Copula Concepts in Financial Markets

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The concept of copula has received growing attention in finance and economics in recent years. From the early days of use in finance over copulas finding their way to Wall Street in a mass market of credit derivatives, this episode of quantitative modelling of markets was also one of euphoria, exaggerations, misperceptions and debates. Bringing tools from the "hard-sciences" like mathematics or physics to a "soft-science", i.e. to use formulas and models for financial markets or the behaviour of market participants always bears a certain portion of model risk. But the complexity and dynamics of financial markets makes it necessary to employ those tools and thereby improve existing methods. The case of copulas in finance is a typical example of how generally appealing and sensible models may cause unforeseen problems and can lead to debates on the applicability of quantitative methods.

With the emergence of the sub-prime crisis and the following credit crunch, academics, practitioners, philosophers and journalists started searching for causes and failures that led to the turmoil and (almost) unprecedented market deteriorations. Several explanations and findings were brought forward in recent months, and while a one-factor-explains-it-all solution will never be found, there is an often-proclaimed perception that risk was not adequately taken into account properly and that it was not measured adequately. However, while one would not reject this notion in general, the arguments against several methods and models used at Wall Street and throughout the world are, in many cases, putting those in the wrong light. Beyond the fact that risks and issues were clouded by the securitization, tranching and packaging of underlyings in the credit markets as well as by the unfortunate and somehow misleading role of rating agencies, mathematical models were used in the markets which are now under fire due to their incapability of capturing risks in extreme market phases.

In this article, we give an overview concerning the copula concept and the several possible forms of it, and give a direct explanation why problems caused by several applications were simply inevitable. We show that copulas can be used to model extreme market and asset interdependencies, i.e. joint tail realizations, a feature that is not obtained when using linear correlations. We explain why a Gaussian (a "normal") copula, which is currently the most commonly used type of copula in finance, is highly inappropriate for most applications aiming at modelling dependencies between financial assets and markets. In addition, we explain why it is generally no real improvement to the linear correlation coefficient concerning extreme dependencies. Furthermore, we show how powerful sensible copula applications may be and what the pro's and con's of the most important copula types in a financial market context are. We first focus on the linear correlation coefficient and the shortcomings of it, as those shortcomings can be seen as the main force for the inclusion of copulas in financial market applications. The usual linear correlation is not a satisfactory measure of the dependence among different securities for several reasons. First, when the variance of returns in those securities tends to be infinite, that is, when extreme events are frequently observed, the linear correlation between these securities is undefined. Second, the correlation is a measure for linear dependence only. Third, the linear correlation is not invariant under nonlinear strictly increasing transformations, implying that returns might be uncorrelated whereas prices are correlated or vice versa. Fourth, linear correlation only measures the degree of dependence but does not clearly discover the structure of dependence. The last caveat has an especially important implication in light of the current crisis. It has been widely observed that market crashes or financial crises often occur in different markets and countries at about the same time period even when the correlation among those markets is fairly low.

The structure of dependence also influences the achieved diversification benefit based on a linear correlation measure. A more prevalent approach that overcomes the disadvantages of linear correlation is to model dependency by using copulas. With the copula method, the nature of dependence that can be modelled is more general and the dependence of extreme events can be considered.

Generally, a copula is used to separate the pure randomness of one variable (for example, a financial asset) from the interdependencies between it and other variables. By doing so, one can model each variable separately and, in addition, have a measure of the relations between those variables in addition. Technically, this means that the univariate probability distribution, being informative on the probabilities of outcomes of one variable can be modelled by a distribution type of choice, while another variable can be modelled using another type of probability distribution. By doing so, one can choose for each and any asset in a spectrum the most appropriate type of distribution, not influencing the interdependencies between those variables/assets. The interdependencies between those variables are represented by a multivariate probability distribution function, which is informative on the joint outcomes of the variables, and this multivariate distribution function is the copula.

We do not need any technical representation to make clear why the interdependencies are modelled fully flexible by a copula: An N-dimensional multivariate distribution representing the copula of N financial variables (for example returns, rates, etc.) has a support on the N-dimensional cube and is a standardized measure being able to capture all possible relationships between the N-variables. To summarize: The use of copula allows the separation of univariate randomness (defined by the individual probability distribution functions of financial random variables) and dependence structure defined by the copula (see Sklar 1959 and 1973).

Along with all the options and flexibilities of the copula come the challenges: One challenge is the choice or estimation of an adequate univariate distribution function that is important to model the randomness of one variable. The other is the choice or estimation of the multivariate dependence of the joint realizations of the cumulative density functions of the variables, i.e. the choice of the copula type. The former problem has been well discussed in recent decades, from the beginning of portfolio theory and Markowitz' work using Gaussian distribution functions to the most recent developments such as stable distributions, heavy-tailed distributions and others (see Rachev and Mittnik 2000, Rachev 2003, Rachev et.al. 2005, Rachev et.al. 2007a and 2007b). Most of these functions were introduced to model the behaviour of financial markets and assets more appropriately, due to the well-known fact that the Gaussian probability distribution is incapable of tail-events, i.e. extreme occurrences far away from the mean of a variable. We will not dig deeper into this topic as the current article aims at providing insight into the nature of copula functions and their application in finance. However, we will soon come back to the inappropriateness of Gaussian distribution types even for the multivariate class.

We will focus on the most common and influential types of copulas, the interested reader is referred to Embrechts et.al. (2003) for a broad overview on copulas. While Archimedean (for example Clayton, Frank or Gumbel) copulas are calculated over a closed-form solution (being very hard to derive for multivariate applications beyond two dimensions however) and do not need to be represented by an application of multivariate distributions using Sklar's theorem, elliptical (for example Gaussian or Student t) copulas are derived via simulations of these multivariate distributions. A caveat of general elliptical copulas is that the upper and lower tail dependence, being informative on joint extreme realizations, is identical, due to the radial symmetric shape of the elliptical copulas. In addition, a Gaussian copula has no tail dependence at all (see Bradley and Taqqu 2003), and this is the main argument against its use in financial market applications.

In the next paragraph we explain why the Gaussian copula is inappropriate for most financial applications due to the aforementioned inability of measuring tail risks. Furthermore, we place this discussion in light of the ongoing debate surrounding copula functions in financial markets and especially during the current credit crisis. The fact that the Gaussian copula has no tail dependence at all is stemming from the fact that a multivariate Gaussian distribution is the n-dimensional version of a Gaussian distribution, which assigns too low probabilities to extreme outcomes. While the use of Gaussian distributions in financial market applications is widely accepted as being flawed due to the fact that this distribution type attributes too low probabilities to extreme observations, the multivariate version was, and is frequently used in copula applications. Especially in credit markets the Gaussian copula served as a method to price risky assets, mainly based on David X. Li's contribution in the *Journal of Fixed Income* dating back to the year 2000.

While it is an obvious mathematical fact that the multivariate Gaussian distribution is not capable of tail dependencies, it was used in growing numbers and frequency in financial markets. Investors in the credit derivatives market used the copula model that was introduced by Li, and the market volume soared along with the use of the model. Hedge funds, banks, traders and rating agencies relied on the methodology in a market that quickly turned out to be huge and dynamic. In Li's model, the Gaussian copula was used to estimate the probability of default of companies in a bond pool, thereby focussing on the joint default probabilities, using the possibility of measuring interdependency as explained above utilizing a copula. However, the wide use of the model and the dramatic increase in market participants and volumes caused massive losses when the markets did not behave in the way indicated by the concept.

We discuss two occurrences that, while being very different in nature have several characteristics in common: First, they show that the model has shortcomings in capturing the risks in the markets. Second, the fact that the occurrences caused so much damage show that market participants were not aware of the shortcomings of the model or chose to ignore them. Third, when searching for the cause of the turmoil, the model or its inventor were more under fire than the people misusing it, resulting in general scepticism concerning quants, i.e. people inventing mathematical and statistical methods for financial markets.

The first time the public became painfully aware of large losses in the credit market and focused on Li's copula model was in Spring 2005. In this period, the default risk of General

Motors and Ford increased rapidly following a credit downgrade of General Motors' corporate debt by a credit rating agency.

Trades and bets based on the assumption of a co-movement in corporate default risk went wrong as the model did not capture the single event of a large company coming under stress separately. However, blaming the model would be going too far, as such isolated events are not predicable and the model is not built to predict them either. This was basically acknowledged in an article in the *Wall Street Journal* on September 12, 2005 by Mark Whitehouse. In addition, the article cited the statements of Li concerning the model that turned out to be often read during subsequent years: "The most dangerous part is when people believe everything coming out of it" and "very few people understand the essence of the model."

It is surprising that this occurrence and the inventors' warnings did not slow down the market or led to the use of new methods. That the market still used the model and relied on it resulting in another episode of even larger losses makes it immediately clear that there is a lack of understanding in the market. In addition, it is questionable whether public discussions of this fact are either helpful or point in the right direction. As an example, the article of Felix Salmon appearing in *Wired* on February 23, 2009 is entitled "Recipe for Disaster: The Formula That Killed Wall Street", implying at first glance that the model was wrong or erroneous. Clearly, this is not the case: It is the usage of a model where it should not be used and the ignorance of that fact which caused the problems. In contrast to the Spring 2005 events surrounding GM, the problem here was not that one asset had a behaviour other than the related ones, but the severity of the market's downturn that was not caught adequately. Recalling the fact that the model is based on a normal copula, which is incapable of measuring extreme interdependencies, this should come as no surprise to us. While the whole discussion surrounding the copula is very often brought in relation to the correlation coefficient (which is not measuring extreme dependency as discussed above) it is of urgent need to use tools that are more sophisticated.

Figure 1 shows the usefulness of the right type of copula and marginal distribution modelling and the drawbacks of using Gaussian copulas in financial markets, here with an example for two stock market indices. Comparing Figures 1a and 1d, it is obvious that a combination of Gaussian copula and Gaussian marginal distribution does a poor job when being fitted to the two stock markets' returns. In addition, it can be seen from Figure 1c that even when modelling the single markets adequately using stable distributions for the univariate randomness of one variable, the occurrences of joint extreme observations is not reflected appropriately when using a Gaussian copula. Figure 1b, in contrast, shows a very good fit to the historical returns for the skewed t copula and marginal distribution combination.

The example with the stock market returns is providing an easy to understand overview of how the different combinations of copulas and marginal distributions work and that it is by no means justifiable to see the copula concept from one single perspective. While the Gaussian copula (like a linear correlation) does a poor job for modelling extreme joint occurrences and the structure thereof, the right copula can be a powerful tool. Skewed t copula methods as the one shown in Figure 1b that was invented by FinAnalytica Inc. are able to catch joint extreme realizations and, in addition, remove the shortcoming of other elliptical copulas through the possibility of differing upper and lower tail dependency.

From the review above it should be clear that discussions surrounding the copula concept should always focus on the type of copula under consideration, rather than on the usefulness of the copula in general. While the right copula concept at the right place can greatly contribute to the set of quantitative models, a wrong copula application may lead to disastrous outcomes and unfortunate surprises. In the near future therefore, we can expect that there will be further inventions in the field of copula and their applications in financial markets. Furthermore, practitioners will be better off utilizing the right concepts instead of those for which the shortcomings have already become clear and, unfortunately, quantifiable in real money during the current crisis.

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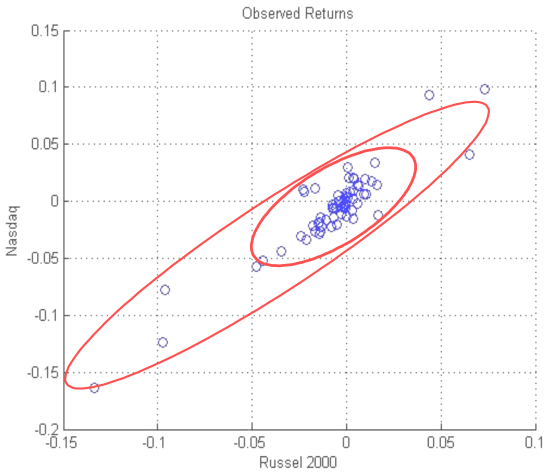

Figure 1a*: Observed returns of NASDAQ and Russell 2000 indices*

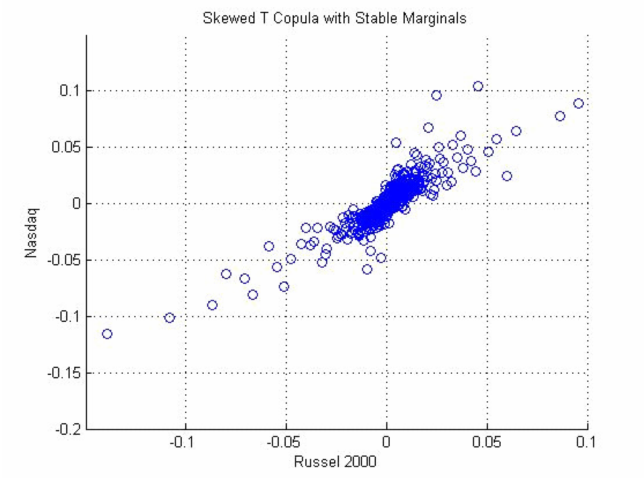

Figure 1b: *Skewed t copula with stable marginal distributions fitted to indices*

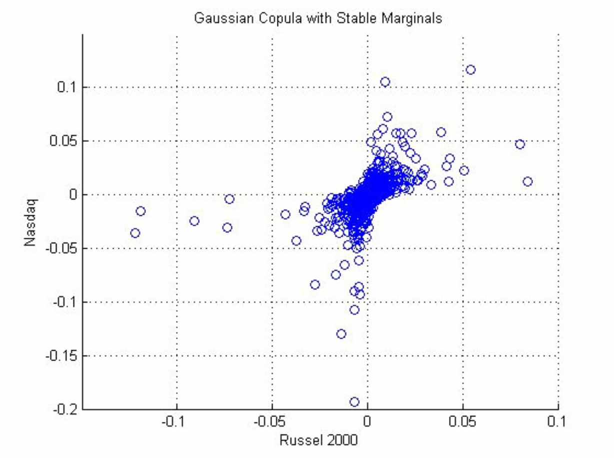

Figure 1c: *Gaussian copula with stable marginal distributions fitted to indices*

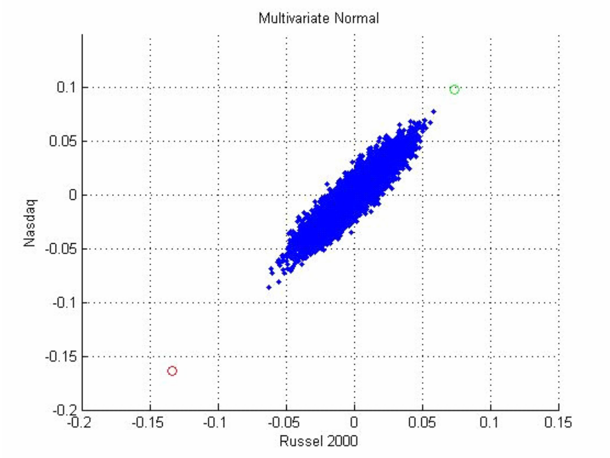

Figure 1d: *Gaussian copula with Gaussian marginal distributions fitted to indices(red and green dot indicate the most extreme historical realizations of the indices for comparison)*

Figure 1: *Historical stock exchange 80 daily returns (August-November 1987 crash) and three different copula and marginal distribution combinations fitted to the returns (2000 simulated values). Source: FinAnalytica (2009)*