Tempered stable and tempered infinitely divisible GARCH models

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Abstract. In this paper, we introduce a new GARCH model with an infinitely divisible distributed innovation, referred to as the rapidly decreasing tempered stable (RDTS) GARCH model. This model allows the description of some stylized empirical facts observed for stock and index returns, such as volatility clustering, the non-zero skewness and excess kurtosis for the residual distribution. Furthermore, we review the classical tempered stable (CTS) GARCH model, which has similar statistical properties. By considering a proper density transformation between infinitely divisible random variables, these GARCH models allow to find the risk-neutral price process, and hence they can be applied to option pricing. We propose algorithms to generate scenario based on GARCH models with CTS and RDTS innovation. To investigate the performance of these GARCH models, we report a parameters estimation for Dow Jones Industrial Average (DJIA) index and stocks included in this index, and furthermore to demonstrate their advantages, we calculate option prices based on these models. It should be noted that only historical data on the underlying asset and on the riskfree rate are taken into account to evaluate option prices.

Keywords: tempered infinitely divisible distribution, tempered stable distribution, rapidly decreasing tempered stable distribution, GARCH model option pricing.

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1 Introduction

The autoregressive conditional heteroskedastic (ARCH) and the generalized ARCH (GARCH) models introduced by Engle (1982) and Bollerslev (1986), respectively, and applied to option pricing by Duan (1995), have become a standard framework to explain the volatility clustering of return processes and volatility smile effect of option prices. However, empirical studies based on GARCH models show that the hypothesis that the distribution of residuals is normally distributed is often rejected (see Duan (1999), and Menn and Rachev (2005a, 2005b)). Duan *et al.* (2006) enhanced the classical GARCH model by adding jumps to the innovation process. Subsequently, Menn and Rachev (2005a, 2005b) introduced both an enhanced GARCH and a nonlinear GARCH model (NGARCH) with innovations which follow the smoothly truncated stable (STS) distribution. Recently, the tempered stable distributions were applied to modeling the residual distribution. For example Kim *et al.* (2008a,2008c) used the tempered stable distributions for fitting residuals of the GARCH model. However, since the convexity correction, which is defined by the log Laplace transform of the innovation distribution, is defined only on a bounded interval, the variance process is artificially restricted.

In this paper, we focus on two different distributional assumptions, the classical tempered stable (CTS) and the rapidly decreasing tempered stable (RDTS). The former belongs to the class proposed by Rosiński (2007) and has been already applied to option pricing with volatility clustering by Kim *et al.* (2008a), the latter belongs to the class proposed by Bianchi *et al.* (2008).

The first objective of the paper is to present this new infinitely divisible (ID) distribution referred to as the RDTS distribution, and to study its mathematical properties. The RDTS distribution is obtained by taking an α-stable law and multiplying the Lévy measure by a moment-generating function of a normal distribution onto each half of the real axis. It has asymmetric properties and fatter tails than the normal distribution. Moreover, its Laplace transform is defined on the entire real line. By following the approach used in Kim *et al.* (2008a), we review an asset price model based on the GARCH model with

CTS distributed innovation, introduce a similar model with RDTS distributed innovation, and compare it with the normal-GARCH case. These non-normal models explain the time-varying property of volatility in asset returns, and describe properties of the empirical residual distribution which cannot be described by the normal distribution including skewness and fat-tail properties. Furthermore, a large scale empirical analysis is considered on the Dow Jones Industrial Average (DJIA) index and stocks included in this index, in order to assess the goodness of fit.

The second objective of the paper is to test the option pricing performance of this approach based on non-normal distributions. Recently, a general idea has been that for the purpose of option valuation, parameters estimated from option prices are preferable to parameters estimated from the underlying returns, see Chernov and Ghysels (2000). Alternatively, the most recent results are based on a different approach. Both historical asset prices and option prices are considered to assess the model performance. Parametric models by Christoffersen *et al.* (2008), Kim *et al.* (2008a), Stentoft (2008), and a nonparametric one by Barone-Adesi *et al.* (2008) have been proposed by connecting the statistical with the risk-neutral measure. Instead of imposing conditions on preferences of investors or the Esscher transform as in Christoffersen *et al.* (2008), by using a density transformation between ID random variables, we can then develop a method for pricing options based on these GARCH models, see also Kim *et al.* (2008a,2008c). It should be noted that only historical data on the underlying asset and on the risk-free rate are considered in obtaining the parameters to be used in option valuation. Instead, to consider a *trader approach*, in which one wants to estimate parameters by using only option prices, we follow the so called *fundamental approach*, that is we calculate option prices by using parameters estimated by fitting the underlying asset process together with a suitable change of measure. Pricing errors on DJIA European call options (DJX) will be computed, in order to analyze the effect of conditional leptokurtosis and skewness on option pricing.

The remainder of this paper is organized as follows. Section 2 reviews the classical tempered stable distribution. The RDTS distribution and its mathematical properties are presented in Section 3. The GARCH model with ID innovations and its CTS and RDTS subclasses are discussed in Section 4. Simulation algorithms for the GARCH models are given in Section 5. The empirical results are reported in Section 6. Section 7 summarizes the principal conclusions of the paper and the appendix contains the proofs of the main theoretical results.

2 Classical tempered stable distribution

Before introducing the RDTS distribution, let us review the CTS distribution. This distribution has been studied under different names including: the *truncated Lévy flight* by Koponen (1995), the *tempered stable* by both Barndorff-Nielsen and Shephard (2001) and Cont and Tankov (2004), the *KoBoL* distribution by Boyarchenko and Levendorskiĭ (2000), and the *CGMY* by Carr *et al.* (2002). The KR distribution of Kim *et al.* (2008b) is an extension of the CTS distribution. Rosiński (2007) generalized the CTS distribution referring to it as the tempered stable distribution.

The CTS distribution is defined as follows:

Definition 2.1. *An infinitely divisible random variable X is said to follow the CTS distribution if its Lévy triplet (σ^2, ν, γ) is given by $\sigma = 0$,*

$$\nu(dx) = \left(C_+ e^{-\lambda_+ x} 1_{x>0} + C_- e^{-\lambda_- |x|} 1_{x<0}\right) \frac{dx}{|x|^{\alpha+1}},$$

and

$$\gamma = m - \int_{|x|>1} x\nu(dx),$$

where $C_+, C_-, \lambda_+, \lambda_- > 0$, $\alpha \in (0,2)$ and $m \in \mathbb{R}$, and we denote $X \sim \text{CTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$. A Lévy process induced from the CTS distribution is called a CTS process *with parameters $(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.*

The characteristic function of $X \sim$ CTS(α, C_+, C_-, λ_+, λ_-, m) is given by

$$
\begin{aligned}
(2.1) \qquad \phi(u;\alpha,C_+,C_-,\lambda_+,\lambda_-,m) = \exp(&ium - iu\Gamma(1-\alpha)(C_+\lambda_+^{\alpha-1} - C_-\lambda_-^{\alpha-1}) \\
&+ C_+\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - {\lambda_+}^\alpha) \\
&+ C_-\Gamma(-\alpha)((\lambda_- + iu)^\alpha - {\lambda_-}^\alpha))).
\end{aligned}
$$

Moreover, ϕ can be extended via analytic continuation to the region $\{z \in \mathbb{C} : \text{Im}(z) \in [-\lambda_-, \lambda_+]\}$. The proof can be found in Carr *et al.* (2002) and Cont and Tankov (2004). Using the characteristic function, we can obtain cumulants

$$
c_n(X) := \frac{1}{i^n}\frac{d^n}{du^n}\log E[e^{iuX}]\big|_{u=0}
$$

of the CTS distributed random variable X such that

$$
\begin{aligned}
c_1(X) &= m, && \text{for } n = 1 \\
c_n(X) &= \Gamma(n-\alpha)\left(C_+\lambda_+^{\alpha-n} + (-1)^n C_-\lambda_-^{\alpha-n}\right), && \text{for } n = 2, 3, \cdots.
\end{aligned}
$$

If we substitute

$$
C = C_+ = C_- = \left(\Gamma(2-\alpha)({\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2})\right)^{-1}
$$

then $X \sim$ CTS(α, C, C, λ_+, λ_-, 0) has zero mean and unit variance. In this case, X is called the *standard CTS distribution* with parameters (α,λ_+,λ_-) and denoted by $X \sim \text{stdCTS}(\alpha, \lambda_+, \lambda_-)$. The log-Laplace transform $\log E[\exp(uX)]$ of the random variable $X \sim \text{stdCTS}(\alpha, \lambda_+, \lambda_-)$ is denoted by $L_{CTS}(u; \alpha, \lambda_+, \lambda_-)$. The function $L_{CTS}(u; \alpha, \lambda_+, \lambda_-)$ is defined on $u \in [-\lambda_-, \lambda_+]$ and we can obtain

$$
\begin{aligned}
(2.2) \qquad & L_{CTS}(u;\alpha,\lambda_+,\lambda_-) \\
&= \frac{(\lambda_+ - u)^\alpha - {\lambda_+}^\alpha + (\lambda_- + u)^\alpha - {\lambda_-}^\alpha}{\alpha(\alpha-1)({\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2})} - \frac{u({\lambda_+}^{\alpha-1} - {\lambda_-}^{\alpha-1})}{(1-\alpha)({\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2})}
\end{aligned}
$$

by the characteristic function (2.1).

We can make use of the following proposition proven in Kim and Lee (2006) to find an equivalent measure for CTS processes.

Proposition 2.2. *Suppose* $(X_t)_{t\in[0,T]}$ *is the CTS process with parameters* $(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ *under* $\mathbb{P}$ *and the CTS process with parameters* $(\tilde{\alpha}, \tilde{C}_+, \tilde{C}_-, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{m})$ *under* $\mathbb{Q}$*. Then* $\mathbb{P}|_{\mathcal{F}_t}$ *and* $\mathbb{Q}|_{\mathcal{F}_t}$ *are equivalent for all* $t > 0$ *if and only if* $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$, *and*

$$\tilde{m} - m = \Gamma(1-\alpha)(C_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - C_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})).$$

When $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent, the Radon-Nikodym derivative is* $\frac{d\mathbf{Q}}{d\mathbf{P}}\big|_{\mathcal{F}_t} = e^{U_t}$ *where* $(U_t, \mathbf{P})$ *is a Lévy process with Lévy triplets* $(\sigma_U^2, \nu_U, \gamma_U)$ *given by*

$$\sigma_U^2 = 0,\ \nu_U = \nu \circ \psi^{-1},\ \gamma_U = -\int_{-\infty}^{\infty} (e^y - 1 - y1_{|y|\le 1})(\nu \circ \psi^{-1})(dy) \tag{2.3}$$

where $\psi(x) = (\lambda_+ - \tilde{\lambda}_+)x1_{x>0} - (\lambda_- - \tilde{\lambda}_-)x1_{x<0}$.

Applying Proposition 2.2 to CTS distributed random variables, we obtain the following corollary.

Corollary 2.3. *(a) Let* $X \sim \text{CTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ *under a measure* $\mathbf{P}$*, and* $X \sim \text{CTS}(\tilde{\alpha}, \tilde{C}_+, \tilde{C}_-, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{m})$ *under a measure* $\mathbf{Q}$*. Then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent if and only if* $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$, *and*

$$\tilde{m} - m = \Gamma(1-\alpha)(C_+\tilde{\lambda}_+^{\alpha-1} - C_-\tilde{\lambda}_-^{\alpha-1} - C_+\lambda_+^{\alpha-1} + C_-\lambda_-^{\alpha-1}).$$

(b) Let $X \sim \text{stdCTS}(\alpha, \lambda_+, \lambda_-)$ *under a measure* $\mathbf{P}$*, and* $(X+k) \sim \text{stdCTS}(\tilde{\alpha}, \tilde{\lambda}_+, \tilde{\lambda}_-)$

under a measure **Q** *for a constant* $k \in \mathbb{R}$*. Then* **P** *and* **Q** *are equivalent if and only if*

$$\begin{cases} \alpha = \tilde{\alpha}, \\ {\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2} = \tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}, \\ k = \dfrac{{\lambda_+}^{\alpha-1} - {\lambda_-}^{\alpha-1} - \tilde{\lambda}_+^{\alpha-1} + \tilde{\lambda}_-^{\alpha-1}}{(1-\alpha)\left({\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2}\right)}. \end{cases} \tag{2.4}$$

2.1 Simulation of the CTS distribution

CTS distributed random numbers can be generated using the subordination method developed by Poirot and Tankov (2006). Here, we will apply the series representation presented by Rosiński (2007) to the CTS distribution instead of the subordination method, see also Asmussen and Glynn (2007).

Consider $\alpha \in (0,2)$, $C > 0$, and $\lambda_+, \lambda_- > 0$. Let $\{v_j\}$ be an independent and identically distributed (i.i.d.) sequence of random variables in $\{\lambda_+, \lambda_-\}$ with $P(v_j = \lambda_+) = P(v_j = -\lambda_-) = 1/2$. Let $\{u_j\}$ be an i.i.d. sequence of uniform random variables on $(0,1)$ and let $\{e_j\}$ and $\{e'_j\}$ be i.i.d. sequences of exponential random variables with parameters 1. Furthermore, we assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{e'_j\}$ are independent. We consider $\gamma_j = e'_1 + \ldots + e'_j$ and, by definition of $\{e'_j\}$, $\{\gamma_j\}$ is a Poisson point process on $(0,\infty)$ with Lebesgue intensity measure. Based on these assumption, we can prove the next theorem.

Theorem 2.4. *Suppose that all the above assumptions are fulfilled. If* $\alpha \in (0,2) \setminus \{1\}$*, the series*

$$S = \sum_{j=1}^{\infty} \left(\left(\frac{\alpha \gamma_j}{C} \right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha} |v_j|^{-1} \right) \frac{v_j}{|v_j|} - \Gamma(1-\alpha) C (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) \tag{2.5}$$

converges a.s. Furthermore, we have that $S \sim CTS(\alpha, C, C, \lambda_+, \lambda_-, 0)$

If $\alpha = 1$*, the series*

$$S = \sum_{j=1}^{\infty} \left(\left(\frac{\alpha \gamma_j}{C} \right)^{-1} \wedge e_j u_j |v_j|^{-1} \right) \frac{v_j}{|v_j|} - C \log \left(\frac{\lambda_+}{\lambda_-} \right) \tag{2.6}$$

converges a.s. and we have $S \sim CTS(1, C, C, \lambda_+, \lambda_-, 0)$.

Proof. This is a particular case of Theorem 5.1 in Rosiński (2007). □

3 Rapidly decreasing tempered stable distribution

In this section, we present an ID distribution which we refer to as the RDTS distribution. This distribution is defined as follows:

Definition 3.1. Let $m \in \mathbb{R}$, $C_+, C_-, \lambda_+, \lambda_- > 0$, $\alpha \in (0, 2)$, and $\alpha \neq 1$. An infinitely divisible distribution is called a RDTS distribution with parameter $(\alpha$, C_+, C_-, λ_+, λ_-, $m)$ if its Lévy triplet (σ^2, ν, γ) is given by $\sigma = 0$,

$$\nu(dx) = (C_+ e^{-\lambda_+^2 x^2/2} 1_{x>0} + C_- e^{-\lambda_-^2 |x|^2/2} 1_{x<0}) \frac{dx}{|x|^{\alpha+1}},$$

and

$$\gamma = m - \int_{|x|>1} x\nu(dx). \tag{3.1}$$

If a random variable X follows the RDTS distribution, then we denote $X \sim$ RDTS$(\alpha$, C_+, C_-, λ_+, λ_-, $m)$.

Remark 3.2. *RDTS distributions are not included in the generalized class of tempered stable distributions by Rosiński (2007), but included in the class of the tempered infinitely divisible distribution (Bianchi et al. (2008)).*

The characteristic function of the RDTS distribution is found in the following proposition and its proof is presented in Appendix A.

Proposition 3.3. *Let*

$$G(x; \alpha, \lambda) := 2^{-\frac{\alpha}{2}-1} \lambda^\alpha \Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)$$
$$+ 2^{-\frac{\alpha}{2}-\frac{1}{2}} \lambda^{\alpha-1} x \Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)$$

where M is the confluent hypergeometric function[1]*. The characteristic function of the RDTS distribution with parameter* $(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ *becomes*

$$\phi(u) = \exp\left(ium + C_+G(iu;\alpha,\lambda_+) + C_-G(-iu;\alpha,\lambda_-)\right) \tag{3.2}$$

for some $m \in \mathbb{R}$. Moreover, $\phi(u)$ is expandable to an entire function on $\mathbb{C}$.

Although the Laplace transform of the CTS distribution is defined on a bounded interval, in the case of the RDTS distribution the Laplace transform is defined on the entire real line.

Proposition 3.4. *Let $X \sim \text{RDTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$. Then the Laplace transform $E[e^{\theta X}] < \infty$ for all $\theta \in \mathbb{R}$. Moreover, the explicit formula of the Laplace transform is given by*

$$E[e^{\theta X}] = \exp\left(\theta m + C_+G(\theta;\alpha,\lambda_+) + C_-G(-\theta;\alpha,\lambda_-)\right).$$

Using the characteristic function (3.2), we can get cumulants of the RDTS distribution.

Proposition 3.5. *The cumulants of $X \sim \text{RDTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ are given by $c_1(X) = m$ and*

$$c_n(X) = 2^{\frac{n-\alpha-2}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-n} + (-1)^nC_-\lambda_-^{\alpha-n}\right), \tag{3.3}$$

for $n = 2, 3, \cdots$.

Proof. Since we have

$$G(\pm iu;\alpha,\lambda_\pm) = \sum_{n=2}^{\infty}\frac{1}{n!}(\pm iu)^n\frac{1}{2}\left(\frac{\lambda_\pm}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right),$$

we deduce

$$\frac{1}{i}\frac{d}{du}G(\pm iu;\alpha,\lambda_\pm)\big|_{u=0} = 0$$

and

$$\frac{1}{i^n}\frac{d^n}{du^n}G(\pm iu;\alpha,\lambda_\pm)\big|_{u=0}=\frac{1}{2}\left(\frac{\lambda_\pm}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right)(-i)^n,$$

if $n=2,3,\cdots$. Hence we obtain the formula (3.3). □

Moreover, we obtain the mean, variance, skewness, and excess kurtosis using the cumulants as given below:

$$\begin{aligned}
E[X]&=c_1(X)=m\\
\text{Var}(X)&=c_2(X)=2^{-\frac{\alpha}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)\\
s(X)&=\frac{c_3(X)}{c_2(X)^{\frac{3}{2}}}=2^{\frac{\alpha}{4}+\frac{1}{2}}\frac{\Gamma\left(\frac{3-\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-3}-C_-\lambda_-^{\alpha-3}\right)}{\left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)\right)^{\frac{3}{2}}}\\
k(X)&=\frac{c_4(X)}{c_2(X)^2}=2^{\frac{\alpha}{2}+1}\frac{\Gamma\left(\frac{4-\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-4}+C_-\lambda_-^{\alpha-4}\right)}{\left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(C_+\lambda_+^{\alpha-2}+C_-\lambda_-^{\alpha-2}\right)\right)^{2}}.
\end{aligned}$$

The parameters λ_+ and λ_- control the rate of decay on the positive and negative tails, respectively. If $\lambda_+>\lambda_-$ ($\lambda_+<\lambda_-$), then the distribution is skewed to the left (right). Moreover, if $\lambda_+=\lambda_-$, then it is symmetric. If we substitute

$$C=C_+=C_-=2^{\frac{\alpha}{2}}\left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)\right)^{-1}$$

then $X\sim RDTS(\alpha,C,C,\lambda_+,\lambda_-,0)$ has zero mean and unit variance. In this case, X is called the *standard RDTS distribution* and denoted by $X\sim \text{stdRDTS}(\alpha,\ \lambda_+,\ \lambda_-)$. Moreover, the log-Laplace transform of X is denoted by $L_{RDTS}(x;\ \alpha,\ \lambda_+,\ \lambda_-)$. By Proposition 3.4, the function $L_{RDTS}(x;\ \alpha,\ \lambda_+,\ \lambda_-)$ is finite for all $x\in\mathbb{R}$, and we have

$$L_{RDTS}(x;\alpha,\lambda_+,\lambda_-)=CG(x;\alpha,\lambda_+)+CG(-x;\alpha,\lambda_-). \tag{3.4}$$

Since the RDTS distribution is infinitely divisible, we can generate a Lévy process called the RDTS process.

Definition 3.6. A Lévy process $X=(X_t)_{t\geq 0}$ is said to be a *RDTS process* with parameters $(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ if $X_1 \sim \text{RDTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.

The parameter α determines the path behavior; that is, the RDTS process has finite variation if $\alpha < 1$ and infinite variation if $\alpha > 1$. The following proposition (which we prove in Appendix A) will be used for determining the equivalent martingale measure.

Proposition 3.7. *Suppose $(X_t)_{t\in[0,T]}$ is the RDTS process with parameters $(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ under $\mathbb{P}$, and the RDTS process with parameters $(\tilde{\alpha}, \tilde{C}_+, \tilde{C}_-, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{m})$ under $\mathbb{Q}$. Then $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are equivalent for all $t > 0$ if and only if $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$, and*

$$\tilde{m} - m = 2^{-\frac{\alpha+1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\left(C_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - C_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})\right).$$

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the Radon-Nikodym derivative is $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{U_t}$ where $(U_t, \mathbb{P})$ is a Lévy process with Lévy triplet $(\sigma_U^2, \nu_U, \gamma_U)$ given by

$$\sigma_U^2 = 0,\ \nu_U = \nu \circ \psi^{-1},\ \gamma_U = -\int_{-\infty}^{\infty}(e^y - 1 - y1_{|y|\le 1})(\nu\circ\psi^{-1})(dy) \tag{3.5}$$

where $\psi(x) = \frac{x^2}{2}(\lambda_+ - \tilde{\lambda}_+)1_{x>0} + \frac{x^2}{2}(\lambda_- - \tilde{\lambda}_-)1_{x<0}$.

Applying Proposition 3.7 to RDTS distributed random variables, we can obtain the following corollary.

Corollary 3.8. *(a) Let $X \sim \text{RDTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ under a measure **P**, and $X \sim \text{RDTS}(\tilde{\alpha}, \tilde{C}_+, \tilde{C}_-, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{m})$ under a measure **Q**. Then **P** and **Q** are equivalent if and only if $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$, and*

$$\tilde{m} - m = 2^{-\frac{\alpha+1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\left(C_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - C_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})\right).$$

*(b) Let $X \sim \text{stdRDTS}(\alpha, \lambda_+, \lambda_-)$ under a measure **P**, and $(X + k) \sim \text{stdRDTS}(\tilde{\alpha}, \tilde{\lambda}_+, \tilde{\lambda}_-)$ under a measure **Q** for a constant $k \in \mathbb{R}$. Then **P** and **Q** are equivalent if and only*

if

$$
\left\{
\begin{aligned}
&\alpha = \tilde{\alpha},\\
&\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} = \tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2},\\
&k = \frac{\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+^{\alpha-1} + \tilde{\lambda}_-^{\alpha-1}\right)}{\sqrt{2}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}\right)}.
\end{aligned}
\right. \tag{3.6}
$$

3.1 Simulation of the RDTS distribution

The RDTS distribution is included in the class of TID distributions. The general method of generating TID distributed random numbers can be found in Bianchi *et al.* (2008) and we summarize it below.

Consider $\alpha \in (0,2) \setminus \{1\}$, $C > 0$, and $\lambda_+, \lambda_- > 0$. Let $\{v_j\}$ be an i.i.d. sequence of random variables in $\{\lambda_+, \lambda_-\}$ with $P(v_j = \lambda_+) = P(v_j = -\lambda_-) = 1/2$. Let $\{u_j\}$ be an i.i.d. sequence of uniform random variables on $(0,1)$ and let $\{e_j\}$ and $\{e_j'\}$ be i.i.d. sequences of exponential random variables with parameters 1 and $1/2$, respectively. Furthermore, we assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$, and $\{e_j'\}$ are independent. We consider $\gamma_j = e_1' + \ldots + e_j'$.

Using Theorems 4.2 and 4.3 of Bianchi *et al.* (2008), we can obtain the following theorem.

Theorem 3.9. *Suppose that all the above assumptions are fulfilled. If $\alpha \in (0,2) \setminus \{1\}$, the series*

$$
X = \sum_{j=1}^{\infty} \left(\left(\frac{\alpha \gamma_j}{C} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} |v_j|^{-1} \right) \frac{v_j}{|v_j|} - \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}} (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) \tag{3.7}
$$

converges a.s.. Furthermore, we have that $X \sim$ RDTS$(\alpha, C, C, \lambda_+, \lambda_-, 0)$.

3.2 Tail properties

Let's look at the probability tails of the CTS and RDTS distributions. Although the exact asymptotic behavior of its tails is difficult to obtain unlike those of the stable distribution, it is possible to calculate the upper and lower bounds.

Proposition 3.10. *If* $X \sim \text{CTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$*, then the following inequality is fulfilled*

$$k\frac{e^{-2\bar{\lambda}y}}{\bar{\lambda}y^{\alpha+1}} \le \mathbb{P}(|X-m| \ge y) \le \frac{K}{y^2}$$

as $y \to \infty$*, where* k *and* K *do not depend on* y *and* $\bar{\lambda} = \min(\lambda_+, \lambda_-)$.

Proposition 3.11. *If* $X \sim \text{RDTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$*, then the following inequality is fulfilled*

$$k\frac{e^{-2\bar{\lambda}^2y^2}}{\bar{\lambda}^2y^{\alpha+2}} \le \mathbb{P}(|X-m| \ge y) \le \frac{K}{y^2}$$

as $y \to \infty$*, where* k *and* K *do not depend on* y *and* $\bar{\lambda} = \min(\lambda_+, \lambda_-)$.

4 GARCH model with infinitely divisible distributed innovations

Our objective in this section is twofold. First, we review the infinitely divisible GARCH (ID-GARCH) model and the CTS-GARCH model which is a subclass of the ID-GARCH model. Second, we construct a new subclass of the ID-GARCH model with standard RDTS distributed innovation. Some details and proofs for the ID-GARCH model and CTS-GARCH model can be found in Kim *et al.* (2008a).

The ID-GARCH stock price model is defined over a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ which is constructed as follows. Consider a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of i.i.d. real random variables on a sequence of probability spaces $(\Omega_t, \mathbf{P}_t)_{t\in\mathbb{N}}$, such that ε_t is an ID distributed random variable with zero mean and unit variance on $(\Omega_t, \mathbf{P}_t)$, and assume that $E[e^{x\varepsilon_t}] < \infty$ where $x \in I$ for some real interval I containing zero. Now we define $\Omega := \prod_{t\in\mathbb{N}} \Omega_t$, $\mathfrak{F}_t := \otimes_{k=1}^t \sigma(\varepsilon_k) \otimes \mathfrak{F}_0 \otimes \mathfrak{F}_0 \cdots$, $\mathfrak{F} := \sigma\left(\cup_{t\in\mathbb{N}}\mathfrak{F}_t\right)$, and $\mathbb{P} := \otimes_{t\in\mathbb{N}}\mathbf{P}_t$, where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ-algebra generated by ε_k on Ω_k.

We first propose the following stock price dynamic:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t\sigma_t - L(\sigma_t) + \sigma_t\varepsilon_t, \;\; t \in \mathbb{N}, \tag{4.1}$$

where S_t is the stock price at time t, r_t, and d_t denote the risk-free and dividend rate for the period $[t-1, t]$, respectively, and λ_t is a $\mathfrak{F}_{t-1}$ measurable random variable. S_0 is the currently observed price. The function $L(x)$ is the *log-Laplace-transform* of ε_t, i.e, $L(x) = \log(E[e^{x\varepsilon_t}])$. If $L(x)$ is defined on the whole real line, then the one-period ahead conditional variance σ_t^2 follows a GARCH(1,1) process, i.e,

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2), \;\; t \in \mathbb{N}, \;\; \varepsilon_0 = 0. \tag{4.2}$$

where α_0, α_1, and β_1 are non-negative, $\alpha_1 + \beta_1 < 1$, and $\alpha_0 > 0$. If $L(x)$ is defined only on a closed interval $[-a, b]$ with $a, b > 0$, then σ_t^2 follows a GARCH(1,1) process with a restriction $0 < \sigma_t \le b$, i.e,

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2) \wedge \rho, \;\; t \in \mathbb{N}, \;\; \varepsilon_0 = 0, \tag{4.3}$$

where $0 < \rho \le b^2$. Clearly, the process $(\sigma_t)_{t\in\mathbb{N}}$ is predictable. In "the normal-GARCH model" introduced by Duan (1995), for example, the Laplace transform of ε_t is defined for every real number and hence σ_t^2 follows (4.2).

4.1 CTS-GARCH Model

Consider the ID-GARCH model with the sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of i.i.d. random variables with $\varepsilon_t \sim \text{stdCTS}(\alpha, \lambda_+, \lambda_-)$ for all $t \in \mathbb{N}$. This ID-GARCH model has been introduced by Kim *et al.* (2008a) under the name *CTS-GARCH model*. Since $E[e^{x\varepsilon_t}] < \infty$ if $x \in [-\lambda_-, \lambda_+]$, ρ has to be in the interval $(0, {\lambda_+}^2]$, and σ_t follows equation (4.3).

By Corollary 2.3 (b), we can prove the following proposition.

Proposition 4.1. *Consider the CTS-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, and*

t a natural number such that $t \le T$. Suppose $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfy the following conditions:

$$(4.4) \qquad \begin{cases} \tilde{\lambda}_+(t)^2 \ge \rho \\ \tilde{\lambda}_+(t)^{\alpha-2} + \tilde{\lambda}_-(t)^{\alpha-2} = {\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2} \\ \dfrac{{\lambda_+}^{\alpha-1} - {\lambda_-}^{\alpha-1} - \tilde{\lambda}_+(t)^{\alpha-1} + \tilde{\lambda}_-(t)^{\alpha-1}}{(1-\alpha)({\lambda_+}^{\alpha-2} + {\lambda_-}^{\alpha-2})} \\ \qquad = \lambda_t + \dfrac{1}{\sigma_t}(L_{CTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) - L_{CTS}(\sigma_t; \alpha, \lambda_+, \lambda_-)) \end{cases} .$$

Then there is a measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$ such that $\varepsilon_t + k_t \sim$ stdCTS$(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t))$ on the measure $\mathbf{Q}_t$ where k_t is the $\mathcal{F}_{t-1}$ measurable random variable given by

$$(4.5) \qquad k_t = \lambda_t + \frac{1}{\sigma_t}(L_{CTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) - L_{CTS}(\sigma_t; \alpha, \lambda_+, \lambda_-)).$$

Suppose $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfy the condition (4.4) in each time $t \le T$. We have the stock price dynamic

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - L_{CTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) + \sigma_t(\varepsilon_t + k_t)$$

where k_t is given by equation (4.5). By Proposition 4.1, there is a measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$ such that $\varepsilon_t + k_t \sim$ stdCTS$(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t))$ on the measure $\mathbf{Q}_t$, and hence we obtain a risk-neutral stock price dynamic

$$(4.6) \qquad \begin{cases} \log\left(\dfrac{S_t}{S_{t-1}}\right) = r_t - d_t - L_{CTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) + \sigma_t \xi_t \\ \xi_t \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) \end{cases} , t \le T$$

having the following variance process

$$(4.7) \qquad \sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho.$$

The risk-neutral stock price dynamic is called the *CTS-GARCH option pricing model*.

Under the CTS-GARCH option pricing model, the stock price S_t at time $t > 0$ is given by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t}\left(r_j - d_j - L_{CTS}(\sigma_j; \alpha, \tilde{\lambda}_+(j), \tilde{\lambda}_-(j)) + \sigma_j \xi_j\right)\right).$$

4.2 RDTS-GARCH Model

Consider the ID-GARCH model with the sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of i.i.d. random variables with $\varepsilon_t \sim \text{stdRDTS}(\alpha, \lambda_+, \lambda_-)$ for all $t \in \mathbb{N}$. We will call the ID-GARCH model the *RDTS-GARCH model*. Since $E[e^{x\varepsilon_t}] < \infty$ for all real number x, the variance process is not artificially restricted; that is, σ_t follows (4.2).

By Corollary 3.8 (b), we can prove the following proposition.

Proposition 4.2. *Consider the RDTS-GARCH model. Let $T \in \mathbb{N}$ be a time horizon, and t a natural number such that $t \le T$. Suppose $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfy the following conditions:*

(4.8)
$$\begin{cases} \tilde{\lambda}_+(t)^{\alpha-2} + \tilde{\lambda}_-(t)^{\alpha-2} = \lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} \\ \dfrac{\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+(t)^{\alpha-1} + \tilde{\lambda}_-(t)^{\alpha-1}\right)}{\sqrt{2}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)} \\ \quad = \lambda_t + \dfrac{1}{\sigma_t}(L_{RDTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) - L_{RDTS}(\sigma_t; \alpha, \lambda_+, \lambda_-)) \end{cases}.$$

Then there is a measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$ such that $\varepsilon_t + k_t \sim \text{stdRDTS}(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t))$ on the measure $\mathbf{Q}_t$ where k_t is the $\mathcal{F}_{t-1}$ measurable random variable given by

(4.9)
$$k_t = \lambda_t + \frac{1}{\sigma_t}(L_{RDTS}(\sigma_t; \alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t)) - L_{RDTS}(\sigma_t; \alpha, \lambda_+, \lambda_-)).$$

Suppose $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfy condition (4.8) in each time $t \le T$. We would then have the stock price dynamic

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t\sigma_t - L_{RDTS}(\sigma_t;\alpha,\lambda_+,\lambda_-) + \sigma_t\varepsilon_t$$
$$= r_t - d_t - L_{RDTS}(\sigma_t;\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t)) + \sigma_t(\varepsilon_t + k_t)$$

where k_t is given by equation (4.9). By Proposition 4.2, there is a measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$ such that $\varepsilon_t + k_t \sim$ stdRDTS$(\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t))$ on the measure $\mathbf{Q}_t$, and hence

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - L_{RDTS}(\sigma_t;\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t)) + \sigma_t\xi_t$$

where $\xi_t \sim$ stdRDTS$(\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t))$. Since $\lambda_t\sigma_t$ disappears in the dynamic on $\mathbf{Q}_t$, λ_t can be interpreted as the market price of risk. Consequently, we deduce the following risk-neutral stock price dynamic from Proposition 4.2

$$(4.10)\qquad \begin{cases} \log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - L_{RDTS}(\sigma_t;\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t)) + \sigma_t\xi_t \\ \xi_t \sim \text{stdRDTS}(\alpha,\tilde{\lambda}_+(t),\tilde{\lambda}_-(t)) \end{cases}, t \le T$$

having the following variance process

$$(4.11)\qquad \sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k_{t-1})^2 + \beta_1\sigma_{t-1}^2).$$

The risk-neutral stock price dynamic is called the *RDTS-GARCH option pricing model*. Under the RDTS-GARCH option pricing model, the stock price S_t at time $t > 0$ is given by

$$S_t = S_0\exp\left(\sum_{j=1}^{t}\left(r_j - d_j - L_{RDTS}(\sigma_j;\alpha,\tilde{\lambda}_+(j),\tilde{\lambda}_-(j)) + \sigma_j\xi_j\right)\right).$$

4.3 Simulation of the risk-neutral stock price processes

Assume that the GARCH parameters (α_0, α_1, and β_1), the standard CTS and standard RDTS parameters (α, λ_+, and λ_-), the constant market price of risk $\lambda_t = \lambda$, and the conditional variance $\sigma_{t_0}^2$ of the initial time t_0 are estimated from historical data. Then we can generate the risk-neutral process for the CTS-GARCH option pricing model by the following algorithm:

1. Initialize $t := t_0$.
2. Find the parameters $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfying condition (4.4).
3. Generate random number $\xi_t \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t))$ using Theorem 2.4.
4. Let $\log\left(\frac{S_t}{S_{t-1}}\right)$ be equal to equation (4.6).
5. Let k_t be equal to equation (4.5).
6. Set $t = t + 1$ and then substitute

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho.$$

7. Repeat 2 through 6 until $t > T$.

We can generate the risk-neutral process for the RDTS-GARCH option pricing model by modifying the above algorithm as follows:

$2'$. Find the parameters $\tilde{\lambda}_+(t)$ and $\tilde{\lambda}_-(t)$ satisfying condition (4.8).

$3'$. Generate random number $\xi_t \sim \text{stdRDTS}(\alpha, \tilde{\lambda}_+(t), \tilde{\lambda}_-(t))$ using Theorem 3.9.

$4'$. Let $\log\left(\frac{S_t}{S_{t-1}}\right)$ be equal to equation (4.10).

$5'$. Let k_t be equal to equation (4.9).

$6'$. Set $t = t + 1$ and then substitute

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k_{t-1})^2 + \beta_1 \sigma_{t-1}^2.$$

5 Market parameter estimation

In this section, we report the maximum likelihood estimation (MLE) of the normal-GARCH, CTS-GARCH, and RDTS-GARCH models using data obtained from Option Metrics's IvyDB in the Wharton Research Data Services. In our empirical study, we use historical prices of the Dow Jones Industrial Average (DJIA) and 29 of its 30-component stocks[2] as of October 2008. First, we consider the time series of the stock prices for the DJIA component companies from October 1, 1997 to December 31, 2006. Then in order to analyze the model performance during that time and to evaluate DJX index options, we consider also the time series of the DJIA index in the time window from January 2, 1996 to June 6, 2007. The analysis of the 29 stocks is totally independent of the analysis of the DJIA and DJX. That is, we study the model performances on stocks, then on the DJIA index together with the corresponding option prices. Since the index composition changes periodically, we prefer to perform the analysis on the current DJIA component stocks. For the daily risk-free rate, we select the appropriate zero-coupon rate obtained from the Ivy DB.

To simplify the estimation, we impose a constant market price of risk λ. We use the total returns data by Ivy DB to estimate the market parameters with the MLE. The total returns are obtained by adjusting prices of indexes and stocks for all applicable splits and dividend distributions. For this reason, we modify the stock price dynamic as follows

$$\log\left(\frac{\hat{S}_t}{\hat{S}_{t-1}}\right) = r_t + \lambda_t\sigma_t - L(\sigma_t) + \sigma_t\varepsilon_t, t \in \mathbb{N}, \tag{5.1}$$

where $\hat{S}_t$ is the adjusted-closing prices.

Our estimation procedure is as follows. First, we estimate the parameters α_0, α_1, β_1, and the constant market price of risk λ from the normal-GARCH model. Second, we fix α_0, α_1, β_1, and λ as parameters estimated in the first step and then estimate α, λ_+, and λ_- from the CTS-GARCH and RDTS-GARCH models under the assumption of $\sigma_0^2 = \alpha_0/(1-\alpha_1-\beta_1)$. For the CTS-GARCH model, we set $\rho = \max\{\sigma_t^2 : t \text{ is the observed date}\}$. We report the estimated GARCH parameters in Table 1, and the parameters for the two standard tempered stable distributions in Table 2 for the DJIA index and 29 component companies.

For the assessment of the goodness-of-fit, we utilize the Kolmogrov-Smirnov (KS) test. We also calculate the Anderson-Darling (AD) statistic to better evaluate the tail fit. We define the null hypotheses as follows:

H_0(Normal-GARCH): The residuals follow the standard normal distribution.

H_0(CTS-GARCH): The residuals follow the standard CTS distribution.

H_0(RDTS-GARCH): The residuals follow the standard RDTS distribution.

Table 3 provides the KS statistic and its p-values. The p-values of the KS statistic are calculated using the calculator designed by Marsaglia *et al.* (2003). Based on the results reported in the table, we conclude that

1. H_0(Normal-GARCH) is rejected at the 5% significance level for 22 of the 29 stocks.
2. H_0(CTS-GARCH) is rejected at the 5% significance level for one stock, DuPont.
3. H_0(RDTS-GARCH) is rejected at the 5% significance level for one stock, DuPont.
4. AD statistic for both CTS-GARCH and RDTS-GARCH are significantly smaller than the AD statistic of the normal-GARCH model.

Furthermore, in order to analyze the model performance during the time, we report the MLE estimate of the normal-GARCH, CTS-GARCH, and RDTS-GARCH models for the DJIA, by considering any Wednesday between January 4, 2006 to June 6, 2007. We consider 75 different time series with daily observations starting from January 2, 1996 and ending on any Wednesday in the time window considered above. This estimations will be also used in the next section for the purpose of option valuation. We report in Table 4 and Figure 1 the normal-GARCH parameters, and in Table 5 and in Figure 2 the market parameters of the innovation processes for the CTS-GARCH and RDTS-GARCH models. In Table 6 and Figure 3, we show the KS, the AD, and the χ^2 statistic with the relative p-value. The empirical study shows that the two non-normal GARCH models largely improve the classical normal-GARCH model. Furthermore, Figure 2 shows that the estimated parameters of the CTS and RDTS innovations do not present large deviations in a time window of more than one year, and, in particular, the RDTS model parameters seem to be more stable.

6 Option prices with GARCH models

In this part of the empirical analysis, we evaluate option prices written on the DJIA (DJX) with different strike prices and maturities. Now, we want to study the effect on option prices when the underlying distribution is skewed and leptokurtic, and compare these models to the normal-GARCH model, which constitute a natural benchmark. European call data on 17 selected Wednesdays (one per month) between January 4, 2006 and May 9, 2007 are considered for a total of $2,670$ option prices. Here, options with a time to maturity more than 250 days are discarded. Option prices and the risk-free rate, calculated from the U.S. Treasury yield curve, are provided by Ivy DB.

Market parameters estimated in the previous section are taken into account in this analysis in order to calculate option prices. We consider the market estimation based on the time series from January 2, 1996 to any corresponding Wednesday in which the European call option is quoted. That is, to price an option quoted on January 11, 2006, we consider the MLE estimated parameters from the time series from January 2, 1996 to January 11, 2006, together with the algorithms in Section 4.3. By repeating the same procedure, we price options for any selected Wednesday, until May 9, 2007.

The Monte Carlo procedure is based on algorithms in Section 4.3 with empirical martingale simulation. This simulation technique, introduced in Duan and Simonato (1998), is a simple way to reduce the variance of the simulated sample and to preserve the martingale property of the simulated risk-neutral process as well, which is in general lost with a crude Monte Carlo method. We point out that for each time step and for each simulated path, we have to solve a nonlinear system, as described in Section 4.3, to find risk-neutral parameters. That is, each random number may have different parameters, which does not occur in the normal case. For this reason, the running time ranges from 10 minutes for the normal case to 42 hours for the RDTS case to simulate 20,000 paths, by using Matlab R2007b on a Xeon Precision at 3.0 GHz with 3GB RAM. Anyway, if one can compute with a cluster, the running time is of minor concern, since the structure of the problem allows one to simulate paths separately. Furthermore, we have to also consider some memory allocation feature in working with an office personal computer such as a Xeon Precision at 3.0 GHz. This is the reason why we consider only 17 Wednesday, one per month, and not all 75 Wednesday as in the market estimation.

To measure the performance of the option pricing model, we consider four statistics (see Schoutens (2003)), described as follows. Let us consider a given market model and observed prices C_i of call options with maturities τ_i and strikes K_i, $i \in \{1, \dots, N\}$, where N is the number of options on a given Wednesday. Let $\overline{C}_i$ be the mean of options prices C_i and $\widehat{C}_i$ be the model price, then we evaluate

1. the average absolute error as a percentage of the mean price (denoted APE)

$$\text{APE} = \frac{1}{\overline{C}_i} \sum_{i=1}^{N} \frac{|C_i - \widehat{C}_i|}{N},$$

2. the average absolute error (denoted AAE)

$$\text{AAE} = \sum_{i=1}^{N} \frac{|C_i - \widehat{C}_i|}{N},$$

3. the root mean square error (denoted RMSE)

$$\text{RMSE} = \sqrt{\sum_{i=1}^{N} \frac{(C_i - \widehat{C}_i)^2}{N}},$$

4. the average relative percentage error (denoted ARPE)

$$\text{ARPE} = \frac{1}{N} \sum_{i=1}^{N} \frac{|C_i - \widehat{C}_i|}{C_i}.$$

Table 7 reports the performance of different option pricing model: the normal-GARCH performs worst than the two others models, as the CTS-GARCH and RDTS-GARCH models have smaller pricing errors.

7 Conclusion

In this paper, we introduce the RDTS distribution. It has statistical properties similar to the CTS distribution, even if the RDTS distribution has finite exponential moment of any order, while the CTS has only some finite exponential moment. Furthermore, we present a discrete time model for stock price log returns driven by a non-normal random variable, that is the RDTS-GARCH model, which allows fat tails, skewness, and volatility clustering. We compare this model to the classical normal-GARCH model and with the CTS-GARCH model, that was introduced by Kim *et al.* (2008a).

Discrete time markets with a continuous return distribution fail to be complete. Consequently, based on a similar argument as in the CTS case as per Kim *et al.* (2008a), the problem of the appropriate choice of the equivalent martingale for the discounted asset price process is solved considering the RDTS innovation assumption. A density transformation between ID random variables allows us to choice a suitable equivalent martingale measure. By the discrete time nature of this setting, the risk-neutral distribution is not always the same for the entire time window, but on each time step it is governed by different parameters. Unfortunately, this approach does not provide analytical solutions to price European options and hence numerical procedures have to be considered. For this reason, algorithms for simulating CTS and RDTS distributions are studied and used to obtain option prices. The use of non-normal GARCH models combined with Monte Carlo simulation methods allows one to obtain very promising results.

For the stocks, the index, and the option prices we analyzed and for the time period studied, the CTS-GARCH and RDTS-GARCH seem to be satisfactory in both market and option analysis, compared to the normal-GARCH model. Consequently, the CTS-GARCH and RDTS-GARCH models explain both the asset price behavior and European option prices better than the normal-GARCH model. Thus, we can say that the skewness and fat-tail properties of the innovation are also important for pricing of European options.

A Appendix

A.1 Proof of Proposition 3.3

Lemma A.1. *If* $\lambda > 0$, *then*

(A.1)
$$\int_0^\infty x^n \frac{e^{-\lambda^2 x^2/2}}{x^{\alpha+1}} dx = \frac{1}{2}\left(\frac{\lambda^2}{2}\right)^{\frac{\alpha-n}{2}} \Gamma\left(\frac{n-\alpha}{2}\right), \quad n \in \mathbb{N}.$$

Proof. By the change of variables, we have

(A.2)
$$\int_0^\infty x^n \frac{e^{-yx^2}}{x^{\alpha+1}} dx = \frac{y^{(\alpha-n)/2}}{2}\Gamma\left(\frac{n-\alpha}{2}\right), \quad n \in \mathbb{N}, y > 0.$$

If we substitute $y = \lambda^2/2$ in (A.2), then we obtain the result. □

Lemma A.2. *Let* $\alpha \in (0, 2)$, $\alpha \neq 1$. *Then we have*

(A.3)
$$\sum_{n=0}^\infty \frac{1}{n!}\Gamma\left(\frac{n-\alpha}{2}\right) x^n = \Gamma\left(-\frac{\alpha}{2}\right) M\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{x}{2}\right)^2\right) + x\Gamma\left(\frac{1-\alpha}{2}\right) M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \left(\frac{x}{2}\right)^2\right).$$

where M *is the confluent hypergeometric function Andrews (1998).*

Proof. We have

$$\sum_{n=0}^\infty \frac{1}{n!}\Gamma\left(\frac{n-\alpha}{2}\right) x^n = \sum_{n=0}^\infty \frac{1}{(2n)!}\Gamma\left(n-\frac{\alpha}{2}\right) x^{2n} + \sum_{n=0}^\infty \frac{1}{(2n+1)!}\Gamma\left(n+\frac{1-\alpha}{2}\right) x^{2n+1}.$$

By the facts that

$$(2n)! = n!2^{2n}\left(\frac{1}{2}\right)_n, \quad (2n+1)! = n!2^{2n}\left(\frac{3}{2}\right)_n,$$

and

$$\Gamma(n+y) = (y)_n \Gamma(y),$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!}\Gamma\left(\frac{n-\alpha}{2}\right) x^n \\
&= \Gamma\left(-\frac{\alpha}{2}\right) \sum_{n=0}^{\infty} \frac{\left(-\frac{\alpha}{2}\right)_n}{n! 2^{2n} \left(\frac{1}{2}\right)_n} x^{2n} + x\Gamma\left(\frac{1-\alpha}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{1-\alpha}{2}\right)_n}{n! 2^{2n} \left(\frac{3}{2}\right)_n} x^{2n} \\
&= \Gamma\left(-\frac{\alpha}{2}\right) M\left(-\frac{\alpha}{2}, \frac{1}{2}; \left(\frac{x}{2}\right)^2\right) + x\Gamma\left(\frac{1-\alpha}{2}\right) M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \left(\frac{x}{2}\right)^2\right)
\end{aligned}$$

□

Proof of Proposition 3.3. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (e^{iux} - 1 - iux 1_{|x|\le 1}) \nu(dx) \\
&= iu \int_{|x|>1} x\nu(dx) + \int_0^{\infty} (e^{iux} - 1 - iux)\nu(dx) + \int_{-\infty}^{0} (e^{iux} - 1 - iux)\nu(dx) \\
&= iu \int_{|x|>1} x\nu(dx) \\
&\quad + C_+ \sum_{n=2}^{\infty} \frac{1}{n!}(iu)^n \int_0^{\infty} x^n \frac{e^{-\lambda_+^2 x^2/2}}{x^{\alpha+1}} dx + C_- \sum_{n=2}^{\infty} \frac{1}{n!}(-iu)^n \int_0^{\infty} x^n \frac{e^{-\lambda_-^2 x^2/2}}{x^{\alpha+1}} dx
\end{aligned}$$

By (A.1) and (A.3), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{1}{n!}(\pm iu)^n \int_0^{\infty} x^n \frac{e^{-\lambda_\pm^2 x^2/2}}{x^{\alpha+1}} dx \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2} \left(\frac{\lambda_\pm}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) (\pm iu)^n \\
&\quad \mp iu \frac{1}{2} \left(\frac{\lambda_\pm}{\sqrt{2}}\right)^{\alpha-1} \Gamma\left(\frac{1-\alpha}{2}\right) - \frac{1}{2} \left(\frac{\lambda_\pm}{\sqrt{2}}\right)^{\alpha} \Gamma\left(\frac{-\alpha}{2}\right) \\
&= 2^{-\frac{\alpha}{2}-1} \lambda_\pm^{\alpha} \left[\Gamma\left(-\frac{\alpha}{2}\right) \left(M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{(\pm iu)^2}{2\lambda_\pm^2}\right) - 1 \right) \right. \\
&\quad \left. + \frac{\pm\sqrt{2} iu}{\lambda_\pm} \Gamma\left(\frac{1-\alpha}{2}\right) \left(M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{(\pm iu)^2}{2\lambda_\pm^2}\right) - 1 \right) \right]
\end{aligned}$$

By Lévy-Khintchine formula and (3.1) in Definition 3.1, we obtain the characteristic function. Moreover, $\phi(u)$ can be extended via analytic continuation to the complex field $\mathbb{C}$. □

A.2 Proof of Proposition 3.7

In this section, we review a general result of equivalence of measures presented by Sato (1999) and then apply it to the RDTS process.

Theorem A.3 (Sato (1999) Theorem 33.1 and 33.2.). *Let $(X_t, \mathbb{P})$ and $(X_t, \mathbb{Q})$ be a Lévy processes on $\mathbb{R}$ with Lévy triplets (σ^2, ν, γ) and $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ respectively. Then $\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are equivalent for all $t > 0$ if and only if the Lévy triplets satisfy*

$$\sigma^2 = \tilde{\sigma}^2, \tag{A.4}$$

$$\int_{-\infty}^{\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty \tag{A.5}$$

with the function $\psi(x) = \ln\left(\frac{\tilde{\nu}(dx)}{\nu(dx)}\right)$ and if $\sigma^2 = 0$ then

$$\tilde{\gamma} - \gamma = \int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx). \tag{A.6}$$

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the Radon-Nikodym derivative is

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{U_t} \tag{A.7}$$

where $(U_t, \mathbf{P})$ is a Lévy process in which the Lévy triplet $(\sigma_U^2, \nu_U, \gamma_U)$ of $(U_t)_{t\in[0,T]}$ is given by

$$\sigma_U^2 = \sigma^2\eta^2, \nu_U = \nu \circ \psi^{-1}, \gamma_U = -\frac{\sigma^2\eta^2}{2} - \int_{-\infty}^{\infty} (e^y - 1 - y1_{|y|\leq 1})\nu_U(dy) \tag{A.8}$$

Here η is such that

$$\tilde{\gamma} - \gamma - \int_{|x|\le 1} x(\tilde{\nu} - \nu)(dx) = \sigma^2 \eta$$

if $\sigma > 0$ and zero if $\sigma = 0$.

Since RDTS distributions are infinitely divisible, we can apply Theorem A.3 to obtain the change of measure.

Proof of Proposition 3.7. Let $(0, \nu, \gamma)$ and $(0, \tilde{\nu}, \tilde{\gamma})$ be Lévy triplets of $(X_t, \mathbb{P})$ and $(X_t, \mathbb{Q})$, respectively. Since the diffusion coefficients of RDTS processes are zero, (A.4) is satisfied. From the definition of Lévy measure $\tilde{\nu}$ and ν, $\psi(x)$ in the condition (A.5) is equal to

$$\begin{aligned}\psi(x) = &\left(\ln\left(\frac{\tilde{C}_+ x^{-\tilde{\alpha}}}{C_+ x^{-\alpha}}\right) + \frac{x^2}{2}(\lambda_+ - \tilde{\lambda}_+)\right) 1_{x>0} \\ &+ \left(\ln\left(\frac{\tilde{C}_- |x|^{-\tilde{\alpha}}}{C_- |x|^{-\alpha}}\right) + \frac{x^2}{2}(\lambda_- - \tilde{\lambda}_-)\right) 1_{x<0}.\end{aligned}$$

Let $k(x) = \frac{\lambda_+^2 x^2}{2} 1_{x>0} + \frac{\lambda_-^2 x^2}{2} 1_{x<0}$ and $\tilde{k}(x) = \frac{\tilde{\lambda}_+^2 x^2}{2} 1_{x>0} + \frac{\tilde{\lambda}_-^2 x^2}{2} 1_{x<0}$. If $\alpha < \tilde{\alpha}$, then we have

$$\lim_{x\to 0^+} \left(\frac{\sqrt{\tilde{C}_+} e^{-\tilde{k}(x)/2}}{x^{(\tilde{\alpha}+1)/2}} - \frac{\sqrt{C_+} e^{-k(x)/2}}{x^{(\alpha+1)/2}}\right)^2 \Big/ \left(\frac{1}{x^{\tilde{\alpha}+1}}\right) = \tilde{C}_+.$$

If $\alpha = \tilde{\alpha}$ but $C_+ \neq \tilde{C}_+$, then we have

$$\lim_{x\to 0^+} \left(\frac{\sqrt{\tilde{C}_+} e^{-\tilde{k}(x)/2}}{x^{(\tilde{\alpha}+1)/2}} - \frac{\sqrt{C_+} e^{-k(x)/2}}{x^{(\alpha+1)/2}}\right)^2 \Big/ \left(\frac{1}{x^{\tilde{\alpha}+1}}\right) = (\sqrt{\tilde{C}_+} - \sqrt{C_+})^2.$$

Hence if $\alpha < \tilde{\alpha}$ or $\alpha = \tilde{\alpha}$ but $C_+ \neq \tilde{C}_+$ then

$$\text{(A.9)} \qquad \begin{aligned}\int_0^\infty \left(e^{\psi(x)/2} - 1\right)^2 \nu(dx) &= \int_0^\infty \left(\frac{\sqrt{\tilde{C}_+} e^{-\tilde{k}(x)/2}}{x^{(\alpha+1)/2}} - \frac{\sqrt{C_+} e^{-k(x)/2}}{x^{(\alpha+1)/2}}\right)^2 dx \\ &\ge K_+ \int_0^1 \frac{1}{x^{\tilde{\alpha}+1}} dx = \infty\end{aligned}$$

for some $K_+ \in \mathbb{R}$. Using similar arguments, we can prove that if $\alpha < \tilde{\alpha}$ or $\alpha = \tilde{\alpha}$ but $C_- \neq \tilde{C}_-$ then

$$\int_{-\infty}^{0} \left(e^{\psi(x)/2} - 1\right)^2 \nu(dx) = \infty. \tag{A.10}$$

By (A.9) and (A.10),

$$\int_{-\infty}^{\infty} \left(e^{\psi(x)/2} - 1\right)^2 \nu(dx) = \infty.$$

Hence the condition (A.5) does not hold. Similarly, we can show that the condition (A.5) does not hold if $\alpha > \tilde{\alpha}$.

Suppose $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$ and, $C_- = \tilde{C}_-$. Then we have

$$\psi(x) = \frac{x^2}{2}(\lambda_+^2 - \tilde{\lambda}_+^2)1_{x>0} + \frac{x^2}{2}(\lambda_-^2 - \tilde{\lambda}_-^2)1_{x<0},$$

and hence

$$\int_0^\infty (e^{\psi(x)/2} - 1)^2 \nu(dx) = C_+ \int_0^\infty \frac{\left(e^{-\frac{\tilde{\lambda}_+^2 x^2}{2}} - e^{-\frac{\lambda_+^2 x^2}{2}}\right)^2}{x^{\alpha+1}} dx.$$

We can show that the right side of the above equation is finite. Using similar arguments, we can prove $\int_{-\infty}^{0} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty$. Thus, condition (A.5) holds if and only if $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$ and, $C_- = \tilde{C}_-$.

Condition (A.6) holds if and only if

$$\int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx) = \tilde{m} - \int_{|x|>1} x\,\tilde{\nu}(dx) - m + \int_{|x|>1} x\,\nu(dx),$$

or

$$\begin{aligned} \tilde{m} - m &= \int_{-\infty}^{\infty} x(\tilde{\nu} - \nu)(dx) \\ &= 2^{-\frac{\alpha+1}{2}} \Gamma\left(\frac{1-\alpha}{2}\right) \left(C_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - C_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})\right). \end{aligned}$$

Hence, $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\alpha = \tilde{\alpha}$, $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$ and

$$\tilde{m} - m = 2^{-\frac{\alpha+1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\left(C_+(\tilde{\lambda}_+^{\alpha-1} - \lambda_+^{\alpha-1}) - C_-(\tilde{\lambda}_-^{\alpha-1} - \lambda_-^{\alpha-1})\right).$$

The Lévy triplet (3.5) can be obtained from (A.8) in Theorem A.3 with $\eta = 0$. □

A.3 Proof of Proposition 3.10 and Proposition 3.11

Lemma A.4. *For $a \in \mathbb{R}_+$, the following equality holds*

$$\int_\beta^\infty s^{-a-1}e^{-s}ds = \beta^{-a-1}e^{-\beta} + o(\beta^{-a-1}e^{-\beta}) \tag{A.11}$$

and

$$\int_\beta^\infty s^{-a-1}e^{-\frac{s^2}{2}}ds = \beta^{-a-2}e^{-\frac{\beta^2}{2}} + o(\beta^{-a-2}e^{-\frac{\beta^2}{2}}) \tag{A.12}$$

as $\beta \to \infty$.

Proof. By integration by parts, if $\beta > 0$, we obtain

$$\int_\beta^\infty s^{-a-1}e^{-s}ds = \beta^{-a-1}e^{-\beta} - (a+1)\int_\beta^\infty s^{-a-2}e^{-s}ds \le \beta^{-a-1}e^{-\beta}$$

and

$$\begin{aligned}\int_\beta^\infty s^{-a-1}e^{-s}ds &= \beta^{-a-1}e^{-\beta} - (a+1)\beta^{-a-2}e^{-\beta} + (a+1)(a+2)\int_\beta^\infty s^{-a-3}e^{-s}ds\\ &\ge \beta^{-a-1}e^{-\beta} - (a+1)\beta^{-a-2}e^{-\beta},\end{aligned}$$

when $\beta \to \infty$, the first result is proved. By integration by parts again, if $\beta > 0$, we obtain

$$\int_\beta^\infty s^{-a-1}e^{-\frac{s^2}{2}}ds = \beta^{-a-2}e^{-\frac{\beta^2}{2}} - (a+2)\int_\beta^\infty s^{-a-3}e^{-\frac{s^2}{2}}ds \le \beta^{-a-2}e^{-\frac{\beta^2}{2}}$$

and

$$\begin{aligned}
&\int_\beta^\infty s^{-a-1}e^{-\frac{s^2}{2}}ds \\
&= \beta^{-a-2}e^{-\frac{\beta^2}{2}} - (a+2)\beta^{-a-4}e^{-\frac{\beta^2}{2}} + (a+2)(a+4)\int_\beta^\infty s^{-a-5}e^{-\frac{s^2}{2}}ds \\
&\geq \beta^{-a-2}e^{-\frac{\beta^2}{2}} - (a+2)\beta^{-a-4}e^{-\frac{\beta^2}{2}},
\end{aligned}$$

when $\beta \to \infty$, the second result is proved. □

We consider the following result:

Proposition A.5. *Let X be an infinitely divisible random variable in $\mathbb{R}$, with Lévy triplet $(\gamma, 0, \nu)$. Then we have*

(A.13) $$\mathbb{P}(|X-m| \geq \lambda) \geq \frac{1}{4}(1-\exp(-\nu(u \in \mathbb{R} : |u| \geq 2\lambda))), \quad \lambda > 0.$$

for all $m \in \mathbb{R}$.

Proof. See Lemma 5.4 of Breton *et al.* (2007). □

Taking into account Proposition A.5 and Lemma A.4, we can prove Proposition 3.10 and Proposition 3.11.

Proof of Proposition 3.10. By Chebyshev's Inequality, the upper bound part can be proved. Applying the following elementary fact

$$1-\exp(-z) \sim z, \quad z \to 0$$

and according to (A.13), we obtain

$$\begin{aligned}
\mathbb{P}(|X-m| \geq \lambda) &\geq \frac{1}{4}\left(1-\exp\left[-C_+\int_{2y}^\infty x^{-\alpha-1}e^{-\lambda_+ x}dx - C_-\int_{2y}^\infty x^{-\alpha-1}e^{-\lambda_- x}dx\right]\right) \\
&\sim \frac{1}{4}\left[C_+\int_{2y}^\infty x^{-\alpha-1}e^{-\lambda_+ x}dx + C_-\int_{2y}^\infty x^{-\alpha-1}e^{-\lambda_- x}dx\right]|x|^{\alpha+1}e^{-\frac{2\lambda}{|x|}}R(dx),
\end{aligned}$$

as $y \to \infty$. By using (A.11) of Lemma A.4, we have

$$\int_{2y}^{\infty} x^{-\alpha-1} e^{-\lambda x} dx \sim (2y)^{-\alpha-1} \lambda^{-1} e^{-2\lambda y}.$$

Hence we obtain

$$\mathbb{P}(|X - m| \geq \lambda) \geq K(2y)^{-\alpha-1} \lambda^{-1} e^{-2\lambda y}$$

for some constant K independent of y and $\bar{\lambda} = \min\{\lambda_+, \lambda_-\}$ as $y \to \infty$ □

Proof of Proposition 3.11. Using the same method in the proof of Proposition 3.10 with (A.12) of Lemma A.4, we can obtain the result. □

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Notes

[1] See Andrews (1998).

[2] Kraft Foods (KFT) is excluded because the time series we employ begins in 1997 but this company was listed on 2001.

Table 1: Estimated normal-GARCH parameters from October 1, 1997 to December 31, 2006 for 29 component companies of the DJIA index.

	ticker	β_1	α_1	α_0	λ
Alcoa Incorporated	AA	0.9599	0.0338	$2.6293E-6$	0.0410
American Express Company	AXP	0.9224	0.0731	$2.1441E-6$	0.0732
Boeing Corporation	BA	0.9325	0.0572	$5.1542E-6$	0.0603
Bank of America Corporation	BAC	0.9550	0.0416	$1.2013E-6$	0.0656
Citigroup Incorporated	C	0.9577	0.0402	$8.3005E-7$	0.0795
Caterpillar Incorporated	CAT	0.9824	0.0152	$8.9119E-7$	0.0626
Chevron	CVX	0.9216	0.0625	$3.9714E-6$	0.0540
DuPont	DD	0.9686	0.0293	$5.6971E-7$	0.0324
Walt Disney Company	DIS	0.9041	0.0852	$6.6607E-6$	0.0471
General Electric Company	GE	0.9606	0.0370	$5.6093E-7$	0.0627
General Motors Corporation	GM	0.9228	0.0585	$9.5254E-6$	0.0275
Home Depot Incorporated	HD	0.9620	0.0362	$9.7257E-7$	0.0675
Hewlett-Packard Company	HPQ	0.9869	0.0111	$1.4125E-6$	0.0405
International Business Machines	IBM	0.9179	0.0794	$2.8849E-6$	0.0658
Intel Corporation	INTC	0.9699	0.0268	$2.2101E-6$	0.0529
Johnson&Johnson	JNJ	0.9181	0.0742	$2.2397E-6$	0.0548
JPMorgan Chase & Company	JPM	0.9432	0.0543	$1.0285E-6$	0.0617
Coca-Cola Company	KO	0.9528	0.0439	$9.0481E-7$	0.0362
McDonald's Corporation	MCD	0.9538	0.0407	$1.8980E-6$	0.0329
3M Company	MMM	0.8478	0.1034	$1.3852E-5$	0.0566
Merck & Company, Incorpoarated	MRK	0.9063	0.0221	$2.6409E-5$	0.0240
Microsoft Corporation	MSFT	0.9348	0.0619	$1.6078E-6$	0.0644
Pfizer Incorporated	PFE	0.8869	0.0887	$1.0399E-5$	0.0326
Procter and Gamble Company	PG	0.9625	0.0360	$3.0415E-7$	0.0673
AT&T Incorporated	T	0.9356	0.0607	$2.2891E-6$	0.0253
United Technologies	UTX	0.8934	0.0994	$4.5332E-6$	0.1027
Verizon Company	VZ	0.9352	0.0614	$1.4839E-6$	0.0352
Wal-Mart Stores Incorporated	WMT	0.9650	0.0335	$4.8725E-7$	0.0458
Exxon Mobil Corporation	XOM	0.9336	0.0559	$2.5577E-6$	0.0602

Table 2: Estimated parameters of the innovation processes for the CTS-GARCH and RDTS-GARCH models from October 1, 1997 to December 31, 2006 for 29 component companies of the DJIA index.

	CTS			RDTS		
Ticker	α	λ_+	λ_-	α	λ_+	λ_-
AA	1.8499	0.3146	7.5000	1.8887	0.2024	10.5721
AXP	1.7500	0.3805	9.7309	1.8803	0.2833	10.0845
BA	1.7329	0.1753	0.5522	1.7461	0.1608	0.4295
BAC	1.7441	0.6178	0.4130	1.7325	0.4828	0.3444
C	1.7376	0.2255	0.9701	1.7430	0.1736	0.6844
CAT	1.7325	0.2531	1.7459	1.7325	0.2305	1.0083
CVX	1.7637	0.6729	2.0788	1.7512	0.4769	1.1658
DD	1.9220	0.0359	7.4336	1.9322	0.1916	11.2821
DIS	1.7325	0.1691	1.0917	1.7325	0.1580	0.7457
GE	1.8965	0.4004	7.5000	1.9195	0.2970	7.5151
GM	1.8545	0.0497	2.8984	1.7336	0.1302	1.2418
HD	1.7500	0.2357	4.6684	1.7752	0.1910	1.8216
HPQ	1.7325	0.0751	0.4124	1.7325	0.0762	0.3574
IBM	1.7325	0.1098	0.5483	1.7325	0.1098	0.4406
INTC	1.8234	0.2091	9.9281	1.9999	0.1784	9.9591
JNJ	1.8322	0.2714	4.2282	1.7689	0.2309	1.4297
JPM	1.7473	0.5283	3.0839	1.7325	0.3847	1.2420
KO	1.7535	0.2020	7.8378	1.7812	0.1566	5.7200
MCD	1.7325	0.1670	0.5328	1.7325	0.1654	0.4167
MMM	1.7325	0.1249	0.7565	1.7325	0.1230	0.5714
MRK	1.7325	0.1158	0.1265	1.7325	0.1163	0.1257
MSFT	1.8710	0.1293	6.0573	1.8547	0.1427	2.5893
PFE	1.7402	0.3854	1.6620	1.7591	0.3072	1.0720
PG	1.7325	0.2770	1.2074	1.7340	0.2530	0.7940
T	1.7325	0.1619	0.4918	1.7325	0.1582	0.3912
UTX	1.8077	0.1455	2.3654	1.7500	0.1931	1.2059
VZ	1.8338	0.2310	5.5116	1.8431	0.1926	3.0450
WMT	1.7325	0.4049	1.7809	1.7327	0.3118	1.0097
XOM	1.7632	0.4682	0.8831	1.8285	0.2816	0.5421

Table 3: Statistic of the goodness of fit tests

Ticker	Model	KS	p-value	AD
AA	Normal-GARCH	0.0285	0.0340	1.3952
	CTS-GARCH	0.0230	0.1408	0.1938
	RDTS-GARCH	0.0230	0.1402	0.1948
AXP	Normal-GARCH	0.0249	0.0886	84.0733
	CTS-GARCH	0.0144	0.6748	0.1090
	RDTS-GARCH	0.0145	0.6655	0.2233
BA	Normal-GARCH	0.0308	0.0173	15.6383
	CTS-GARCH	0.0202	0.2575	0.0698
	RDTS-GARCH	0.0196	0.2884	0.0850
BAC	Normal-GARCH	0.0266	0.0240	0.3805
	CTS-GARCH	0.0144	0.5390	0.0359
	RDTS-GARCH	0.0138	0.5918	0.0639
C	Normal-GARCH	0.0298	0.0073	160.1880
	CTS-GARCH	0.0205	0.1400	0.1798
	RDTS-GARCH	0.0215	0.1072	0.4389
CAT	Normal-GARCH	0.0319	0.0124	1.9053
	CTS-GARCH	0.0248	0.0914	0.1492
	RDTS-GARCH	0.0245	0.0995	0.1406
CVX	Normal-GARCH	0.0177	0.4113	0.1066
	CTS-GARCH	0.0143	0.6843	0.0950
	RDTS-GARCH	0.0135	0.7535	0.0970
DD	Normal-GARCH	0.0354	0.0037	1.4710
	CTS-GARCH	0.0284	0.0347	0.1104
	RDTS-GARCH	0.0344	0.0053	0.1773
DIS	Normal-GARCH	0.0381	0.0014	281.9976
	CTS-GARCH	0.0265	0.0592	0.1241
	RDTS-GARCH	0.0262	0.0649	0.2034
GE	Normal-GARCH	0.0243	0.1033	0.3035
	CTS-GARCH	0.0187	0.3436	0.1701
	RDTS-GARCH	0.0188	0.3364	0.1782
GM	Normal-GARCH	0.0428	0.0002	17852.7859
	CTS-GARCH	0.0197	0.2837	0.1788
	RDTS-GARCH	0.0211	0.2131	0.1993
HD	Normal-GARCH	0.0338	0.0066	1.2829
	CTS-GARCH	0.0126	0.8194	0.1547
	RDTS-GARCH	0.0120	0.8620	0.1452
HPQ	Normal-GARCH	0.0506	0.0000	3476.9698
	CTS-GARCH	0.0205	0.2438	0.0810
	RDTS-GARCH	0.0213	0.2072	0.0986
IBM	Normal-GARCH	0.0554	0.0000	99.1506
	CTS-GARCH	0.0219	0.1797	0.0962
	RDTS-GARCH	0.0222	0.1682	0.0940
INTC	Normal-GARCH	0.0266	0.0579	17.7457
	CTS-GARCH	0.0160	0.5435	0.1512
	RDTS-GARCH	0.0266	0.0580	5.7968

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Ticker	Model	KS	p-value	AD
JNJ	Normal-GARCH	0.0405	0.0005	0.7086
	CTS-GARCH	0.0262	0.0647	0.1181
	RDTS-GARCH	0.0201	0.2626	0.1180
JPM	Normal-GARCH	0.0323	0.0009	1.3883
	CTS-GARCH	0.0191	0.1363	0.1826
	RDTS-GARCH	0.0182	0.1727	0.1733
KO	Normal-GARCH	0.0379	0.0015	1.2765
	CTS-GARCH	0.0146	0.6633	0.1520
	RDTS-GARCH	0.0151	0.6168	0.1476
MCD	Normal-GARCH	0.0403	0.0006	5.9558
	CTS-GARCH	0.0110	0.9215	0.0577
	RDTS-GARCH	0.0119	0.8678	0.0765
MMM	Normal-GARCH	0.0503	0.0000	1.1493
	CTS-GARCH	0.0165	0.5057	0.0861
	RDTS-GARCH	0.0167	0.4876	0.0918
MRK	Normal-GARCH	0.0535	0.0000	1334.3847
	CTS-GARCH	0.0147	0.6519	0.0354
	RDTS-GARCH	0.0146	0.6606	0.0394
MSFT	Normal-GARCH	0.0379	0.0015	2.2925
	CTS-GARCH	0.0210	0.2170	0.2039
	RDTS-GARCH	0.0198	0.2786	0.2402
PFE	Normal-GARCH	0.0233	0.1328	0.9711
	CTS-GARCH	0.0161	0.5332	0.1184
	RDTS-GARCH	0.0162	0.5262	0.1326
PG	Normal-GARCH	0.0277	0.0428	1.6565
	CTS-GARCH	0.0115	0.8959	0.1423
	RDTS-GARCH	0.0116	0.8873	0.1601
T	Normal-GARCH	0.0341	0.0000	36014.5914
	CTS-GARCH	0.0121	0.4690	0.1089
	RDTS-GARCH	0.0128	0.3976	0.2522
UTX	Normal-GARCH	0.0456	0.0001	0.0990
	CTS-GARCH	0.0209	0.2240	0.0640
	RDTS-GARCH	0.0193	0.3109	0.0535
VZ	Normal-GARCH	0.0383	0.0013	0.3669
	CTS-GARCH	0.0219	0.1807	0.1478
	RDTS-GARCH	0.0215	0.1973	0.1528
WMT	Normal-GARCH	0.0257	0.0729	0.2594
	CTS-GARCH	0.0121	0.8568	0.0650
	RDTS-GARCH	0.0116	0.8878	0.0657
XOM	Normal-GARCH	0.0232	0.1361	0.5962
	CTS-GARCH	0.0122	0.8508	0.0611
	RDTS-GARCH	0.0122	0.8500	0.0593

Table 4: DJIA index estimated normal-GARCH parameters from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007. Dates are in the form *yyyymmdd*.

Date	β_1	α_1	α_0	λ
20060104	0.903133393	0.085430921	0.000001560	0.060129047
20060111	0.903899223	0.085094959	0.000001501	0.061413483
20060118	0.904257327	0.084812044	0.000001490	0.060306713
20060125	0.903934244	0.084314520	0.000001584	0.058942811
20060201	0.904144067	0.084163556	0.000001572	0.059975492
20060208	0.904345327	0.083994304	0.000001567	0.059315554
20060215	0.904623506	0.083762386	0.000001555	0.060203581
20060222	0.904872184	0.083650176	0.000001533	0.060626208
20060301	0.904926629	0.083600987	0.000001528	0.060042349
20060308	0.904991033	0.083821527	0.000001497	0.059438035
20060315	0.905403570	0.083536155	0.000001476	0.060515481
20060322	0.905665145	0.083491712	0.000001444	0.061302821
20060329	0.905868280	0.083414634	0.000001428	0.060638761
20060405	0.905466068	0.083870138	0.000001420	0.060896764
20060412	0.906315853	0.083252675	0.000001387	0.059510284
20060419	0.905440562	0.083505992	0.000001473	0.061118625
20060426	0.906571170	0.082599797	0.000001423	0.060935127
20060503	0.907238968	0.082327226	0.000001375	0.061185589
20060510	0.907541644	0.082041724	0.000001369	0.062400962
20060517	0.905627280	0.083107200	0.000001487	0.059829448
20060524	0.907225942	0.081906920	0.000001426	0.059112607
20060531	0.906909700	0.082078405	0.000001450	0.059121953
20060607	0.906945963	0.082005507	0.000001458	0.058054978
20060614	0.907287348	0.081730294	0.000001444	0.057353045
20060621	0.907149109	0.081668771	0.000001462	0.058230227
20060628	0.907412084	0.081433349	0.000001450	0.057809489
20060705	0.907567297	0.081424284	0.000001449	0.058474573
20060712	0.907882902	0.081144161	0.000001437	0.057810391
20060719	0.907207521	0.081558251	0.000001479	0.057789919
20060726	0.907983981	0.081017279	0.000001446	0.058371012
20060802	0.908077524	0.080940851	0.000001437	0.058321366
20060809	0.907363364	0.081494704	0.000001453	0.058200539
20060816	0.907353197	0.081329080	0.000001464	0.058310165
20060823	0.907132183	0.081788698	0.000001433	0.058839002
20060830	0.905911773	0.082806522	0.000001449	0.059615938
20060906	0.905876432	0.083177981	0.000001421	0.059604668
20060913	0.906141835	0.082952224	0.000001412	0.060016695
20060920	0.905911312	0.083431383	0.000001387	0.060114661
20060927	0.906337830	0.083096664	0.000001370	0.060719289
20061004	0.906392666	0.083068975	0.000001365	0.061431208
20061011	0.906898792	0.082916270	0.000001321	0.061530415
20061018	0.906977589	0.082981544	0.000001307	0.061819964
20061025	0.907535726	0.082680552	0.000001269	0.063199949
20061101	0.907964439	0.082471880	0.000001238	0.063083118
20061108	0.908256665	0.082214211	0.000001230	0.063403110

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20061115	0.908603981	0.082163206	0.000001200	0.063636590
20061122	0.909216905	0.082012625	0.000001145	0.064181416
20061129	0.908643826	0.081862611	0.000001223	0.063137579
20061206	0.909327202	0.081524806	0.000001185	0.063215486
20061213	0.910077129	0.081194900	0.000001129	0.063372442
20061220	0.910395984	0.081110766	0.000001101	0.064276960
20061227	0.910170083	0.081188649	0.000001118	0.064276967
20070103	0.910728597	0.080880770	0.000001086	0.064049842
20070110	0.911259701	0.080646861	0.000001049	0.063648369
20070117	0.911677497	0.080467826	0.000001019	0.064199176
20070124	0.911109046	0.080910692	0.000001041	0.063985713
20070131	0.911345423	0.080545067	0.000001045	0.064172538
20070207	0.912454661	0.079996173	0.000000978	0.064213568
20070214	0.912293093	0.080054217	0.000000989	0.064578837
20070221	0.912878969	0.079749900	0.000000955	0.063792703
20070228	0.907712192	0.080889040	0.000001417	0.062291435
20070307	0.908990823	0.079857765	0.000001381	0.061764192
20070314	0.909243931	0.079590626	0.000001385	0.061167423
20070321	0.909385517	0.079234126	0.000001386	0.061954361
20070328	0.909744245	0.079098044	0.000001365	0.061544456
20070404	0.909621305	0.079172161	0.000001362	0.062314976
20070411	0.909290646	0.079480096	0.000001362	0.062107768
20070418	0.908848424	0.079885589	0.000001364	0.063504981
20070425	0.909236929	0.079510540	0.000001361	0.064247765
20070502	0.909228228	0.079572615	0.000001349	0.065073633
20070509	0.908949832	0.080007948	0.000001331	0.065964853
20070516	0.909345811	0.079531912	0.000001338	0.065845470
20070523	0.909494162	0.079563098	0.000001314	0.066272392
20070530	0.909527828	0.079539061	0.000001312	0.066581613
20070606	0.909457567	0.079615742	0.000001313	0.065829470
Average	0.907736375	0.081812917	0.000001351	0.061523578

Table 5: DJIA market parameters of the innovation processes for the CTS-GARCH and RDTS-GARCH models. The DJIA time series from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007 are considered.

	CTS				RDTS			
Date	C	λ_-	λ_+	α	C	λ_+	λ_-	α
20060104	0.1145	0.3314	1.0346	1.7613	0.0786	0.9436	0.2905	1.8193
20060111	0.1160	0.3354	1.0466	1.7588	0.0792	0.9518	0.2922	1.8180
20060118	0.1135	0.3293	1.0361	1.7634	0.0781	0.9453	0.2894	1.8206
20060125	0.1248	0.3523	1.0329	1.7400	0.0836	0.9349	0.2996	1.8062
20060201	0.1232	0.3474	1.0461	1.7436	0.0831	0.9473	0.2982	1.8076
20060208	0.1203	0.3402	1.0428	1.7493	0.0819	0.9472	0.2952	1.8106
20060215	0.1204	0.3403	1.0448	1.7491	0.0821	0.9489	0.2954	1.8103
20060222	0.1207	0.3407	1.0474	1.7486	0.0822	0.9511	0.2956	1.8100
20060301	0.1200	0.3395	1.0497	1.7500	0.0818	0.9529	0.2949	1.8111
20060308	0.1210	0.3417	1.0419	1.7477	0.0823	0.9459	0.2958	1.8096
20060315	0.1196	0.3378	1.0546	1.7508	0.0818	0.9574	0.2945	1.8110
20060322	0.1201	0.3393	1.0611	1.7501	0.0820	0.9619	0.2953	1.8106
20060329	0.1190	0.3369	1.0600	1.7524	0.0815	0.9619	0.2943	1.8120
20060405	0.1193	0.3381	1.0629	1.7519	0.0816	0.9636	0.2948	1.8119
20060412	0.1185	0.3361	1.0631	1.7535	0.0811	0.9640	0.2937	1.8130
20060419	0.1235	0.3530	0.9878	1.7417	0.0822	0.8984	0.2979	1.8092
20060426	0.1239	0.3544	0.9930	1.7411	0.0822	0.9011	0.2981	1.8092
20060503	0.1247	0.3561	0.9909	1.7394	0.0826	0.8990	0.2988	1.8081
20060510	0.1250	0.3568	0.9920	1.7389	0.0827	0.8999	0.2990	1.8079
20060517	0.1338	0.3801	1.0156	1.7222	0.0860	0.9094	0.3091	1.7996
20060524	0.1341	0.3801	1.0057	1.7212	0.0861	0.9016	0.3084	1.7991
20060531	0.1370	0.3873	1.0295	1.7164	0.0872	0.9173	0.3117	1.7968
20060607	0.1390	0.3930	1.0357	1.7127	0.0879	0.9201	0.3141	1.7951
20060614	0.1360	0.3857	1.0310	1.7185	0.0867	0.9191	0.3113	1.7980
20060621	0.1346	0.3829	1.0280	1.7212	0.0862	0.9182	0.3103	1.7993
20060628	0.1341	0.3820	1.0246	1.7222	0.0859	0.9154	0.3095	1.8002
20060705	0.1327	0.3795	1.0093	1.7243	0.0854	0.9049	0.3084	1.8013
20060712	0.1319	0.3781	1.0119	1.7263	0.0848	0.9071	0.3075	1.8028
20060719	0.1301	0.3753	1.0079	1.7297	0.0840	0.9054	0.3063	1.8050
20060726	0.1282	0.3706	1.0054	1.7336	0.0833	0.9055	0.3047	1.8068
20060802	0.1283	0.3702	1.0047	1.7333	0.0834	0.9051	0.3045	1.8064
20060809	0.1288	0.3721	0.9934	1.7319	0.0835	0.8960	0.3050	1.8060
20060816	0.1278	0.3683	0.9996	1.7341	0.0833	0.9020	0.3039	1.8065
20060823	0.1301	0.3759	0.9951	1.7294	0.0841	0.8957	0.3069	1.8045
20060830	0.1340	0.3867	0.9960	1.7219	0.0854	0.8922	0.3112	1.8011
20060906	0.1336	0.3857	0.9974	1.7227	0.0854	0.8939	0.3113	1.8012
20060913	0.1318	0.3806	1.0051	1.7264	0.0848	0.9018	0.3095	1.8029
20060920	0.1338	0.3854	1.0060	1.7225	0.0856	0.9008	0.3116	1.8007
20060927	0.1315	0.3799	1.0127	1.7273	0.0848	0.9079	0.3097	1.8031
20061004	0.1304	0.3772	1.0135	1.7295	0.0845	0.9098	0.3090	1.8040
20061011	0.1327	0.3833	1.0093	1.7250	0.0852	0.9042	0.3112	1.8019
20061018	0.1330	0.3830	1.0119	1.7242	0.0856	0.9068	0.3113	1.8010
20061025	0.1339	0.3867	1.0138	1.7228	0.0858	0.9068	0.3128	1.8006
20061101	0.1335	0.3868	1.0077	1.7236	0.0854	0.9020	0.3126	1.8015
20061108	0.1326	0.3847	1.0059	1.7252	0.0851	0.9017	0.3119	1.8021
20061115	0.1323	0.3837	1.0112	1.7260	0.0851	0.9061	0.3117	1.8024
20061122	0.1347	0.3903	1.0141	1.7213	0.0859	0.9056	0.3142	1.8003
20061129	0.1398	0.4015	1.0253	1.7117	0.0880	0.9104	0.3191	1.7949
20061206	0.1398	0.4005	1.0249	1.7115	0.0882	0.9105	0.3188	1.7944

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20061213	0.1428	0.4090	1.0232	1.7056	0.0891	0.9057	0.3218	1.7919
20061220	0.1439	0.4115	1.0269	1.7037	0.0896	0.9078	0.3229	1.7909
20061227	0.1393	0.4003	1.0301	1.7127	0.0881	0.9147	0.3192	1.7949
20070103	0.1402	0.4027	1.0253	1.7109	0.0883	0.9099	0.3199	1.7942
20070110	0.1420	0.4077	1.0258	1.7075	0.0888	0.9081	0.3214	1.7930
20070117	0.1432	0.4104	1.0337	1.7053	0.0893	0.9129	0.3226	1.7918
20070124	0.1420	0.4066	1.0377	1.7077	0.0890	0.9178	0.3214	1.7926
20070131	0.1424	0.4087	1.0464	1.7075	0.0889	0.9239	0.3221	1.7931
20070207	0.1454	0.4164	1.0471	1.7016	0.0899	0.9209	0.3248	1.7904
20070214	0.1406	0.4046	1.0454	1.7110	0.0884	0.9245	0.3210	1.7945
20070221	0.1422	0.4085	1.0433	1.7077	0.0889	0.9210	0.3221	1.7929
20070228	0.1123	0.2817	0.9029	1.7566	0.0799	0.8606	0.2542	1.8108
20070307	0.1102	0.2756	0.8981	1.7609	0.0791	0.8590	0.2513	1.8129
20070314	0.1144	0.2857	0.9087	1.7522	0.0809	0.8639	0.2557	1.8083
20070321	0.1145	0.2869	0.9191	1.7525	0.0808	0.8709	0.2564	1.8086
20070328	0.1140	0.2848	0.9075	1.7531	0.0807	0.8631	0.2552	1.8088
20070404	0.1153	0.2877	0.9110	1.7505	0.0812	0.8648	0.2564	1.8073
20070411	0.1161	0.2900	0.9067	1.7486	0.0815	0.8606	0.2572	1.8065
20070418	0.1170	0.2917	0.9185	1.7470	0.0819	0.8694	0.2581	1.8056
20070425	0.1150	0.2867	0.9219	1.7512	0.0813	0.8741	0.2565	1.8074
20070502	0.1157	0.2881	0.9237	1.7500	0.0815	0.8750	0.2570	1.8068
20070509	0.1179	0.2937	0.9263	1.7452	0.0824	0.8752	0.2591	1.8044
20070516	0.1183	0.2943	0.9430	1.7449	0.0826	0.8879	0.2596	1.8043
20070523	0.1194	0.2970	0.9382	1.7425	0.0830	0.8834	0.2605	1.8031
20070530	0.1180	0.2933	0.9445	1.7456	0.0825	0.8894	0.2592	1.8046
20070606	0.1180	0.2939	0.9435	1.7457	0.0823	0.8887	0.2591	1.8050
Average	0.1278	0.3574	1.0032	1.7330	0.0842	0.9095	0.2975	1.8037

Table 6: Goodness of fit statistics. KS, AD, and χ^2 with the relative p-value for the normal-GARCH, CTS-GARCH and RDTS-GARCH models from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007. Values are in average.

	KS	AD	χ^2(p-value)
Normal-GARCH	0.0347	14.5832	185.0790(0.0016)
CTS-GARCH	0.0327	0.0689	149.7951(0.0732)
RDTS-GARCH	0.0328	0.0694	151.2415(0.0569)

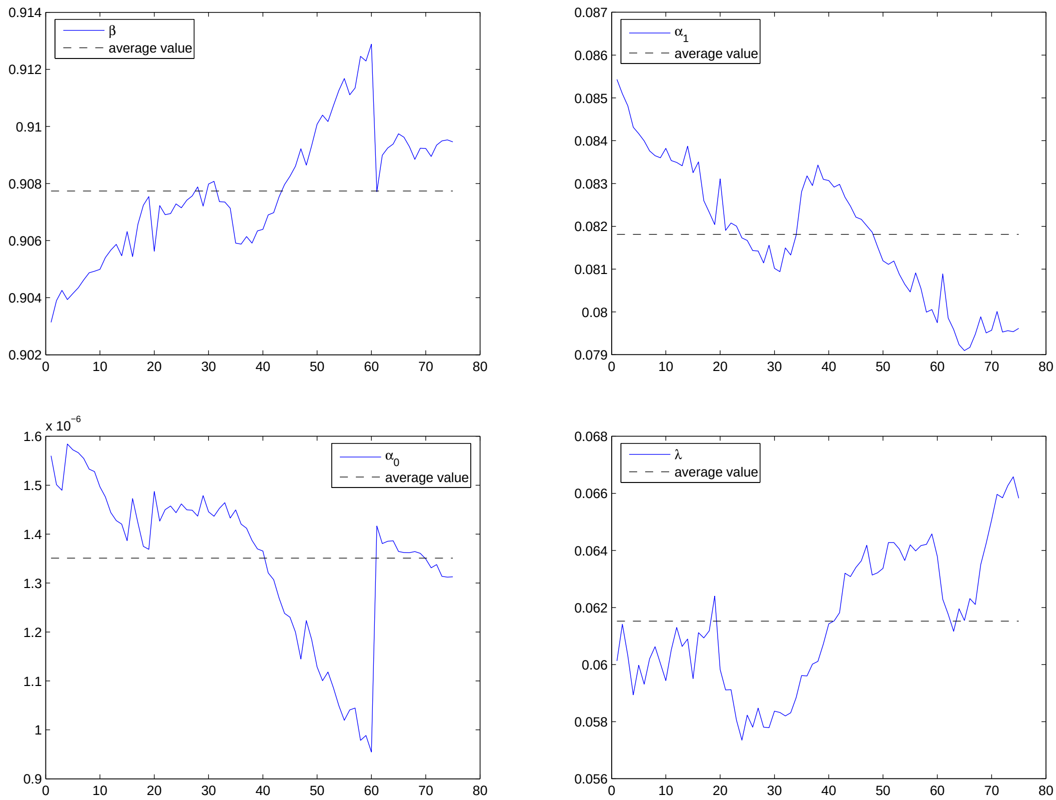

Figure 1: DJIA estimated market parameters for the normal-GARCH model from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007.

Figure 2: DJIA estimated market parameters for the CTS-GARCH and RDTS-GARCH models from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007.

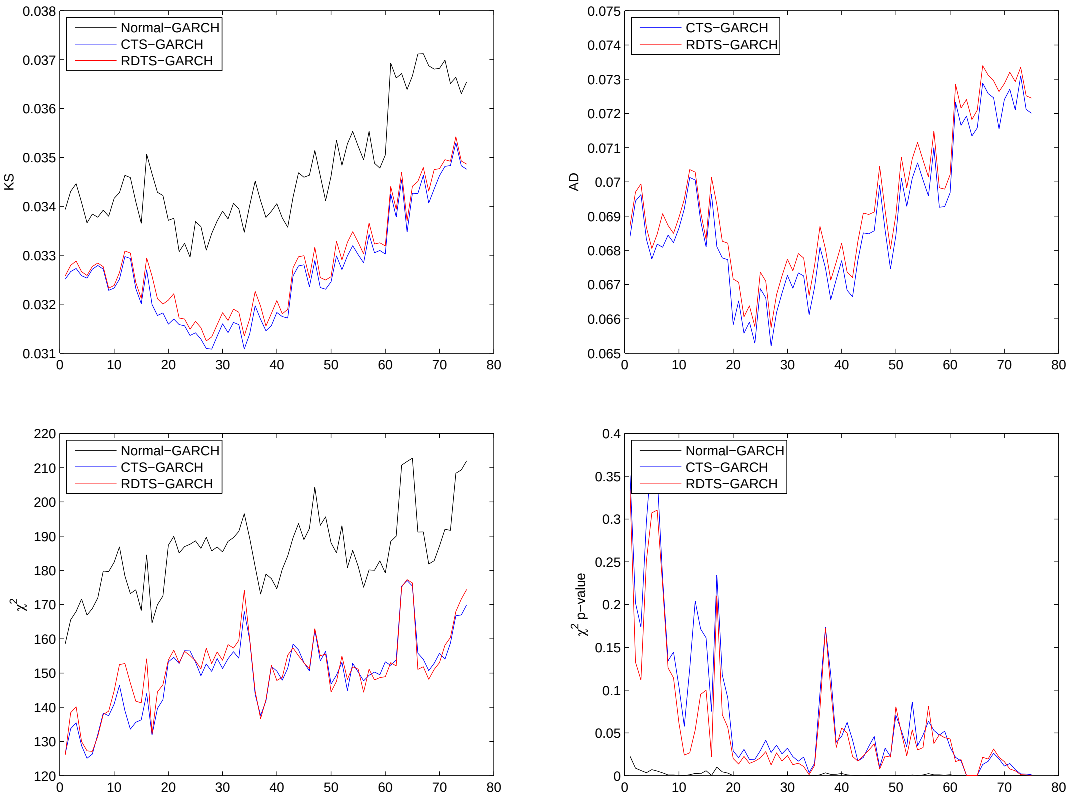

Figure 3: Goodness of fit. KS, AD and χ^2 with the relative p-value for the normal-GARCH, CTS-GARCH and RDTS-GARCH models from January 2, 1996 to any Wednesday from January 4, 2006 to June 6, 2007. The AD statistic for the normal-GARCH is not comparable, since it is always greater than 9.1474.

Table 7: Option pricing performance for 17 selected Wednesday (one per month) between January 4, 2006 and May 9, 2007.

		APE	AAE	RMSE	ARPE
Normal-GARCH	20060111	0.0988	0.6381	0.8394	0.6289
CTS-GARCH		0.0489	0.3161	0.4268	0.2524
RDTS-GARCH		0.0470	0.3035	0.4417	0.1441
Normal-GARCH	20060208	0.1069	0.6513	0.7851	0.9685
CTS-GARCH		0.0540	0.3292	0.4139	0.3883
RDTS-GARCH		0.0486	0.2958	0.4157	0.1733
Normal-GARCH	20060315	0.0489	0.4487	0.5815	0.3760
CTS-GARCH		0.0253	0.2321	0.3209	0.1286
RDTS-GARCH		0.0238	0.2188	0.3230	0.0513
Normal-GARCH	20060412	0.0731	0.5483	0.7627	0.4077
CTS-GARCH		0.0363	0.2726	0.4462	0.1401
RDTS-GARCH		0.0355	0.2661	0.4678	0.0816
Normal-GARCH	20060510	0.0591	0.6099	0.7897	0.5024
CTS-GARCH		0.0347	0.3584	0.4700	0.1747
RDTS-GARCH		0.0316	0.3260	0.4583	0.0739
Normal-GARCH	20060614	0.0510	0.3229	0.4081	0.9875
CTS-GARCH		0.0460	0.2913	0.3837	0.3488
RDTS-GARCH		0.0548	0.3470	0.4563	0.1981
Normal-GARCH	20060712	0.0802	0.5860	0.7886	1.1047
CTS-GARCH		0.0284	0.2076	0.3321	0.3963
RDTS-GARCH		0.0258	0.1881	0.3268	0.1966
Normal-GARCH	20060809	0.0545	0.4186	0.5991	0.7688
CTS-GARCH		0.0252	0.1938	0.3067	0.2580
RDTS-GARCH		0.0282	0.2164	0.3342	0.1666
Normal-GARCH	20060913	0.0541	0.4947	0.6194	0.6993
CTS-GARCH		0.0293	0.2677	0.3718	0.3047
RDTS-GARCH		0.0248	0.2268	0.3602	0.1429
Normal-GARCH	20061011	0.0476	0.5284	0.7378	0.4608
CTS-GARCH		0.0240	0.2666	0.4404	0.1791
RDTS-GARCH		0.0228	0.2527	0.4453	0.0917
Normal-GARCH	20061108	0.1232	0.8606	1.0311	0.7525
CTS-GARCH		0.0716	0.5002	0.6108	0.3714
RDTS-GARCH		0.0615	0.4297	0.5800	0.1842
Normal-GARCH	20061213	0.0291	0.3631	0.4954	0.3599
CTS-GARCH		0.0182	0.2275	0.3481	0.1637
RDTS-GARCH		0.0176	0.2196	0.3473	0.0747
Normal-GARCH	20070110	0.0273	0.3313	0.4568	0.2751
CTS-GARCH		0.0188	0.2279	0.3588	0.1383
RDTS-GARCH		0.0202	0.2446	0.3742	0.0962
Normal-GARCH	20070207	0.0391	0.5965	0.7901	0.3261
CTS-GARCH		0.0253	0.3856	0.5312	0.1688
RDTS-GARCH		0.0227	0.3460	0.5219	0.0944
Normal-GARCH	20070314	0.0748	0.6228	0.8485	3.0362
CTS-GARCH		0.0287	0.2385	0.3386	1.0589
RDTS-GARCH		0.0243	0.2021	0.3101	0.5639
Normal-GARCH	20070411	0.0389	0.3809	0.5487	1.6838
CTS-GARCH		0.0212	0.2080	0.3205	0.6683
RDTS-GARCH		0.0210	0.2051	0.3268	0.3803
Normal-GARCH	20070509	0.0765	0.7750	0.9636	0.8979
CTS-GARCH		0.0432	0.4370	0.6169	0.3702
RDTS-GARCH		0.0433	0.4380	0.6492	0.2103