated, and the dependence structure can be represented by a copula. Embrechts et al. [Emb03] argue that copulas provide a natural way to study and measure dependence between random variables. Exploiting copula theory, Patton [Pat04] construct models of the time-varying dependence structure that allow for different dependence during bear markets than bull markets. Stock returns appear to be more highly correlated during market downturns than during market upturns. Patton [Pat06] extended the theory of copulas to allow for conditioning variables for evaluating asymmetry in dependence, and employed it to construct flexible models of conditional dependence structure in the joint density of the some exchange rates. Two different copulas were estimated: the copula associated with the bivariate normal distribution and the symmetrized Joe-Clayton copula, which allows for general asymmetric dependence. Time variation in the dependence structure between the two exchange rates was captured by allowing the parameters of the two copulas to vary over the sample period. He found evidence that the mark-dollar and yen-dollar exchange rates are more correlated when they are depreciating against the dollar than when they are appreciating.

5.3. Dynamics of Volatility and Correlation Series

For this empirical section, it is attempted to understand in details some facts of human activity in the financial markets that are particularly well documented for financial data and more recently for high-frequency time series. An important task of related investigations is to unveil several stylized facts of financial markets, which consistently appear on different markets and in different periods of time, and that any candidate model should convincingly explain. With regards to the fact that realized volatility and correlation estimators are in principle model free, in bellow the presence or absence of some of these stylized facts in time-varying volatilities and correlations series is empirically investigated.

Either a slow decay in autocorrelation of volatility series can be reported or not, left panels on Figures 5 and 6 exhibit the autocorrelation plots for CAC and FTSE. Autocorrelations for squared based volatility series in both CAC and FTSE last to almost 20 lags, almost one calendar month (regarding to almost 22 working days in a month), in the meantime autocorrelations in absolute based volatility series still remain meaningful until more than 40 lags, almost two calendar months. This important fact conveys that a shock in the volatility process will have a long-lasting impact. As a matter of fact, autocorrelation may be an indication of a long memory process. As such, long memory may be an interesting signature for series dynamics. If the decay in the autocorrelation function is slower than a hyperbolic rate, i.e., the correlation function decreases algebraically with increasing integer lag, then the time series would possess a long memory. Thus, it makes sense to investigate the decay on a double logarithmic scale and to estimate the decay exponent. The right panels on the figures display long memory behavior of the estimators in volatilities constructed on minute-by-minute frequency series. Graphically, if the time series exhibits long memory behavior, it can be seen as a straight line in the plot on the right panels of Figures 5 and 6. Because of longer meaningful autocorrelation in absolute based series, it is visible that the corresponding log-log plots of long memory decays slower than that of squared based estimator. Reported in Table 4, the Hurst Exponent estimated through the Whittle estimation for two-scale squared based, absolute based, and two-scale absolute based estimators are equal to 0.541, 0.572, and 0.576 and to 0.551, 0.591, and 0.596 for CAC and FTSE respectively. The values in parenthesis are standard deviation. Consistent with Andersen et al. [And99], there is indeed an evidence to suggest that volatility is a long memory process.

Figure 7 seeks for possible regular patterns in dependency structure across equity markets among the correlation estimators. If the dependency structure between the markets obeys squared based model of correlation, then there does not exist autocorrelation and long memory in dependency of the markets, as the top plots of the figure state. Meanwhile, if the dependency patterns are ruled by absolute based models of correlations in real world, although not too long but the correlation series include somewhat autocorrelation and long memory process implying regular behavior of dependency. This can be seen in lower plots of the figure. Reported in Table 5, the Hurst Exponents with the values of 0.513, 0.558, and 0.563 for squared, absolute and two-scale absolute based correlations indicate differences.

6. Conclusion and Discussion

The consistency of volatility estimators differs given they are constructed differently. This fact is valid for correlation estimators as well. Actually, squared based estimators are more sensitive to the larger values. Moreover, adopting the two-scale approach when the construction of the estimators corrects the bias of the estimation. Therefore, the performance of the two-scale power estimator dominates the performance of the others. The simulation studies revealed these facts.

Different assumptions about the type of microstructure noise have a meaningful impact on the accuracy and bias of the estimators nested in a continuous GARCH process. All estimators are affected by the assumptions and almost all the estimators react identically to the change in assumptions. This implies that all the estimators are almost identically sensitive to the changes in assumptions about the type of noise and they act in the same way when the assumptions change. Meanwhile, all estimators perform the best when the fractional stable noise assumption is adopted and the worst when the i.i.d. assumption about noise is made. This turns out that the underlying microstructure noise process actually follows fractals and a stable distribution. As such, there exists self-similarity in the real world microstructure noise including long range dependence as well as heavy tailedness in minute-by-minute stock index data.

Heavy tailedness in volatility estimators is empirically obvious. The volatility estimators unveil some dynamic behavior investigated by the Hurst parameter. In particular, absolute based correlation estimators are consistent with the other studies described here to reveal negative asymmetry in dependence structure. Like copulas, the absolute based correlations indicate a negative asymmetry in dependence between the markets conveying that negative shocks in the returns have greater impact than positive shocks. In the other word, downside comoves are greater than upside comoves between the markets.

In our simulations using minute-by-minute frequency data, the most realistic noise assumption was revealed to be the fractional stable noise. But, if market microstructure noise behaves differently at different frequencies, the problem of microstructure noise would be more complicated. For example, tick-by-tick frequency data might show another type of noise process and might not include fractals. Even the coarser frequencies might approach to other types of noise. These need further investigations.

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Figure 4. Some characteristics of distribution of realized correlations are graphically depicted. Obviously realized correlations, based on first row plots, oscillate over the time. Negative asymmetry is present in absolute based correlations. Longer left tail in dependence structure of the absolute based correlations is obvious considering density plots.

Figure 5. Autocorrelation function and long memory autocorrelation function plots of volatility series, estimated based on CAC time series. For autocorrelation functions and long memory autocorrelation function, the number of lags is arbitrarily selected equal to 200. Left plots are autocorrelation functions and right ones are long memory autocorrelation functions. Obviously plots suggest that realized volatility is a long memory process, although absolute based estimators confirm this suggestion stronger.

Figure 6. Autocorrelation function and long memory autocorrelation function plots of volatility series, estimated based on FTSE time series. For autocorrelation functions and long memory autocorrelation function, the number of lags is arbitrarily selected equal to 200. Obviously, plots suggest that realized volatility is a long memory process, although absolute based estimators confirm this suggestion stronger.

Figure 7. Autocorrelation function and long memory autocorrelation function plots (ACF and log-log) of correlations between CAC and FTSE. For autocorrelation function and long memory autocorrelation function, the number of lags is arbitrarily selected equal to 200. The top row plots, which belong to squared based correlation, do not indicate autocorrelation and long memory while the others do somewhat.