Calibrating affine stochastic mortality models using insurance contracts premiums

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Abstract

In this paper, we focus on the calibration of affine stochastic mortality models using insurance contracts premiums. Viewing insurance contracts as "market products," we propose fitting stochastic models on the quotes of insurance policies. For this purpose, insurance contracts are viewed as a "swap" in which policyholders exchange cash flows (premiums vs. benefits) with an insurer analogous to a generic interest rate swap or credit default swap. Using a simple bootstrapping procedure, we derive the term structure of mortality rates from a stream of contract quotes with different maturities. This term structure is used to calibrate the parameters of affine stochastic mortality models where the survival probability is expressed in closed form. The Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models are considered for fitting the survival probabilities term structure. An evaluation of the performance of these models is provided with respect to representative premiums of Italian insurance contracts for individuals at different ages.

keyword: affine stochastic models, bootstrapping, calibration, stochastic force of mortality, mortality risk, insurance contracts, Vasicek model, Cox-Ingersoll-Ross model, jump-extended Vasicek model.

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Contents

1 Introduction

Recently, stochastic mortality models have received increased attention among practitioners and academic researchers. The introduction of market-consistent accounting and risk-based solvency requirements has called for the integration of mortality risk analysis into stochastic valuation models.[1] Furthermore, the issuance of mortality-linked securities requires stochastic models to price financial instruments related to demographic risks.

Several proposals for modeling and forecasting mortality rates have appeared in the literature. The leading statistical model of mortality forecasting in the literature is the one proposed by Lee and Carter (1992). The use of the Lee-Carter model or one similar to it was recommended by two U.S. Social Security Technical Advisory Panels. There is further support for this model in other countries.[2]

However, an important body of literature regarding models that describe death arrival as the first jump time of a Poisson process with stochastic force of mortality appeared since the turn of the century.[3] In these models, the same mathematical tools used in interest rate and credit risk modeling are applied, yielding a closed-form solution for survival probabilities. Milevsky and Promislow (2001) were the first to propose a stochastic force of mortality model and several stochastic affine models are considered assuming mean-reverting characteristics.[4] However, Cairns, Blake and Dowd (2006) suggest that affine stochastic models need to incorporate non-mean reverting elements; Luciano and Vigna (2005) propose non-mean reverting affine processes for modeling the force of mortality. In these models, the deterministic part of the mortality rate process increases exponentially in a consistent manner with the exponential growth that is the main feature of the Gompertz model.[5]

A fundamental issue in the use of any stochastic mortality model deals with the quantification of the parameters. Parameter estimation can be based on historical data employing various statistical procedures.

In a distinctive manner with respect to the historical estimation approach, we develop a new procedure to calibrate the parameters of affine stochastic mortality models using insurance contracts premiums. The approach that we propose is new in that it employees an historical estimation procedure. The fundamental idea is using real quotes of mortality-dependent products to provide the parameters calibration starting from the quotes of simple insurance products such as term assurance, pure endowment, and life annuities contracts. We consider these insurance contracts as "market products" where the pricing function is

[1] See European Community (2009), IASB (2007), and IAIS (2006) for further information.

[2] For alternative models, see Lee (2000), Yang (2001), Cairns, Blake, and Dowd (2005), Lin and Cox (2005), Renshaw and Haberman (2003), Brouhns, Denuit and Vermunt (2002), and Giacometti, Ortobelli and Bertocchi (2009).

[3] In actuarial science, force of mortality represents the instantaneous rate of mortality at a certain age measured on an annualized basis. It is identical in concept to the failure rate or hazard function.

[4] See Dahl (2004), Biffis (2005), Denuit and Devolder (2006), and Schrager (2006).

[5] The model is based on the Gompertz law (1825) founded on the biological concept of organism senescence, in which mortality rates increase exponentially with age.

similar to the pricing function of an interest rate swap or credit default swap.

An important step in the proposed model involves deriving the term structure of mortality rates by means of a new bootstrapping technique, a procedure similar to bootstrapping of the yield curve. Then, the term structure of mortality rates is used to calibrate the parameters of affine stochastic mortality models by means of an optimization procedure because the survival probability implied in the affine models is expressed in closed form. For this purpose, the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models, without mean reversion, are used in the calibration procedure in which the deterministic part of the model increases exponentially. Representative premiums of Italian insurance contracts for individuals at different ages 20, 40, and 60 are used for our empirical testing.

In Section 2, we recall several mortality measures and review some methods developed to model mortality risk. The proposed model is described in Section 3 and the empirical results are presented in Section 4. The conclusions are provided in Section 5.

2 Modeling of mortality risk

A standard measure of mortality is the probability in t that an individual aged x dies within the period $[t, T]$ with $t < T$. This probability is denoted by $D_x(t, T)$. Given the probability in t that an individual aged x dies within the period $[T_{i-1}, T_i]$ holds that,

$$D_x(T_{i-1}, T_i) = D_x(t, T_i) - D_x(t, T_{i-1}). \tag{1}$$

Starting from the probability $D_x(t, T)$, it is possible to define the survival probability, denoted by $S_x(t, T)$, which reflects the probability that an individual aged x survives over T. The survival probability is such that

$$S_x(t, T) = 1 - D_x(t, T). \tag{2}$$

Another measure of mortality is the so-called *force of mortality*. This is the instantaneous death rate for lives aged x. Whereas $D_x(t, T)$ is the death rate over the discrete period $[t, T]$, the force of mortality is the instantaneous death rate. The concept of force mortality is identical in concept to the failure rate or hazard function.
In this paper, reduced form models are considered to model the death event, describing death arrival as the first jump time of a Poisson process with deterministic and stochastic force of mortality.

2.1 Time-homogeneous Poisson process: deterministic and piecewise constant force of mortality or mortality rate

Time-homogeneous Poisson process can be considered to model mortality risk, where the force of mortality is deterministic and piecewise constant. This quantity is noted in literature also as *mortality rate*.
Considering an individual aged x at time t and denoted the mortality rate with

maturity T by $\mu_x(t,T)$, the survival probability at time t with respect to the interval $[t,T]$ is,

$$S(t,T) = e^{-\mu_x(t,T)\tau(t,T)}, \tag{3}$$

where the quantity $\tau(t,T)$ is the time measure as year fraction between the dates t and T according to same convention.

Consequently, it is possible to express the mortality rate as,

$$\mu_x(t,T) = -\frac{log[S_x(t,T)]}{\tau(t,T)}. \tag{4}$$

2.2 Double stochastic Poisson process (Cox process): stochastic force of mortality

Assuming *Double stochastic Poisson process (Cox process)*, the force of mortality is stochastic.

Considering an individual aged x at time t and denoting with $\mu_x(t)$ the *stochastic force of mortality*, the survival probability at time t of an individual aged x will be,

$$S(t,T) = E\left[exp\left(-\int_t^T \mu_x(u)du\right)\middle|G_t\right], \tag{5}$$

where G_t describe the information at time t.

Provided that affine stochastic mortality models are used in modeling of mortality risk, the approach leads to an analytical representation of survival probability.

2.3 Affine stochastic models (without mean reversion) as mortality models

We employ the same affine stochastic models used in interest rate and credit risk modeling to model the force of mortality.[6] Affine stochastic models such as the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models, incorporating non-mean reversion, are analyzed in the calibration procedure. In fact, as suggested by Cairns, Blake, and Dowd (2006) and as demonstrated by Luciano and Vigna (2005), non-mean reverting affine processes with a deterministic component that increases exponentially seems to be an appropriate description of human mortality where the exponential growth is consistent with the Gompertz model.

In our analysis, non-mean reverting affine models are considered by imposing (1) a long-term average rate parameter equal to zero and (2) a mean-reversion speed parameter constrained to be strictly negative. It is important to note that modeling the force of mortality as an affine function leads to the analytical

[6] See Brigo and Mercurio (2006) and Nawalkha, Beliaeva and Soto (2007) for further details.

representations of survival probabilities with closed-form solution.

We provide a calibration procedure for each of the models that are explained below employing the related closed formula for survival probability.

2.3.1 Vasicek model

For the Vasicek model, we assume that the force of mortality follows the stochastic differential equation

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dW(t),$$

with $\mu_x(0)$ and σ positive constants, θ equal to zero, and k constrained to be strictly negative. The main drawback of this process is that the force of mortality can be negative with positive probability. For this model, the survival probability can be obtained by

$$S_x(t,T) = A(t,T)e^{-B(t,T)\mu_x(t)}$$

$$B(t,T) = \frac{1}{k}\left[1 - e^{-2k\tau(t,T)}\right]$$

$$A(t,T) = e^{\left(\theta - \frac{\sigma^2}{2k^2}\right)\left(B(t,T) - \tau(t,T)\right) - \frac{\sigma^2}{4k}B(t,T)^2}$$

2.3.2 Cox-Ingersoll-Ross model

Assuming the dynamic of the Cox-Ingersoll-Ross (CIR) model, the force of mortality $\mu_x(t)$ satisfies

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dW(t),$$

with $\mu_x(0)$ and σ positive constants, θ equal to zero, and k constrained to be strictly negative. The principal advantage of the CIR model over the Vasicek model is that the hazard rate is guaranteed to remain non-negative. However, the condition $2k\theta > \sigma^2$ is not applicable and the hazard rates can be equal to zero with positive probability. Survival probabilities can still be computed analytically and are given by

$$S_x(t,T) = A(t,T)e^{-B(t,T)\mu_x(t)}$$

$$\gamma = \sqrt{k^2 + 2\sigma^2}$$

$$A(t,T) = \left[\frac{2\gamma e^{\frac{1}{2}(k+\gamma)\tau(t,T)}}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}$$

$$B(t,T) = \frac{2(e^{\gamma\tau(t,T)} - 1)}{2\gamma + (k+\gamma)(e^{\gamma\tau(t,T)} - 1)}$$

2.3.3 Jump-extended Vasicek model

The jump-extended Vasicek model implies the following process to model the force of mortality

$$d\mu_x(t) = k(\theta - \mu_x(t))dt + \sigma dW(t) + dJ(t),$$

where J is a pure jump process with jump arrival rate λ and exponential distribution with mean η. In this model, $\mu_x(0)$, σ and λ are positive constants, θ is equal to zero, and k is constrained to be strictly negative.[7] Also in this case, survival probabilities can be computed analytically as

$$S_x(t,T) = e^{A(t,T)-B(t,T)\mu_x(t)}$$

$$A(t,T) = \big[\tau(t,T) - B(t,T)\big]\big(\frac{\sigma^2}{2k^2}\big) +$$
$$-\frac{\sigma^2 B(t,T)^2}{4k} - \lambda\tau(t,T) + \frac{\lambda\eta}{k\eta+1}\log\left|\left(1+\frac{1}{k\eta}\right)e^{k\tau(t,T)} - \frac{1}{k\eta}\right| - \theta\tau(t,T)$$

$$B(t,T) = B(t,T) = \frac{1}{k}\left[1 - e^{-2k\tau(t,T)}\right]$$

3 Proposed model for calibrating affine stochastic mortality models using insurance contracts premiums

We develop in this section our model to calibrate affine stochastic mortality models using quotes of insurance contracts. Viewing these contracts as "market products," we propose a new bootstrapping procedure to derive the term structure of mortality rates implied in the contracts. For this purpose an insurance contract is expressed as a swap in which the policyholder (or the investor) exchanges with the insurer (or a new counterparty) the premium payments against the contingent benefit payment.[8] The term structure of mortality rates is then used to calibrate the parameters of affine stochastic mortality models by means of an optimization procedure.

In summary, our proposed model involves:

- starting from real quotes of mortality-sensitive products;
- re-writing the pricing function of these contracts in terms of a swap;

[7] The model allows jumps with a positive size, in which case the mortality increases (in the case of wars, for instance), or jumps with a negative size, in which case mortality decreases (in the case of medical advancements, for instance).

[8] In the "Life Settlement" market a policy can be sold by the contract's policyholder to a third party (an investor). Transactions of this type, referred to as viatical settlements, have been available in the United States since 1911; the volume of these transactions was roughly $18 to $19 billion in 2009. This form of investment is still underdeveloped in Europe, where it is accessible only through hedge funds, structured products, and funds of funds for qualified investors. See *The Economist* (2009) and United States Senate (2009) for further information.

- providing a bootstrapping procedure to construct the mortality rates term structure;
- calibrating the parameters of affine stochastic mortality models on the bootstrapped term structure.

3.1 Insurance contracts as swap: pricing functions

3.1.1 Term assurance

Term life insurance or term assurance is a life insurance contract which provides coverage with respect to payments for a limited period of time. Although this form of insurance can have a fixed payment or one that changes over time, here we only consider the fixed payment case. If the insured dies during the term, the death benefit C will be paid to the beneficiary; the policy does not provide any benefit if the insured does not dies during the term.

As explained above, a term assurance can be considered a swap in which policyholders exchange cash flows (premiums vs. benefits) with an insurer just as with a generic interest rate swap or credit default swap. The policyholder pays to an insurer a constant annual premium Q (or a single premium U) to ensure the life of an individual aged x (ensured) against the death event during a certain number of years. The beneficiary of the contract receives a fixed amount C in the case of the insured's death. We assume that the payment related to the effective death time is postponed to the first discrete time T_i.

Given a set of n annual payments at discrete time $T_1, T_2, ...T_n$, the present value of the swap at time $t < T_1$ is the difference between the present value of the premium leg, indicated with $PV^{ta}_{pml}(t)$, and the present value of the protection leg, indicated by $PV^{ta}_{prl}(t)$,

$$PV^{ta}_{pml}(t) = Q \sum_{i=1}^{n-1} P(t, T_i)\tau(T_{i-1}, T_i)S_x(t, T_i), \tag{6}$$

and

$$PV^{ta}_{prl}(t) = C \sum_{i=1}^{n} P(t, T_i)D_x(T_{i-1}, T_i), \tag{7}$$

where $P(t, T_i)$ is the risk-free discount factor in the time horizon $[t, T_i]$.[9]

Consequently, the value of the annual premium will be such that,

$$Q = \frac{C \sum_{i=1}^{n} P(t, T_i)D_x(T_{i-1}, T_i)}{1 + \sum_{i=1}^{n-1} P(t, T_i)\tau(T_{i-1}, T_i)S_x(t, T_i)}. \tag{8}$$

At the valuation date, the value of the swap, denoted by $PV_{ta}(t)$, is exactly $-Q$ for the policyholder, namely, in the case of annual payments, the upfront swap payment corresponds to the first premium,

$$PV_{ta}(t) = PV^{ta}_{pml}(t) - PV^{ta}_{prl}(t) = -Q. \tag{9}$$

[9]For the sake of simplicity, although deterministic interest rates are considered, the model can be generalized to the case where interest rates are stochastic.

Considering a contract with a single premium U, the upfront swap payment is equal to the present value of the protection leg,

$$U = PV_{prl}^{ta}(t) = C \sum_{i=1}^{n} P(t, T_i) D_x(T_{i-1}, T_i). \tag{10}$$

Then, for the policyholder (or the investor), a term assurance contract can be viewed as a long position on the death rate to n years: if the n years death rate increases, the fair value of the contract increases for the policyholder. From the prospective of the insurer (or another counterparty), the contract can be viewed as a short position on the n years death rate; consequently, if the death rate decreases, the fair value of the contract increases for the insurer.

3.1.2 Pure endowment

Pure endowment is a life insurance contract which provides coverage with respect to payments for a limited period of time. In this case, if the insured does not dies during the term, a benefit C is payed at the end of the period. Also in this case, although this form of insurance can have a fixed payment or one that changes over time, here we only consider the fixed payment case.

A pure endowment can be considered as a swap where the policyholder, during a period of n years, pays to an insurer a constant annual premium Q (or a single premium U) to ensure the payment of the fixed amount C if the insured of age x survives up to the year n.

Given a set of n annual payments the present value of the premium leg and the present value of a protection leg are

$$PV_{pml}^{pe}(t) = Q \sum_{i=1}^{n-1} P(t, T_i) \tau(T_{i-1}, T_i) S_x(t, T_i), \tag{11}$$

and

$$PV_{prl}^{pe}(t) = P(t, T_n) S_x(t, T_n) C. \tag{12}$$

So, the value of the annual premium will be such that,

$$Q = \frac{P(t, T_n) S_x(t, T_n) C}{1 + \sum_{i=1}^{n-1} P(t, T_i) \tau(T_{i-1}, T_i) S_x(t, T_i)}. \tag{13}$$

Consequently, the present value of the swap at time t, denoted by $PV_{pe}(t)$, is the difference between the present value of the premium leg and the present value of a protection leg,

$$PV_{pe}(t) = PV_{pml}^{pe}(t) - PV_{prl}^{pe}(t) = -Q. \tag{14}$$

The value of the swap at time t for the policyholder is exactly $-Q$, namely, in case of annual payments, the upfront swap payment corresponds to the first premium.

Also in this case, considering a contract with a single premium U, the upfront swap payment is equal to the present value of the protection leg,

$$U = PV_{prl}^{pe}(t) = P(t, T_n) S_x(t, T_n) C. \tag{15}$$

3.1.3 Life annuity

Life annuity is an insurance contract according to which an insurer makes a series of future payments to an insured in exchange for the immediate payment of a lump sum (single-payment annuity) or a series of regular payments (regular-payment annuity), prior to the onset of the annuity.
As in the cases of term assurance and pure endowment, an annuity can be considered as a swap.
For simplicity, we consider a temporary annuity for n years (at the beginning of the year), with respect to a life aged x and a fixed benefit equal to A, where the insurer makes the payments starting from the issue date of the contract.
The value of the protection leg, at the issue date t, will be,

$$PV_{prl}^{la}(t) = A + A\sum_{i=1}^{n-1} P(t,T_i)\tau(T_{i-1},T_i)S_x(t,T_i), \tag{16}$$

In case of a single premium U, we obtain,

$$PV_{pml}^{la}(t) = PV_{prl}^{la}(t) = U. \tag{17}$$

Consequently, the value of the swap is zero,

$$PV_{la}(t) = PV_{pml}^{la}(t) - PV_{prl}^{la}(t) = 0. \tag{18}$$

3.2 Bootstrapping the term structure of mortality rates

3.2.1 Bootstrapping procedure using term assurance premiums

In this section, we derive a term structure of mortality rates from a vector of premiums related to term assurance quotes for different discrete maturities and for each x.

A stream of mortality rates related to the respective maturities represents the term structure of mortality rates. It also can be expressed in terms of the survival probabilities computed according to the relation (3).

From a series of maturities, $T_1, T_2, ...T_n$, we developed a new bootstrapping procedure to obtain a vector of mortality rates, $\mu_x(t,T_1), \mu_x(t,T_2), ...\mu_x(t,T_n)$, that represents the term structure of mortality rates. Suppose that a set of n term assurance contracts is quoted in terms of their annual premiums $Q_1, Q_2, ...Q_n$ with respect to the maturities $T_1, T_2, ...T_n$.[10]
Starting from a term assurance contract with maturities T_1 and setting $T_0 = t$, the pricing formula is

$$-Q_1 = -P(t,T_1)D_x(t,T_1)C.$$

After setting

$$D_x(t,T_1) = 1 - S_x(t,T_1) = 1 - e^{-\mu_x(t,T_1)\tau(t,T_1)},$$

[10]Pure premiums are considered in order to obtain the term structure of mortality rates. In practice, only the part of the premium which is sufficient to pay losses and loss adjustment expenses is considered, but not other expenses. The various types of loading (commission, expenses, taxes, and so on) are ignored.

it follows that

$$-Q_1 = -P(t,T_1)\Big[1 - e^{-\mu_x(t,T_1)\tau(t,T_1)}\Big]C.$$

Solving with respect to $\mu_x(t,T_1)$ we obtain

$$\mu_x(t,T_1) = -\frac{\ln\left[1 - \frac{Q_1}{P(t,T_1)C}\right]}{\tau(t,T_1)}. \tag{19}$$

So, considering a contract with maturity T_2, it is possible to compute $\mu_x(t,T_2)$ given $\mu_x(t,T_1)$ as an input of the following pricing function

$$-Q_2 = P(t,T_1)\tau(t,T_1)S_x(t,T_1)Q_2 - C\sum_{i=1}^{2} P(t,T_i)D_x(T_{i-1},T_i). \tag{20}$$

Based on expressions (2) and (3), expression (20) becomes

$$-Q_2 = P(t,T_1)\tau(t,T_1)e^{\left[-\mu_x(t,T_1)\tau(t,T_1)\right]}Q_2 - CP(t,T_1)\Big[1 - e^{\left[-\mu_x(t,T_1)\tau(t,T_1)\right]}\Big] +$$
$$-CP(t,T_2)\Big[e^{\left[-\mu_x(t,T_1)\tau(t,T_1)\right]} - e^{\left[-\mu_x(t,T_2)\tau(t,T_2)\right]}\Big]. \tag{21}$$

Solving with respect to $\mu_x(t,T_2)$ we obtain

$$\mu_x(t,T_2) =$$
$$-\frac{\left[-Q_2 - P(t,T_1)\tau(t,T_1)Q_2 + P(t,T_1)C + P(t,T_1)C - P(t,T_2)C\right]}{e^{\left[\mu_x(t,T_1)\tau(t,T_1)\right]}CP(t,T_2)\tau(t,T_2)} \tag{22}$$

Iterating the procedure described above up to n, the term structure of mortality rates, $\mu_x(t,T_1), \mu_x(t,T_2), ..., \mu_x(t,T_n,)$, is obtained with respect to the maturities $T_1, T_2, ...T_n$.

3.2.2 Bootstrapping procedure using pure endowment premiums

In this case, we suppose that a set of n pure endowment contracts is quoted in terms of their annual premiums $Q_1, Q_2, ...Q_n$ with respect to the maturities $T_1, T_2, ...T_n$.

Starting from a pure endowment contract with maturities T_1 and setting $T_0 = t$, the pricing formula is

$$-Q_1 = -P(t,T_1)S_x(t,T_1)C.$$

After setting

$$S_x(t,T_1) = e^{-\mu_x(t,T_1)\tau(t,T_1)},$$

and solving with respect to $\mu_x(t,T_1)$, we obtain,

$$\mu_x(t,T_1) = -\frac{\ln\left[\frac{Q_1}{P(t,T_1)C}\right]}{\tau(t,T_1)}. \tag{23}$$

Iterating the procedure described above up to n, the term structure of mortality rates, $\mu_x(t,T_1), \mu_x(t,T_2), ..., \mu_x(t,T_n,)$, is obtained with respect to the maturities $T_1, T_2, ...T_n$ in analogous way with the term assurance case.

3.2.3 Bootstrapping procedure using life annuity single premiums

In this section, we consider a stream of life annuity single premiums, $U_1, U_2, ...U_n$, relative to contracts with different maturities, $T_1, T_2, ...T_n$; in this case the bootstrapping procedure start with the contract with maturity T_2.
Setting $n = 2$ and $T_0 = t$, the pricing formula is

$$U_2 - A - P(t, T_1)\tau(t, T_1)S_x(t, T_1)A = 0.$$

After setting

$$S_x(t, T_1) = e^{-\mu_x(t,T_1)\tau(t,T_1)},$$

and solving with respect to $\mu_x(t, T_1)$, we obtain,

$$\mu_x(t, T_1) = -\frac{\ln\left[\frac{U_2 - A}{P(t,T_1)\tau(t,T_1)A}\right]}{\tau(t, T_1)}. \tag{24}$$

Also in this case, iterating the procedure described above up to n, the term structure of mortality rates, $\mu_x(t, T_1), \mu_x(t, T_2), ..., \mu_x(t, T_n,)$, is obtained with respect to the maturities $T_1, T_2, ...T_n$ in analogous way with the term assurance and pure endowment case.

3.3 Model calibration

We calibrate each model's parameters by minimizing the sum of squares relative differences between mortality rates implied in the quotes and mortality rates implied by a specific affine model. This calibration technique is analogous to the calibration of affine stochastic interest rates models with respect to the term structure of interest rates.

The relative error is defined as

$$\varepsilon_i(\beta) = \frac{\mu_x^M(t, T_i) - \mu_x(t, T_i)}{\mu_x^M(t, T_i)}, \tag{25}$$

where $\mu_x^M(t, T_i)$ is the mortality rate implied in the contracts and $\mu_x(t, T_i)$ is the mortality rate of the considered model.

Denoting by β the set of the parameters of the affine model, the calibration procedure is such that

$$\widehat{\beta} = arg \min_{\beta} \varepsilon'(\beta)\varepsilon(\beta). \tag{26}$$

4 Empirical results

In this section, some numerical results related to a representative term assurance pure premiums of Italian insurance companies in force at the end of 2008 are presented. We have used premiums with respect to male aged 20, 40, and 60. The premiums are denominated in euros and are related to an insured amount of euro 1.000. For each age, contracts with a maturity from 5 years to 20 years were used.

4.1 Bootstrapping procedure

Commencing with a stream of 16 contracts with different discrete maturities (from 5 years to 20 years), the term structure of mortality rates, $\mu_x(t, T_i)$, with $i = 5, 6, ..., 20$, is derived from a vector of premiums related to term assurance with different time horizons and for each x.

The three panels in Tables 1 report the results of the bootstrapping procedure for the three ages ($x = 20, 40, 60$). For simplicity, a deterministic and flat term structure of interest rates is assumed with $\bar{r} = 0.05$.[11] The first column in the table contains the maturity (in years) of each contract while the second column contains the constant pure premiums. The term structure of mortality rates is reported in the third column. Survival and death probabilities are shown for each time horizon in the remain columns.

We can see from Table 1 that the term structure of mortality rates increases exponentially across time for each age and this result is consistent with the assumption that the dynamic of the death rate can be explained by means of non-mean-reverting affine models. Furthermore, the term structure is increasing across the ages; this result is consistent with the biological concept of organism senescence.

4.2 Calibration procedure

In this section, we discuss the results of the calibration procedure. Beginning with the term structure of mortality rates bootstrapped as explained in Section 3.2, the Vasicek, CIR, and jump-extended Vasicek models (without mean reversion) were calibrated. The mean-square error and the optimal values of the parameters for each model are reported in the three panels in Table 2. The mean-square errors are very low in all the models considered, indicating a satisfactory fitting of the survival probability implied in the quotes. The jump-extended Vasicek model shows the best calibration results.

Figure 1 shows the results for the fitting procedure graphically for the three models. Note that the goodness of fit improves as the age increases.

5 Conclusions

In this paper, we demonstrate how the use of insurance contracts premiums can be utilized to derive the term structure of implied mortality rates and how to calibrate the parameters of affine stochastic mortality models. We provide a new estimation procedure of affine models based on the premiums for insurance contracts in a distinctive manner with respect to the historical-estimation approach based on statistical procedures.

Viewing insurance contracts as market products, we explain how to fit stochastic models to policy quotes. For this purpose, insurance contracts are conceptually

[11] We assume continuous compounding and the ACT/365 day-count convention with $P(t,T) = e^{-\bar{r}\tau(t,T)}$.

viewed as a swap in which a policyholder (or the investor) exchanges cash flows with an insurer (or a new counterparty) as in a generic interest rate swap or credit default swap. By employing a bootstrapping procedure that we proffer, the term structure of mortality rates can be derived from the premiums of insurance contracts. The technique is analogous to the bootstrapping procedure used to generate the term structure of interest rates.

Different types of affine models are investigated for fitting the term structure of mortality rates: the Vasicek, Cox-Ingersoll-Ross, and jump-extended Vasicek models. Because the survival probability implied in the affine models is expressed in closed form, by minimizing the differences between mortality rates implied in the quotes and the theoretical ones, we derive the value of the parameters of affine stochastic mortality models using the calibration procedure described in this paper. Using Italian term assurance premiums and the results of the proposed calibration procedure, we find support for fitting the term structure of mortality rates.

Our model represents an alternative approach to estimate the parameters of affine stochastic mortality models in a consistent manner with respect to the death probabilities implied in insurance contracts. So, the calibrated affine models can be used in the mortality scenario generation in order to improve models for the pricing of mortality-linked securities, as well as to provide a more complete picture of the impact of mortality risks with respect to a market-consistent accounting framework and risk-based solvency requirements.

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TABLE 1 - Bootstrapping procedure results

a. Age = 20

$\tau(t,T_i)$	Q	$\mu_{20}(t,T_i)$	$S_{20}(t,T_i)$	$D_{20}(t,T_i)$	$D_{20}(T_{i-1},T_i)$
0	-	-	1.000000	0.004518	-
5	0.906000	0.000906	0.995482	0.004518	0.004518
6	0.917000	0.000918	0.994506	0.005494	0.000976
7	0.931000	0.000934	0.993481	0.006519	0.001025
8	0.946000	0.000952	0.992414	0.007586	0.001067
9	0.963000	0.000972	0.991289	0.008711	0.001125
10	0.982000	0.000995	0.990096	0.009904	0.001193
11	0.999000	0.001016	0.988883	0.011117	0.001213
12	1.015000	0.001036	0.987641	0.012359	0.001242
13	1.028000	0.001053	0.986409	0.013591	0.001232
14	1.037000	0.001064	0.985220	0.014780	0.001190
15	1.045000	0.001074	0.984026	0.015974	0.001194
16	1.051000	0.001081	0.982857	0.017143	0.001169
17	1.056000	0.001087	0.981695	0.018305	0.001161
18	1.062000	0.001095	0.980491	0.019509	0.001205
19	1.069000	0.001104	0.979236	0.020764	0.001255
20	1.076000	0.001114	0.977958	0.022042	0.001278

b. Age = 40

$\tau(t,T_i)$	Q	$\mu_{40}(t,T_i)$	$S_{40}(t,T_i)$	$D_{40}(t,T_i)$	$D_{40}(T_{i-1},T_i)$
0			1.000000	0.008081	
5	1.624000	0.001623	0.991919	0.008081	0.008081
6	1.697000	0.001706	0.989817	0.010183	0.002103
7	1.782000	0.001804	0.987451	0.012549	0.002366
8	1.882000	0.001922	0.984743	0.015257	0.002707
9	1.989000	0.002050	0.981720	0.018280	0.003024
10	2.098000	0.002183	0.978407	0.021593	0.003312
11	2.207000	0.002318	0.974820	0.025180	0.003588
12	2.318000	0.002459	0.970918	0.029082	0.003902
13	2.435000	0.002612	0.966614	0.033386	0.004303
14	2.564000	0.002785	0.961758	0.038242	0.004857
15	2.703000	0.002977	0.956324	0.043676	0.005433
16	2.854000	0.003192	0.950208	0.049792	0.006117
17	3.015000	0.003428	0.943386	0.056614	0.006822
18	3.187000	0.003688	0.935766	0.064234	0.007620
19	3.369000	0.003972	0.927304	0.072696	0.008462
20	3.563000	0.004285	0.917863	0.082137	0.009441

c. Age = 60

$\tau(t,T_i)$	Q	$\mu_{60}(t,T_i)$	$S_{60}(t,T_i)$	$D_{60}(t,T_i)$	$D_{60}(T_{i-1},T_i)$
0			1.000000	0.061855	
5	12.852000	0.012770	0.938145	0.061855	0.061855
6	13.496000	0.013519	0.922086	0.077914	0.016059
7	14.158000	0.014303	0.904729	0.095271	0.017357
8	14.818000	0.015101	0.886203	0.113797	0.018526
9	15.485000	0.015930	0.866433	0.133567	0.019769
10	16.159000	0.016793	0.845410	0.154590	0.021023
11	16.871000	0.017740	0.822721	0.177279	0.022689
12	17.583000	0.018720	0.798800	0.201200	0.023922
13	18.311000	0.019765	0.773410	0.226590	0.025389
14	19.049000	0.020871	0.746623	0.253377	0.026788
15	19.808000	0.022066	0.718215	0.281785	0.028407
16	20.592000	0.023368	0.688056	0.311944	0.030160
17	21.406000	0.024802	0.655971	0.344029	0.032085
18	22.226000	0.026337	0.622463	0.377537	0.033508
19	23.057000	0.028001	0.587415	0.412585	0.035047
20	23.905000	0.029832	0.550660	0.449340	0.036755

TABLE 2 - Models and related parameters

Note: VAS = Vasicek model; CIR = Cox, Ingersoll, and Ross model; JVAS = jump-extended Vasicek model.

a. Age = 20

Model	μ_0	k	θ	σ	τ	η	Error
VAS	0.000797	-0.051085	0	0.001343	-	-	0.000382
CIR	0.000859	-0.027046	0	0.003082	-	-	0.003380
JVAS	0.000299	-0.092501	0	0.000924	0.000532	0.003010	0.000555

b. Age = 40

Model	μ_0	k	θ	σ	τ	η	Error
VAS	0.001217	-0.106695	0	0.000199	-	-	0.000597
CIR	0.001217	-0.106574	0	0.003802	-	-	0.000588
JVAS	0.001007	-0.116136	0	0.000198	0.000242	-0.001005	0.000284

c. Age = 60

Model	μ_0	k	θ	σ	τ	η	Error
VAS	0.010054	-0.095001	0	0.001071	-	-	0.000180
CIR	0.010069	-0.094490	0	0.006830	-	-	0.000182
JVAS	0.010066	-0.094655	0	0.000918	0.000001	0.121655	0.000179

FIGURE 1 - Graphical presentation for the results of the fitting procedure