Applied Probability Trust (16 December 2010)

ON A CLASS OF DISTRIBUTIONS STABLE UNDER RANDOM SUMMATION

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Abstract

We investigate a family of distributions having a property of stability-under-addition, provided that the number ν of added-up random variables in the random sum is also a random variable. We call the corresponding property a ν-stability and investigate the situation with the semigroup generated by the generating function of ν is commutative.

Using results from the theory of iterations of analytic functions, we show that the characteristic function of such a ν-stable distribution can be represented in terms of Chebyshev polynomials, and for the case of ν-normal distribution, the resulting characteristic function corresponds to the hyperbolic secant distribution.

We discuss some specific properties of the class and present particular examples.

Keywords: Stability, random summation, characteristic function, hyperbolic secant distribution

2000 Mathematics Subject Classification: Primary 60E07

Secondary 60E10

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1. Introduction

In many applications of probability theory certain specific classes of distributions have become very useful, usually called "fat tailed" of "heavy tailed" distributions. The *Stable distributions* that originate from the Central Limit problem, are probably most popular among the heavy tailed distributions, however there is a wide collection of classes of distributions, all related to Stable ones in many various ways, often these relations are not at all obvious.

Besides, certain generalizations of stable distributions are known, using sums of random numbers of random variables (instead of sums with deterministic number of summands), see e.g. Gnedenko [4], Klebanov, Mania, Melamed [9], for the examples of such, including the so-called *ν-stable* distributions, introduced independently by Klebanov and Rachev [10] and Bunge [1].

In the present paper, we focuse on presenting further examples of strictly ν-stable random variables, that could be useful in practical applications, including applications in financial mathematics.

2. Definition of strictly ν-stable r.v.'s, properties and examples

In the present section, we give a general insight on strictly ν-stable distributions and describe some examples.

2.1. Basic definitions

Let $X, X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables, and let $\{\, \nu_p\,, \, p \in \Delta\,\}$ be a family of some discrete r.v.'s taking values in the set of natural numbers $\mathbb{N}$. Assume that this family does not depend on the sequence $\{X_j,\, j \geq 1\}$, and that, for $\Delta \subset (0,1)$,

$$\mathbf{E}\nu_p = \frac{1}{p}, \;\; \forall\, p \in \Delta\,. \tag{1}$$

Definition 1. We say that the r.v. X has a strictly ν–stable distribution, if $\forall\, p \in \Delta$ it holds that

$$X \stackrel{d}{=} p^{1/\alpha} \sum_{i=1}^{\nu_p} X_j\,,$$

where $\alpha \in (0,2]$ is called the index of stability.

After this general definition, a narrower class is defined for $\alpha = 1/2$.

Definition 2. We call the r.v. X a strictly ν–normal r.v., if $\mathbf{E}X = 0$, $\mathbf{E}X^2 = \infty$, and the following holds:

$$X \stackrel{d}{=} p^{1/2} \sum_{i=1}^{\nu_p} X_j \,, \quad \forall\, p \in \Delta.$$

Closely related to stability property is the property of infinite divisibility, so we also give the following definition.

Definition 3. X has a strictly ν–infinitely divisible distribution, if for any $p \in \Delta$, there exists a r.v. $Y^{(p)}$, s.t.

$$X \stackrel{d}{=} \sum_{j=1}^{\nu_p} Y_j^{(p)} \,, \quad \text{with } Y^{(p)}, Y_1^{(p)}, \ldots, Y_n^{(p)}, \ldots \text{ being iid r.v.'s}$$

A powerful tool for investigating distributions' properties is the *generating function*, so the generating function of the r.v. ν_p, will be denoted by $\mathcal{P}_p(z) := \mathbf{E}\,[z^{\nu_p}]$. Moreover, we denote by $\mathcal{A}$ the semigroup generated by the family $\{\,\mathcal{P}_p\,,\ p \in \Delta\}$, with the operation of the functions' composition.

2.2. Summary of the known results

With regards to the definitions above, the following results are known (see e.g. [1], [8], [10] for proofs and details).

Theorem 2.1. *For the family* $\{\,\mathcal{P}_p\,,\ p \in \Delta\}$*, with* $\mathbf{E}\,[\nu_p] = \frac{1}{p}$ *, there exists a strictly* ν*-normal distribution, iff the semigroup* $\mathcal{A}$ *is commutative.*

Suppose that we have a commutative semigroup $\mathcal{A}$. Then the following statements (that we refer to in the sequel as *Properties*) are known to be true (see e.g. [5] for proofs and details):

1. The system

$$\varphi(t) = \mathcal{P}_p(\varphi(pt)), \ \ \forall\, p \in \Delta\,. \tag{2}$$

of functional equations has a solution that satisfies the initial conditions

$$\varphi(0) = 1\,, \ \ \varphi'(0) = -1\,. \tag{3}$$

The solution is unique. In addition, there exists a distribution function (cdf) $A(x)$ (with $A(0)=0$) such that

$$\varphi(t) = \int_0^\infty e^{-tx} dA(x). \tag{4}$$

2. The characteristic function (ch.f.) of the strictly ν-normal distribution has the form

$$f(t) = \varphi(at^2), \quad a > 0. \tag{5}$$

3. A ch.f. $g(t)$ is a ch.f. of a ν-infinitely divisible r.v., iff there exists a chf $h(t)$ of an infinitely divisible (in the usual sense) r.v., such that

$$f(t) = \varphi(-\ln h(t)). \tag{6}$$

The relation (6) allows obtaining explicit representations of ch.f. of strictly ν-stable distributions. Clearly, they are obtained through applying (6) to the ch.f. ($h(t)$) of strictly stable (in the usual sense) distributions. Moreover, note that the ch.f. $\varphi(ait)$, $a \in \mathbb{R}^1$, is the ch.f. of an analogue of the degenerate r.v., and that for the r.v. with such ch.f. the following analogue of the Law of Large Numbers exists.

Theorem 2.2. *Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of iid random variables with the finite absolute value of the first moment, and $\{\nu_p, p \in \Delta\}$ a family of r.v.'s taking values in $\mathbb{N}$, independent of the sequence $\{X_j, j = 1, 2, \dots\}$. Assume that $\mathbf{E}[\nu_p] = \frac{1}{p}$ and that the semigroup $\mathcal{A}$ is commutative.*

Then the series $p\sum_{j=1}^{\nu_p} X_j$ is convergent is distribution, as $p \to 0$, and the limit of convergence is a r.v. having the ch.f. $\varphi(ait)$.

The proof of this theorem follows straightforwardly from the *Property 1* outlined above and from the *Transfer Theorem* of Gnedenko (see, [4]).

In the following paragraph we discuss several particular examples of strictly ν-normal and strictly ν-stable distributions.

2.3. Examples and the outline of the problem

Example 2.1. *The usual stability.*

Assume the following setup: $\nu_p = \frac{1}{p}$ with probability 1, where $p \in \Delta = \left\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$, and so $\mathcal{P}_p(z) = z^{1/p}$.

Clearly, here the semigroup $\mathcal{A}$ is commutative.

Furthermore, $\varphi(t) = \exp\{-t\} = \int\limits_0^\infty e^{-tx} dA(x)$, where $A(x)$ is a cdf with a single unit-sized jump at $x = 1$. In this setup the strictly ν-normal ch.f. is the ch.f. of the normal (in the usual sense) r.v. with the zero mean.

Example 2.2. *The geometric summation scheme.*

Suppose, ν_p is the r.v. having a geometric distribution

$$\mathbf{P}\{\nu_p = k\} = p(1-p)^{k-1}\,, \;\; k = 1, 2, \dots \;\; , \, p \in (0,1)\,.$$

Clearly, here $\mathbf{E}\,[\nu_p] = \frac{1}{p}$, and $\mathcal{P}_p(z) = \frac{pz}{1-(1-p)z}$, $p \in (0,1)$. It is quite straightforward to check that $\mathcal{A}$ is commutative.

Moreover, a direct calculation gives $\varphi(t) = \frac{1}{1+t} = \int\limits_0^\infty e^{-tx} e^{-x} dx$, i.e. $A(x)$ is the cdf of the exponential distribution. So that a ν-analogue of the strictly normal distribution is the Laplace distribution with the ch.f. $f(t) = \frac{1}{1+at^2}$.

Example 2.3. *Branching process scheme.*

Let $\mathcal{P}(z)$ be some generating function, with $\mathcal{P}'(1) = \frac{1}{p_0} > 1$ (so that the introduced notation is $p_0 = 1/\mathcal{P}'(1)$, with the condition $p_0 < 1$).

Consider now a family given by $\mathcal{P}^{0\,n}(z) = \mathcal{P}^{0(n-1)}\left(\mathcal{P}(z)\right)$, $n = 1, 2, \dots$. Related to that is another family of the r.v.'s $\nu_p : \mathcal{P}_p(z) = \mathcal{P}^{0\,n}(z)$, $p \in \left\{\frac{1}{p_0^n}\,,\; n = 1, 2, \dots\right\} =: \Delta$.

Clearly, the semigroup $\mathcal{A}$ coincides with the family $\{\mathcal{P}_p \; p \in \Delta\}$. The ch.f. $\varphi(t)$ is a solution of the functional equation $\varphi(t) = \mathcal{P}\left(\varphi(p_0 t)\right)$.

It can be noted that the content of the paper by Mallows and Shepp [12] is actually based on considering an example identical to the Example 2.3 above. Probably, neither the authors of that work nor its reviewers were familiar with the works by Klebanov and Rachev [10] and Bunge [1], which had dealt with exactly the same example a number of years earlier.bf

Like mentioned in Introduction, in the present work we aim in widening the collection of examples that involve random summation with the commutative semigroup $\mathcal{A}$. For that reason, we address the description of pairs of certain commutative generating functions $\mathcal{P}$ and $\mathcal{Q}$, i.e. the ones for which the balance equality $\mathcal{P} \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{P}$ holds, – but including only the case when *there exists no* such function $\mathcal{H}$ such that $\mathcal{P} = \mathcal{H}^{0k}$ and $\mathcal{Q} = \mathcal{H}^{0m}$ for some $k, m \in \mathbb{N}$ (which would be exactly the case of the *Example 2.3*).

In a general setting, the problem of describing all such commutative pairs of generating functions appears, unfortunately, far involved to approach. However, certain special cases are rather straightforward for consideration. In order to approach the problem, we will use certain notions typical for the theory of iterations of analytic functions, that we outline in the separate section below.

3. Theoretic justification via iterations of analytic functions

Let $\mathcal{P}$ be a rational function with (deg) ≥ 2. Denote by $\mathcal{P}^{0n}$ its nth iteration. The functions $\mathcal{P}$ and $\mathcal{Q}$ are called *conjugates*, if there exists a linear-fractional function R, such that $\mathcal{P} \circ R = R \circ \mathcal{Q}$.

A subset E of the extended complex plane $\overline{\mathbb{C}}$ is called *completely invariant*, if its complete inverse image $\mathcal{P}^{-1}(E)$ coincides with E. The maximal finite completely invariant set $E(\mathcal{P})$ exists and is called the *exceptional set* of the function $\mathcal{P}$. It is always the case that $\operatorname{card} E(\mathcal{P}) \leq 2$. Moreover, if $\operatorname{card} E(\mathcal{P}) = 1$ then the function $\mathcal{P}$ is a conjugate to a polynomial, while for $\operatorname{card} E(\mathcal{P}) = 2$ the function $\mathcal{P}$ is a conjugate to $\mathcal{Q}(z) = z^n$, $n \in \mathbb{Z}\backslash\{0, 1\}$. Clearly, $E(\mathcal{Q}) = \{0, \infty\}$.

If $\mathcal{P}$ is a rational function, then it is known (see e.g. [2]) that there us a finite number of open sets F_i, $i = 1, \ldots, r$, which are *left invariant* by the operator $\mathcal{P}$ and are such that (in the sequel, we will refer to the two points below as *Conditions*)

1. the union $\bigcup_{i=1}^{r} F_i$ is *dense* on the plane
2. $\mathcal{P}$ behaves *regularly* on each of F_i.

The latter means that the termini of the sequences of iterations generated by the points of F_j are either precisely the same set, which is then a finite cycle, or they are finite cycles of finite or annular shaped sets that are lying concentrically. In the first case the cycle is *attracting*, in the second one it is *neutral*.

The sets F_j are the *Fatou domains* of $\mathcal{P}$, and their union is the *Fatou set* $F(\mathcal{P})$ of

$\mathcal{P}$.

The complement of $F(\mathcal{P})$ is the *Julia set* $\mathcal{J}(\mathcal{P})$ of $\mathcal{P}$. Note that $\mathcal{J}(\mathcal{P})$ is either a nowhere dense set (that is, without interior points) and an uncountable set (of the same cardinality as the real numbers), or $\mathcal{J}(\mathcal{P}) = \overline{\mathbb{C}}$. Like $F(\mathcal{P})$, $\mathcal{J}(\mathcal{P})$ is left invariant by $\mathcal{P}$, and on this set the iteration is *repelling*, meaning that $|\,\mathcal{P}(z) - \mathcal{P}(w)\,| > |\,z - w\,|$ for all elements w in a neighborhood of z (within $\mathcal{J}(\mathcal{P})$). This means that $\mathcal{P}(z)$ behaves chaotically on the Julia set. Although there are points in the Julia set whose sequence of iterations is finite, there is only a countable number of such points (and they make up an infinitely small part of the Julia set). The sequences generated by points outside this set behave chaotically, a phenomenon called *deterministic chaos.* Let z_0 be a repelling fixed point of the function $\mathcal{P}$, and let $\lambda = \mathcal{P}'(z_0)$. Define $\Lambda : \ z \to \lambda z$. Then there exists a unique solution of the Poincaré equation

$$F \circ \Lambda = \mathcal{P} \circ F \,, \quad F(0) = z_0 \,, \quad F'(0) = 1 \,,$$

that is meromorphic in $\overline{\mathbb{C}}$.

Now let

$$\mathcal{I}(\mathcal{P}) = F^{-1}(\mathcal{J}(\mathcal{P}))\,.$$

If for two functions $\mathcal{P}$ and $\mathcal{Q}$ we have $\mathcal{P} \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{P}$, then they have the same function F.

There are the two following possibilities:

1. $\mathcal{I}(\mathcal{P}) = \mathbb{C}$, in which case $\mathcal{J}(\mathcal{P}) = \overline{\mathbb{C}}$,.

2. $\mathcal{I}(\mathcal{P})$ is nowhere dense and consists of analytic cuvrves.

Fatou [3], and Julia [6] investigated the case. It turned out that is this case $\mathcal{P}$ and $\mathcal{Q}$ can be reduced by a conjugancy either to the form $\mathcal{P}(z) = z^m$ and $\mathcal{Q}(z) = z^n$ or to the form $\mathcal{P}(z) = T_m(z)$ and $\mathcal{Q}(z) = T_n(z)$, where T_k is the Chebyshev polynomial determined by the equation $\cos(k\zeta) = T_k(\cos\zeta)$.

4. Main results

4.1. A new example

Let us return to the study of ν-normal and ν-stable random variables. Recall that we deal with the family $\{\nu_p\,,\ p\in\Delta\}$ taking its values in $\mathbb{N}=\{1,2,\dots\}$. As before, we work with the generating function $\mathcal{P}_p(z)=\mathbf{E}\,[\,z^{\nu_p}\,]$ of ν_p. The important result that we stressed says the a strictly ν-normal (resp. strictly ν-stable) r.v. exist iff the semigroup $\mathcal{A}$ generated by $\{\mathcal{P}_p\,,\ p\in\Delta\}$ is commutative. If $\mathcal{P}_p\,,\ p\in\Delta\,,$ is a rational function (with deg ≤ 2) satisfying *Condition 2* of the above section, then either $\mathcal{P}_p(z)$ is reduced to a form $\widetilde{\mathcal{P}}_p(z)=z^{1/p}\,,\ p\in\left\{\frac{1}{n}\,,\ n=1,2,\dots\right\}$, and then we deal, in fact, with the classical (deterministic) summation scheme, or $\mathcal{P}_p(z)$ is reduced to the form $\mathcal{P}_p(z)=T_{1/\sqrt{p}}(z)\,,\ p\in\left\{\frac{1}{n^2}\,,\ n=1,2,\dots\right\}$. Clearly, the polynomial $T_m(z)$ is not a generating function itself, however a function to which it is a conjugate, specifically the function

$$\mathcal{P}_p(z)=\frac{1}{T_{1/\sqrt{p}}(1/z)}\ ,\ \ p\in\left\{\frac{1}{n^2}\,,\,n=1,2,\dots\right\}\ , \tag{7}$$

is indeed a generating function, – the fact that we prove below. Moreover, below we consider in some details a family of r.v.'s $\left\{\nu_p\,,\,p\in\left\{\frac{1}{n}\,,\,n=1,2,\dots\right\}\right\}$ that have generating functions of the form (7), and investigate the corresponding strictly ν-normal and strictly ν-stable distributions.

Lemma 1. *Let $P_n(x)$ be a polynomial with $\deg P_n=n$ by to the even powers of x, and whose zeros are all within the interval $(-1,1)$. Let $P_n(1)=1$ and polynomial's coefficient with with x^n be positive. Then for any natural number k, the function*

$$\mathcal{P}(x)=\frac{x^k}{P_n(\frac{1}{x})}$$

is a generating function.

Proof. Represent $P_n(x)$ as

$$P_n(x)=b_0+b_1x+\cdots+b_nx^n=b_n\prod_{j=1}^{n}(x-a_j)\ ,$$

where a_j $(j=1,\dots,n)$ are the zeros of the polynomial P_n sorted in the order of ascendance. As P_n is a polynomial by the even powers of x, then if a_j is a zero of P_n, then $-a_j$ is also a zero of P_n. Therefore,

$$\begin{aligned}
\frac{1}{P_n(\frac{1}{x})} &= \frac{1}{b_n \prod\limits_{j=1}^{n} \left(\frac{1}{x} - a_j\right)} \\
&= \frac{1}{b_n \prod\limits_{j=1}^{n/2} \left(\frac{1}{x} - a_j\right)\left(\frac{1}{x} + a_j\right)} \\
&= \frac{1}{b_n} \prod_{j=1}^{n/2} \frac{1}{\left(\frac{1}{x} - a_j\right)\left(\frac{1}{x} + a_j\right)} \\
&= \frac{1}{b_n} \prod_{j=1}^{n/2} \frac{x^2}{1 - a_j^2 x^2} \qquad (8)
\end{aligned}$$

Obviously,

$$\frac{x^2}{1 - a_j^2 x^2} = \sum_{k=0}^{\infty} a_j^{2k} x^{2k+2}$$

is a series with positive (non-negative) coefficients, converging when $|x| \le 1$. From (8), it now follows that $\mathcal{P}(x) = \frac{x^k}{P_n(\frac{1}{x})}$ is a series also convergent when $|x| \le 1$, having non-negative coefficients, and $\mathcal{P}(1) = 1$. Hence, $\mathcal{P}(x)$ is a generating function of some random variable.

Corollary 1. *Let $T_n(x)$ be a Chebyshev polynomial of degree n. Then*

$$\mathcal{P}(x) = \frac{1}{P_n(\frac{1}{x})}$$

is a generating function of some r.v. which takes values in $\mathbb{N}$.

Proof. When n is an even number, the result follows directly from *Lemma 1* and from the properties of Chebyshev polynomials. For odd n, consider the representation $T_n(x) = xP_{n-1}(x)$, where $P_{n-1}(x)$ is a polynomial by the even degrees of x, satisfying the conditions of *Lemma 1.*

Let us now set $\Delta := \left\{\frac{1}{n^2}\,,\ n = 1, 2, \dots\right\}$. Consider the family of generating functions

$$\mathcal{P}_p(z) = \frac{1}{T_{1/\sqrt{p}}(1/z)}\,, \quad p \in \Delta\,.$$

Clearly, $\mathcal{P}_{p_1} \circ \mathcal{P}_{p_2} = \mathcal{P}_{p_2} \circ \mathcal{P}_{p_1}$ for all $p_1, p_2 \in \Delta$, due to the well known property of Chebyshev polynomials stating that $T_n(T_m(x)) = T_{n \cdot m}(x)$. In other words, semigroup generated by the family $\{\mathcal{P}_p\,,\ p \in \Delta\}$ is commutative. It follows (see e.g. [8]) that there exists a solution to the system of equations

$$\varphi(t) = \mathcal{P}_p(\varphi(pt))\ ,\ \ p \in \Delta\,, \tag{9}$$

satisfying initial conditions

$$\varphi(0) = 1\ ,\ \ \varphi'(0) = -1\,, \tag{10}$$

and the solution is unique.

Since $T_n(x) = \cos(n \cdot \arccos x) = \cosh(n \cdot \operatorname{arccosh} x)$, the direct plugging gives that the function

$$\varphi(t) = 1/\cosh\left(\sqrt{2t}\right) \tag{11}$$

satisfies the system (9), as well as the conditions (10). Hence, the function

$$f(t) = \frac{1}{\cosh(at)}\ ,\ \ a > 0 \tag{12}$$

is actually a ch.f. of a strictly ν-normal r.v.. The ch.f. (12) is, in fact, well known – it is the ch.f. of the *hyperbolic secant distribution*. Clearly, a here is the scale parameter. When $a = 1$, it is the case of the *standard hyperbolic secant distribution*, whose pdf has the form

$$p(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi x}{2}\right)\,,$$

while the cdf is

$$F(x) = \frac{2}{\pi}\arctan\left[\exp\left(\frac{\pi x}{2}\right)\right]\,.$$

Furthermore, in order to obtain the expression for the ch.f. of strictly ν-stable distributions, one just needs to apply the relation (6) to the strictly stable (in the usual sense) ch.f. h.

4.2. An interesting property

Note that the function φ, as represented by (11), can be viewed somewhat interesting on its own, and so we shall address its properties and consider its cdf $A(x)$ (which corresponds to $\varphi(t)$ via (4)).

Let $W_1(t)$ and $W_2(t)$, $t \geq 0$, be two independent Wiener processes. Consider a r.v.

$$\xi = \int_0^1 W_1^2(t)dt + \int_0^1 W_2^2(t)dt\,. \tag{13}$$

This r.v. is well studied, and it is known that its Laplace transform equals to

$$\mathbf{E}\left[e^{-t\xi}\right] = \frac{1}{\cosh\left(\sqrt{2t}\right)},$$

which coincides with $\varphi(t)$ as given by (11).

Hence $A(x)$ is the cdf of the r.v. ξ. On the other hand, as follows from Gnedenko's Transfer Theorem,

$$A(x) = \lim_{p\to 0} \mathbf{P}\left\{ p\,\nu_p < x \right\}.$$

Consequently, the following theorem is valid.

Theorem 4.1. *Let* $\{\nu_p < x\,,\; p \in \Delta\}$ *be is family of r.v.'s having generating functions*

$$\mathcal{P}_p(z) = \frac{1}{T_{1/\sqrt{p}}(\frac{1}{z})}\,, \quad p \in \Delta = \left\{ \frac{1}{n^2}\,,\, n = 1, 2, \ldots \right\}.$$

Then

$$\lim_{p\to 0} \mathbf{P}\left\{ p\,\nu_p < x \right\} = \mathbf{P}\{\xi < x\},$$

where the r.v. ξ *is the one defined via (13).*

Theorem 4.1 may be reformulated in the following way.

Let

$$\frac{1}{T_n(\frac{1}{z})} = \sum_{k=0}^{\infty} p_k(n) z^k.$$

Then

$$\lim_{n\to\infty} \sum_{k=0}^{[n^2 x]} p_k(n) = \mathbf{P}\{\xi < x\}.$$

On Figure 1, the plot of the $\sum_{k=0}^{[n^2 x]} p_k(n)$ is given as a function of n starting with $n = 2$ until $n = 50$. We see that the functions attains the constant level rather quickly, and therefore it is possible to use the asymptotic result for $n > 25$.

Corollary 2. *Let* X *be a r.v. having the standard hyperbolic secant distribution. Then its distribution can be represented in the form of a scale mixture of normal distributions with zero mean and standard deviation* $\sqrt{\xi}$*, where* ξ *is defined via (13).*

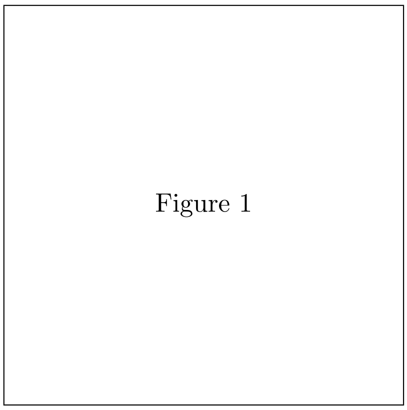

FIGURE 1: Plot of the $\sum_{k=0}^{[n^2 x]} p_k(n)$ as the function of $n = 2, \dots, 50$

To prove the above, one just needs to write the ch.f. of X in the form $\int\limits_0^\infty e^{-t^2 x} dA(x)$, and note that $e^{-t^2 x}$ is actually the ch.f. of the standard Normal r.v. $N(0, \sigma^2)$ $(\sigma^2 = x)$, while $A(x)$ is the cdf of ξ.

Note that there is a certain analogy between the representation $A(x)$ as the cdf of the r.v. ξ from (13) and the corresponding result in the scheme of the random summation with geometric distribution. Specifically, considering the family $\{\nu_p\,,\ p \in (0,1)\}$ having the geometric distribution $\mathbf{P}\{\nu_p = k\} = p(1-p)^{k-1}$, $k = 1, 2, \dots$, the function φ turns into

$$\varphi(t) = \frac{1}{1+t} = \int\limits_0^\infty e^{-tx} dA_1(x)\,,$$

where $A_1(x)$ is the cdf of the exponential distribution, i.e. $A_1(x) = 1 - e^{-x}$ for $x > 0$ and $A_1 = 0$ for $x \leq 0$. It can be checked that if η_1 and η_2 are two independent standard Normal r.v.'s, then A_1 is a cdf of the r.v. $\xi_1 = \eta_1^2 + \eta_2^2$, which is, in a way, related to (13).

4.3. Characterizations

Let us now turn to the characterizations of the distribution of the r.v. (13) and of the hyperbolic secant distribution.

Theorem 4.2. *Let $X_1, \dots, X_n, \dots$ be a sequence of non-negative iid random variables, and ν_p, $p \in \left\{\frac{1}{n^2}\,,\ n = 2, \dots\right\}$, is a family of the r.v.'s having the generating function*

$\mathcal{P}_p(z) = \frac{1}{T_{1/\sqrt{p}}(\frac{1}{z})}$, *independent of the sequence* $\{X_j \, , \; j \geq 1\}$.

If, for some fixed $p \in \Delta$,

$$X_1 \stackrel{d}{=} p \sum_{j=1}^{\nu_p} X_j \; , \tag{14}$$

(where "$\stackrel{d}{=}$" *is the equality in distribution), then* X_1 *has the distribution whose Laplace transform is*

$$\mathbf{E}\, e^{-tX} = \frac{1}{\cosh\left(\sqrt{at}\right)} \; , \; a > 0\, . \tag{15}$$

Proof. The equality (14), in terms of the Laplace transform $\Psi(t) = \mathbf{E}\, e^{-tX}$, can be represented as

$$\Psi(t) = \mathcal{P}_p\left(\Psi(pt)\right) \, . \tag{16}$$

Clearly, the function

$$\Psi_a(t) = \frac{1}{\cosh\left(\sqrt{at}\right)}$$

satisfies (16) for any $a > 0$ and, moreover, it is analytic in the strip $|\, t\,| < r$ $(\, r > 0\,)$.

In the following, we use the results of the book by Kakosyan, Klebanov and Melamed [7]. Example 1.3.2 of this book shows that $\{\, \Psi_a \, , \; a > 0\,\}$ forms a strongly $\mathcal{E}$-positive family on the set $\mathcal{C}$ of restrictions of Laplace transforms of probability distributions given in R_+ on an interval $[0, T]$ $(0 < T < r)$.

Clearly, the operator $A \, : \, f \, \to \, \mathcal{P}_p(f(pt))$ on $\mathcal{C}$ is intensively monotone.

The result follows from Theorem 1.1.1 of the above mentioned book (page 2).

Theorem 4.3. *Let* $X_1, \ldots, X_n, \ldots$ *be a sequence of non-negative iid random variables, having a symmetric distribution, while* $\{\nu_p \, , \; p \in \Delta\}$ *is the same family as in the previous Theorem.*

If, for some fixed $p \in \Delta$,

$$X_1 \stackrel{d}{=} p^{1/2} \sum_{j=1}^{\nu_p} X_j \; , \tag{17}$$

then X_1 *has the hyperbolic secant distribution whose ch.f. is*

$$f(t) = \frac{1}{\cosh\left(at\right)} \; , \; a > 0\, . \tag{18}$$

Proof. Quite analogous to the proof of the previous Theorem, with the difference that instead of Example 1.3.2, the use of the Example 1.3.1 from [7] is sufficient.

5. On other random sums of random number of summands with rational generating functions

In Section 3 it was mentioned that in the case described there, if two functions $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathcal{P} \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{P}$, then $\mathcal{P}$ and $\mathcal{Q}$ can be reduced by a conjugancy either to the form $\mathcal{P}(z) = z^m$ and $\mathcal{Q}(z) = z^m$ or to the form $\mathcal{P}(z) = T_m(z)$ and $\mathcal{Q}(z) = T_m(z)$. Therefore, the following question arises:

Let R be a fraction-linear function. Put $\mathcal{P} = R^{-1} \circ S \circ R$, where S is either z^m ($m > 1$), or $T_m(z)$. Is there a function $R(z) \neq a * z$ for which P is a generating function?

Here we will show the for the case $S(z) = z^m$ the answer is negative, while for the case $S(z) = T_n(m)$ the answer is yes.

5.1. Case $S(z) = z^m$

Consider linear-fractional function

$$R(z) = \frac{az+b}{cz+d}, \quad c \neq 0. \tag{19}$$

Because $\mathcal{P} = R^{-1} \circ S \circ R$, we have

$$\mathcal{P}_m(z) = P(z) = \frac{d(az+b)^m - b(cz+d)^m}{a(cz+d)^m - c(az+b)^m}. \tag{20}$$

However, $\mathcal{P}$ has to be a generating function of an integer-valued random variable $\nu \geq 1$, and therefore we must have

$$\mathcal{P}_m(1) = 1, \quad \mathcal{P}_m(0) = 0,$$

i.e.

$$\begin{cases} db^m = bd^m, \\ (a+b)^m(c+d) = (a+b)(c+d)^m. \end{cases} \tag{21}$$

The system (21) leads to six sub-cases:

$$\begin{cases} a+b = 0, \\ d = 0, \end{cases} \tag{22}$$

$$\begin{cases} a+b = 0, \; b = d \end{cases} \tag{23}$$

$$\begin{cases} c+d=0, \\ b=0, \end{cases} \tag{24}$$

$$\begin{cases} c+d=0, \\ b=d, \end{cases} \tag{25}$$

$$\begin{cases} a+b=c+d, \\ b=0, \end{cases} \tag{26}$$

$$\begin{cases} a+b=c+d, \\ d=0. \end{cases} \tag{27}$$

All the sub-cases have to be considered separately, but the method of consideration is similar for all of them, therefore we consider here one of them only. Let it be the case (26).

In the case (26) the generating function $\mathcal{P}_m$ has the form

$$\mathcal{P}_m(z) = \frac{d(c+d)^m z^m}{(c+d)(cz+d)^m - c(c+d)^m z^m} \tag{28}$$

We may suppose that $cd(c+d) \neq 0$. Denoting $p_1 = c/(c+d)$ rewrite (28) in the following form

$$\mathcal{P}_m(z) = \frac{q_1 z^m}{(p_1 z + q_1)^m - p_1 z^m}, \tag{29}$$

where $q_1 = 1 - p_1$. It is clear that $\mathcal{P}$ is a generating function if and only if

$$\mathcal{Q}_m(z) = \frac{q_1}{(p_1 z + q_1)^m - p_1 z^m}$$

is also a generating function. However,

$$\mathcal{Q}_m(z) = \frac{q_1}{p_1^m - p_1} \frac{1}{\prod_{k=1}^m (z - z_k)},$$

where z_k (k=1,2, …,m) are the zeros of the polynomial $(p_1 z + q_1)^m - p_1 z^m$. it is easy to find these zeros. We consider two cases:

a) $p_1 > 0$

b) $p_1 < 0$.

Let us start with the case a). In this case the zeros of the polynomial $(p_1 z+q_1)^m - p_1 z^m$ have the form

$$z_k = \frac{q_1}{p_1^{1/m}\varepsilon_m^{(k)} - p_1}, \quad k=1,2,\dots,m, \tag{30}$$

where $\varepsilon_m^{(k)}$ $(k=1,2,\dots,m)$ are roots of order m from 1. In other words,

$$\varepsilon_m^{(k)} = \cos\frac{2(k-1)\pi}{m} + i\sin\frac{2(k-1)\pi}{m}, \quad k=1,2,\dots,m.$$

Using partial fraction decomposition let us write the function $\mathcal{Q}_m$ in the form

$$\mathcal{Q}_m(z) = \frac{q_1}{p_1^m - p_1}\sum_{k=1}^{m}\frac{A_k}{z-z_k}, \tag{31}$$

where $A_k = 1/\prod_{j\neq k}(z_k - z_j)$.

Now it is easy to find the expression of $\mathcal{Q}_m(z)$ in the form of power series. Namely,

$$\mathcal{Q}_m(z) = \frac{q}{p_1 - p_1^m}\sum_{s=0}^{\infty}\Big(\sum_{k=1}^{m}\frac{1}{\prod_{j\neq k}(z_k - z_j)}\frac{1}{z_k^{s+1}}\Big)z^s. \tag{32}$$

Because $\mathcal{Q}_m$ is a generating function, the series in (32) must converge for all complex z under condition $|z|\leq 1$, the nearest to zero singular point of the series has to lie on positive semi-axes, and the coefficients of the series have to be non-negative. Moreover, because $\mathcal{Q}_m \circ \mathcal{Q}_m = \mathcal{Q}_{m^2}$, the same properties have to hold not only for one fixed m, but for the sequence m^l, $l=1,2,\dots$, i.e. for all functions $\mathcal{Q}_{m^l}$, $l=1,2,\dots$.

The property of the convergence of the series inside the closed unit circle implies, that $|z_k|>1$ for all $k=1,2,\dots,m$, i.e. we must have

$$\cos^2\frac{2\pi n}{m} + p_1^{1+1/m}\cos\frac{2\pi n}{m} \geq 2p_1 + p_1^{2/m}\cos^2\frac{2\pi n}{m}, \; n=0,1,\dots m-1.$$

The last inequality has to hold not only for one fixed value of m, but for a sequence $m_l \to \infty$ as $l\to\infty$ (for example, for $m_l = m^l$). Passing to limit as $m\to\infty$ in the case of $n=0$ we get

$$1 \geq 1 + p_1,$$

which is wrong. So, in the case a), the function $\mathcal{P}_m$ cannot be a generation function.

Let us now consider the case b) $p_1 < 0$. Denote $p_2 = -p_1 > 0$, $q_1 = 1-p_1 = 1+p_2$. the polynomial $(p_1 z + q_1)^m - p_1 z^m = (-p_2 z + (1+p_2))^m + p_2 z^m$ has the form

$$z_k = \frac{1+p_2}{p_2^{1/m}\delta_m^{(k)} + p_2}, \quad k=1,2,\dots,m, \tag{33}$$

where

$$\delta_m^{(k)} = \cos\frac{(2k-1)\pi}{m} + i\sin\frac{(2k-1)\pi}{m}, \quad k = 1, 2, \ldots, m.$$

It is easy to calculate that

$$z_k = \frac{(1+p_2)(p_2^{1/m}\ cos\frac{(2k-1)\pi}{m} + p_1 - i\sin\frac{(2k-1)\pi}{m})}{p_2^{2/m} + 2p_2^{1+1/m}\cos\frac{(2k-1)\pi}{m} + p_2^2}$$

$$k = 1, 2, \ldots, m.$$

Therefore,

$$|z_k| = (1+p_2)\frac{1}{\sqrt{p^{2/m} + 2p_2^{1+1/m}\cos\frac{(2k-1)\pi}{m} + p_2^2}}, \ k = 1, 2, \ldots, m. \tag{34}$$

If the function $\mathcal{Q}_m$ would be a generating function, then $|z_k|$ with minimal absolute value has to lie on positive semi-axis, but, as (34) shows, it is not true. Therefore in the case b), the function $\mathcal{P}_m$ cannot be a generation function either.

To summarize: as we have seen, $\mathcal{P}_m$ cannot be a generation function in the case (26). The cases (22)-(25) and (27) can be considered in similar way.

Finally, we see that *there is no generating functions, which are conjugate to power z^m and to equal to this power.*

5.2. Case $S = T_m$

For any $a \in [1/2, 1]$ and any integer $m > 1$ let us define the following function

$$\mathcal{P}_m(z) = \frac{1}{aT_m(1/(az) - (1-a)/a) + (1-a)}. \tag{35}$$

Hypothesis *For any $a \in (1/2, 1]$ and any integer $m > 1$ the function $P_m(z)$ defined by (35) is a generating function.*

Unfortunately, we cannot prove this Hypothesis in full. However, we will give the proof for the case of even m.

Theorem 5.1. *For any $a \in (1/2, 1]$ and any integer even $m > 1$ the function $\mathcal{P}_m(z) defined by (35)$ is a generating function.*

Proof. Consider the equation

$$aT_m(x) + (1-a) = 0.$$

Its roots are

$$x_k = \cos\Big(\frac{1}{m}\arccos\frac{a-1}{a} + \frac{2\pi k}{m}\Big), \quad k = 0, 1, \dots, m-1,$$

and $|x_k| < 1$. In view of the fact that m is an even number, the roots are symmetric around zero. Therefore

$$aT_m(x) + (1-a) = a2^{m-1}\prod_{k=0}^{m-1}(x - x_k) = a2^{m-1}\prod(x^2 - x_k^2),$$

where the latter product is taken over such k for which $x_k > 0$. Now we see, that

$$\mathcal{P}_m(z) = \Big(\frac{az}{1-(1-a)z}\Big)^m \frac{1}{\lambda\prod(1 - x_k^2(az/(1-(1-a)z))^2)},$$

where $\lambda = a2^{m-1}$. The statement now follows from the fact, that

$$\frac{1}{1 - x_k^2(az/(1-(1-a)z))^2)} = \sum_{j=0}^{\infty}(x_k)^{2j}(az/(1-(1-a)z))^{2j},$$

and $az/(1-(1-a)z)$ is the sum of a geometric progression with denominator $(1-a)$.

The family of functions (35) for any fixed $a \in [1/2, 1]$ is commutative with respect to convolution, i.e.

$$\mathcal{P}_m(\mathcal{P}_n(z)) = \mathcal{P}_n(\mathcal{P}_m(z)) = \mathcal{P}_{mn}(z),$$

and consequently, there exists ν-Gaussian distribution, where the family $\{\nu_p,\ p \in \Delta\}$ is defined by the family of corresponding generating functions $\mathcal{P}_m(z),\ m = 2, 4, 6, \dots$, and the parameter p is defined by m through the relation

$$p = p(m) = \frac{1}{\mathcal{P}'_m(1)}.$$

We will not study here corresponding characteristic functions. It can be done using the general approach described above.

6. Examples with non-rational generating functions

There exist examples of the pairs of commutative functions, which are *not rational.* Here we refer to the two classes of such functions, the first of which was investigated by Melamed [11] and the second appears at first in the present work.

Example 6.1. (See Melamed [11] for detailed study)

Consider the family of generating functions

$$\mathcal{P}_p(z) = \frac{p^{1/m}\, z}{(1-(1-p)\, z^m)^{1/m}} \ , \tag{36}$$

where $p \in (0,1)$, and m is a fixed positive integer. Obviously, in the case $m = 1$, $\mathcal{P}_p(z)$ reduces to the generating function of the geometric distribution, and has already been mentioned this case above. Hence, assume that $m \geq 2$. In that case, it is easy to check that

$$\varphi(t) = \frac{1}{(1+mt)^{1/m}} \ , \tag{37}$$

and therefore the ch.f. of the strictly ν-normal distribution (for the family $\{\nu_p \, , \ p \in \Delta\}$ having the generating function (36)) has the form

$$\varphi(t) = \frac{1}{(1+mat^2)^{1/m}} \ ,$$

with a parameter $a > 0$.

Example 6.2. Consider the family of functions

$$\mathcal{P}_p(z) = \frac{1}{\left(T_{1/\sqrt{p}}\left(\frac{1}{z^m}\right)\right)^{1/m}} \ , \tag{38}$$

where $p \in \left\{\frac{1}{n^2} \, , \, n = 2, \dots\right\}$, and $m \geq 1$ (an integer).

Using a slightly modified version of the proof of *Lemma 1*, it is easy to check that $\mathcal{P}_p(z)$ is a generating function of some r.v. ν_p, $p \in \left\{\frac{1}{n^2} \, , \, n = 2, \dots\right\}$ for any fixed whole number $m \geq 1$ (surely, both $\mathcal{P}_p$ and ν_p both depend on m, but we omit this dependence in the notation).

The case $m = 1$ has already been considered above. For $m \geq 2$ analogous methods are applicable, and so will shall only refer to the results. Specifically,

$$\varphi(t) = \frac{1}{\left(\cosh\sqrt{2mt}\right)^{1/m}} \ , \tag{39}$$

while the ch.f. of the corresponding strictly ν-normal distribution has the form

$$f(t) = \frac{1}{\left(\cosh at\right)^{1/m}} \ , \tag{40}$$

where $a > 0$.

Note that in the case $m = 2$, we have the following expressions for the distributions whose Laplace transforms are (37) and (39).

For $m = 2$, the formula (37) gives

$$\varphi(t) = \frac{1}{\sqrt{1+2t}} .$$

This function is the Laplace transform of the distribution of the r.v. X^2, with X being the standard Normal r.v.

In a similar way, (39) gives for $m = 2$

$$\varphi(t) = \frac{1}{\sqrt{\cosh\sqrt{4t}}} .$$

This function is the Laplace transform of the distribution of the r.v. $I = \int_0^1 X^2(t)dt$, where $X(t)$ is the standard Wiener process.

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