Lecture 10: Performance measures

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Introduction

- A key step in the investment management process is measurement and evaluation of portfolio performance.
- Usually, the performance of the portfolio is measured with respect to the performance of some benchmark portfolio which can be a broad-based market index, a specialized index, or a customized index. In recent years, some defined benefit plans have developed liability-driven indexes.
- The formula which quantifies the portfolio performance is called a performance measure.

Introduction

- A widely used measure for performance evaluation is the Sharpe ratio introduced by Sharpe (1966).
- The Sharpe ratio calculates the adjusted return of the portfolio relative to a target return.
- It is the ratio between the average active portfolio return and the standard deviation of the portfolio return.
- In this way, it is a reward-to-variability ratio in which the variability is computed by means of the standard deviation.

Introduction

- Recall that standard deviation penalizes both the upside and the downside potential of portfolio return, thus it is not a very appropriate choice as a measure of performance.

Many alternatives to the Sharpe ratio have been proposed in the literature.

- Some of them are reward-to-variability ratios in which a downside dispersion measure is used in the denominator. One example is the Sortino ratio, in which the downside semi-standard deviation is used as a measure of variability.

Introduction

Other types of performance measures are the reward-to-risk ratios. These ratios calculate the risk-adjusted active reward of the portfolio.

- For example, the Sortino-Satchell ratio calculates the average active return divided by a lower partial moment of the portfolio return distribution.
- The STARR calculates the average active return divided by average value-at-risk (AVaR) at a given tail probability.

There are examples in which a reward measure is used instead of the average active return.

- For instance, a one-sided variability ratio introduced by Farinelli and Tibiletti (2002) is essentially a ratio between an upside and a downside partial moment of the portfolio return distribution.
- The Rachev ratio (R-ratio) is a ratio between the average of upper quantiles of the portfolio return distribution and AVaR.

Introduction

- Measuring a strategies performance is an ex-post analysis. The performance measure is calculated using the realized portfolio returns during a specified period back in time (e.g. the past one year).
- Alternatively, performance measures can be used in an ex-ante analysis, in which certain assumptions for the future behavior of the assets are introduced. In this case, the general goal is to find a portfolio with the best characteristics as calculated by the performance measure.

Introduction

- The performance measure problems of the *ex-ante* type can be related to the efficient frontier generated by mean-risk analysis, and more generally by reward-risk analysis.
- We'll consider these two types of performance measures and their relationship to the efficient frontier.
- We'll provide examples of frequently used performance measures and remark on their advantages and disadvantages.
- Finally, we'll consider the capital market line in the case of the general reward-risk analysis with a risk-free asset added to the investment universe.

Reward-to-risk ratios

- One general type of a performance measure is the reward-to-risk (RR) ratio. It is defined as the ratio between a reward measure of the active portfolio return and the risk of active portfolio return,

$$RR(r_p) = \frac{\nu(r_p - r_b)}{\rho(r_p - r_b)}, \tag{1}$$

where

$r_p - r_b$	is the active portfolio return
$r_p = w'X$	denotes the return of the portfolio with weights w and assets returns described by the random vector X
r_b	denotes the return of the benchmark portfolio;
$\nu(r_p)$	is a reward measure of r_p
$\rho(r_p)$	calculates the risk of r_p

Reward-to-risk ratios

- The benchmark return r_b can either be a fixed target, for instance 8% annual return, or the return of another portfolio or reference interest rate meaning that r_b can also be a r.v.
- We consider a simpler version of the reward-to-risk ratio in which the reward functional is the expected active portfolio return,

$$RR(r_p) = \frac{E(r_p - r_b)}{\rho(r_p - r_b)}. \tag{2}$$

Reward-to-risk ratios

- In the ex-post analysis, equation (2) is calculated using the available historical returns in a certain period back in time.
- In this case, the numerator is the average of the realized active return and the denominator is the risk estimated from the sample.
- The past performance of different portfolios can be compared by the resulting ratios.

$\Rightarrow$ The portfolio with the highest RR ratio is said to have the best performance in terms of this measure.

- In the ex-ante analysis, the joint distribution of the portfolio return and the benchmark return is hypothesized.
- The parameters of the assumed distribution are estimated from the historical data and the RR ratio is calculated from the fitted distribution.
- In this setting, the portfolio manager is interested in finding a feasible portfolio with highest RR ratio as this portfolio is expected to have the highest return for a unit of risk in its future performance. Formally,

$$\begin{array}{rl} \max\limits_{w} & \dfrac{E(r_p - r_b)}{\rho(r_p - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{3}$$

- On condition that the risk measure is a convex function of portfolio weights, the objective function has nice mathematical properties which guarantee that the solution to (3) is unique.

RR ratios and the efficient portfolios

We discuss here the relationship of the solution to problem (3) with the efficient frontier generated by mean-risk (M-R) analysis. We provide examples of RR ratios and discuss their properties.

- The principle of M-R analysis is introduced in *Lecture 8*. According to it, from all feasible portfolios with a lower bound on expected return, we find the portfolio with minimal risk. This portfolio represents the optimal portfolio given the constraints of the problem.
- By varying the lower bound on the expected return, we obtain a set of optimal portfolios which are called efficient portfolios. Plotting the expected return of the efficient portfolio versus their risk we derive the efficient frontier.

RR ratios and the efficient portfolios

We'll demonstrate that the portfolio with maximal RR ratio, i.e. the solution to problem (3), is among the efficient portfolios when the benchmark return is a constant target.

- If benchmark return is the return of another portfolio, then r_b is a r.v. and the RR ratio cannot be directly related to the efficient frontier resulting from M-R analysis. Nonetheless, it can be related to the efficient frontier of a benchmark tracking type of optimal portfolio problem.
- We start the analysis assuming that the benchmark return is equal to zero. Then the maximal ratio portfolio is the solution to problem

$$\begin{array}{rl} \max\limits_{w} & \dfrac{w'\mu}{\rho(r_p)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{4}$$

which is derived from (3) by setting $r_b = 0$ and making use of the equality $E(r_p) = w'\mu$.

RR ratios and the efficient portfolios

- Consider the efficient frontier generated by the optimal portfolio problem **(12)** given in *Lecture 8*. The shape of the efficient frontier is as the one plotted in *Figure 1*; that is, we assume that the risk measure is a convex function of portfolio weights.
- Each feasible portfolio in the mean-risk plane is characterized by the RR ratio calculated from its coordinates.
- The RR ratio equals the slope of the straight line passing through the origin and the point corresponding to this portfolio in the mean-risk plane.
- All portfolios having equal RR ratios lie on a straight line passing through the origin. Therefore, the portfolio with the largest RR ratio lies on the straight line passing through the origin which is tangent to the efficient frontier. This line is also called the tangent line.

RR ratios and the efficient portfolios

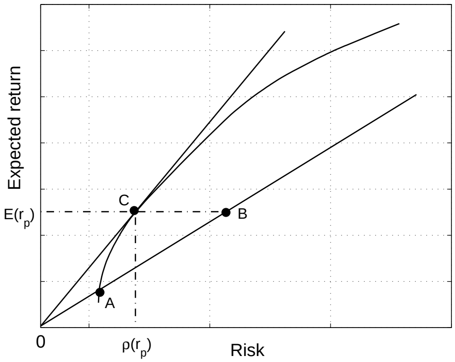

Figure 1. The efficient frontier and the tangent portfolio.

RR ratios and the efficient portfolios

- In Figure 1, the portfolio C has the largest RR ratio. Portfolios A and B have equal RR ratios.
- Portfolio A is an efficient portfolio since it lies on the efficient frontier and portfolio B is sub-optimal.
- Portfolio C is also called ρ-tangent portfolio to emphasize that it is the tangent portfolio to the efficient frontier generated by a risk measure $\rho(X)$.

$\Rightarrow$ This analysis demonstrates that if $r_b = 0$, then the portfolio with the highest RR ratio is one of the efficient portfolios as it lies on the efficient frontier and coincides with the tangent portfolio.

RR ratios and the efficient portfolios

- As a second case, suppose that the benchmark return is a constant.
- Then, under the additional assumption that the risk measure $\rho(X)$ satisfies the invariance property

$$\rho(X + C) = \rho(X) - C,$$

where C is a constant, the maximal RR ratio portfolio is a solution to the optimization problem

$$\begin{array}{rl} \max\limits_{w} & \dfrac{w'\mu - r_b}{\rho(r_p) + r_b} \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \tag{5}$$

- The additional assumption on the risk measure is satisfied by all coherent risk measures and all convex risk measures and, therefore, it is not restrictive.

RR ratios and the efficient portfolios

- Consider the efficient portfolios generated by problem **(12)** given in *Lecture 8* and the corresponding efficient frontier in which the risk coordinate is replaced by the shifted risk defined by the sum $\rho(r_p) + r_b$.
- The RR ratio of any feasible portfolio is equal to the slope of a straight line passing through the point with zero shifted risk and expected return equal to r_b and the point in the mean-shifted risk plane corresponding to the feasible portfolio. (See the illustration in *Figure 2*).

RR ratios and the efficient portfolios

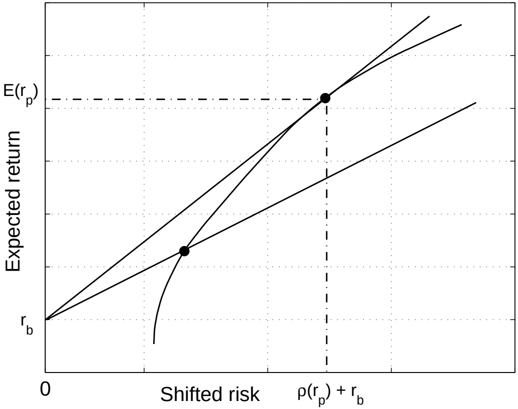

Figure 2. The efficient frontier and the tangent portfolio in the mean-shifted risk plane.
⇒ The portfolio with the highest RR ratio in this case is also an efficient portfolio and it is a ρ-tangent portfolio in the mean-shifted risk plane.

RR ratios and the efficient portfolios

The analysis corresponding to $r_b = 0$ can be obtained as a special case from (5).

- If the benchmark return is equal to 0, then (5) is the same as (4).
- Geometrically, starting from $r_b = 0$ and increasing r_b continuously means that we shift the efficient frontier in *Figure 1* to the right while moving upwards the crossing point between the tangent line and the vertical axis.
- As a result, the tangent portfolio moves away from the minimum risk portfolio and gets closer to the global maximum expected return portfolio.
- The slope of the tangent line decreases. At the limit, when the benchmark return equals the expected return of the global maximum expected return portfolio, the tangent line becomes parallel to the horizontal axis.

RR ratios and the efficient portfolios

- Conversely, starting from $r_b = 0$ and decreasing continuously r_b, we shift the efficient frontier in *Figure 1* to the left while moving downwards the crossing point between the tangent line and the vertical axis.
- The tangent portfolio moves towards the minimum risk portfolio.
- At the limit, when the benchmark return equals the negative of the risk of the global minimum risk portfolio, the tangent line becomes coincident with the vertical axis.
- In this case, the slope of the tangent line is not defined as the RR ratio explodes because the denominator turns into zero.

$\Rightarrow$ This scenario can be considered as a limit case in which the optimal RR ratio portfolio approaches the global minimum risk portfolio.

- In summary, when the benchmark return varies from the negative of the risk of the global minimum risk portfolio to the expected return of the global maximum performance portfolio, the solutions to (5) describe the entire efficient frontier.
- In this analysis, we have tacitly assumed that the risk of all feasible portfolios is non-negative and that ρ is a coherent risk measure which is needed in order for the efficient frontier to have the nice concave shape as plotted in *Figure 1*.

RR ratios and the efficient portfolios

- The general case, in which r_b is the return of a benchmark portfolio, is more complicated and it is not possible to link the solution of (3) to the efficient frontier obtained without the benchmark portfolio because r_b is a r.v.
- It is possible to simplify the optimization problem at the cost of introducing an additional variable and provided that the risk measure satisfies the positive homogeneity property and a few other technical conditions.

RR ratios and the efficient portfolios

- Stoyanov et al. (2007) demonstrate that the following problem

$$\begin{array}{rl} \min\limits_{v,t} & \rho(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & E(v'X - tr_b) = 1 \\ & v \geq 0, t \geq 0, \end{array} \tag{6}$$

  where $v'X$ denotes the returns of a portfolio with scaled weights and $t \in \mathbb{R}$ is an additional variable, is equivalent to problem (3) in the sense that if $(\bar{v}, \bar{t})$ is a solution to (6), then $\bar{v}/\bar{t}$ is a vector of weights solving (3).
- This equivalence holds only if the optimal ratio problem (3) is well-defined; that is, for all feasible portfolios the risk $\rho(r_p - r_b)$ is strictly positive and there are feasible portfolios with positive mean active return.

Limitations in the application of reward-to-risk ratios

- The risk of a random variable as calculated by a risk measure may not always be a positive quantity.
- In *Lecture 6*, we considered the coherent risk measures, which satisfy the invariance property.
- The rationale behind the invariance property is the interpretation of the risk measure in terms of capital requirements. Investments with a zero or negative risk are acceptable in the sense that no capital reserves are required to insure against losses.
- If a portfolio has risk equal to zero, then its RR ratio is not defined.

Limitations in the application of reward-to-risk ratios

This observation has more profound consequences in the ex-ante analysis.

- Suppose that the set of feasible portfolios contains one portfolio with risk equal to zero. Then, problem (3) becomes unbounded and cannot be solved.
- In practice, it is difficult to assess whether the set of feasible portfolios contains a portfolio having zero risk.
- The global minimum risk portfolio and the global maximum return portfolio can be used to construct a criterion. If the former has a negative risk and the risk of the latter is positive, then the feasible set contains a portfolio with zero risk and problem (3) is unbounded.

Limitations in the application of reward-to-risk ratios

A feasible portfolio with a zero risk or a negative risk is not uncommon.

- For example, if we choose AVaR as a risk measure, $\rho(X) = AVaR_{\epsilon}(X)$, then for any portfolio with a positive expected return, there exists a tail probability ϵ^* such that $AVaR_{\epsilon^*}(X) = 0$.
- AVaR is a continuous non-increasing function of the tail probability and is not below the negative of the mathematical expectation of the portfolio return distribution.
- If $\epsilon_1 \leq \epsilon_2$, then

$$AVaR_{\epsilon_1}(r_p) \geq AVaR_{\epsilon_2}(r_p) \geq -E(r_p)$$

- Under these assumptions, if for some small tail probability AVaR is positive, then there exists a tail probability ϵ^* such that $AVaR_{\epsilon^*}(r_p) = 0$.

$\Rightarrow$ The AVaR of any portfolio with positive expected return may become equal to zero. It depends on the choice of the tail probability.

Limitations in the application of reward-to-risk ratios

- A way to avoid the issue of an unbounded ratio is through the linearized forms of RR ratios.
- Comparing 2 investments with equal expected return but different risks, we prefer the investment with the larger RR ratio.
- If M-R analysis is consistent with second-order stochastic dominance (SSD), then the ratio is also consistent with SSD,

$$w'X \succeq_{SSD} v'X \quad \Longrightarrow \quad \frac{v'\mu - r_b}{\rho(v'X) + r_b} \leq \frac{w'\mu - r_b}{\rho(w'X) + r_b}.$$

where r_b is a constant benchmark the values of which are in the range discussed in the previous chapter, v and w denote the compositions of the two portfolios, and X stands for the vector of random returns of the assets in the portfolios.

Limitations in the application of reward-to-risk ratios

- The following functional, which is also consistent with SSD, is called a linearized form of a RR ratio

$$LRR(w,\lambda) = w'\mu - \lambda\rho(r_p), \tag{7}$$

where $\lambda \geq 0$ is a risk-aversion coefficient.

- The consistency with SSD is a consequence of the corresponding consistency of M-R analysis,

$$w'X \succeq_{SSD} v'X \qquad \Longrightarrow \qquad LRR(v,\lambda) \leq LRR(w,\lambda).$$

Limitations in the application of reward-to-risk ratios

- Equation (7) coincides with the objective function of problem **(18)** in *Lecture 8*. We remarked that by varying λ and solving **(18)**, we obtain the efficient frontier.
- Since the solution to the ratio problem (5) is also a portfolio on the efficient frontier, then there exists a particular value of $\lambda = \lambda_{r_b}$ such that using $LRR(w, \lambda_{r_b})$ as the objective function of **(18)**, we obtain the portfolio solving the ratio problem (5).

Limitations in the application of reward-to-risk ratios

- The linearized form $LRR(w, \lambda)$ is capable of describing the efficient frontier without any requirements with respect to $\rho(X)$.
- The risk measure can become equal to zero, or turn negative for a sub-set of the feasible portfolios, without affecting the properties of $LRR(w, \lambda)$.

$\Rightarrow$ Provided that the risk-aversion can be appropriately selected, the linearized form $LRR(w, \lambda)$ can be used as a performance measure.

The STARR

- The performance ratio in which AVaR is selected as a risk measure is called STARR, stable tail-adjusted return ratio.
- The concept behind STARR can be translated to any distributional assumption. Formally, STARR is defined as

$$STARR_\epsilon(w) = \frac{E(r_p - r_b)}{AVaR_\epsilon(r_p - r_b)}. \tag{8}$$

- If r_b is a constant benchmark return, then STARR equals

$$STARR_\epsilon(w) = \frac{w'\mu - r_b}{AVaR_\epsilon(r_p) + r_b}. \tag{9}$$

The STARR

Suppose that our goal is ranking the past performance of several portfolios by STARR using a constant benchmark return.
The available data consist of the observed returns of the portfolios in the past 12 months.

- As a first step, the tail probability of AVaR is chosen. The chosen value of ϵ depends on the extent to which we would like to emphasize the tail risk in the comparison.
- A small value of ϵ, for instance $\epsilon = 0.01$, indicates that we compare the average realized active portfolio return per unit of the extreme average realized losses.
- In contrast, if $\epsilon = 0.5$, then we compare the average realized active portfolio return per unit of the total average realized loss. In this case, we include all realized losses and not just the extreme ones.

The STARR

- Having selected the tail probability, the empirical AVaR for each portfolio can be calculated using, for example, formula **(6)** given in *Lecture 7*.
- The numerator of (9) contains the average realized active return of each portfolio which can be calculated by subtracting the constant benchmark return from the average portfolio return.
- Finally, dividing the observed average outperformance of the benchmark return by the empirical portfolio AVaR, we obtain the ex-post STARR of each portfolio.
- If all empirical AVaRs are positive, then the portfolio with the highest STARR had the best performance in the past 12 months with respect to this performance measure.

The STARR

- If a portfolio has a positive expected return, then it is always possible to find a tail probability at which the portfolio AVaR is negative.
- Alternatively, for a fixed tail probability, the portfolio AVaR can become negative if the expected return of the portfolio is sufficiently high.

The STARR

This can be demonstrated in the following way.

- In *Lecture 6*, we discussed a link between the coherent risk measures and dispersion measures according to which an expectations bounded coherent risk measure can be decomposed into two parts one of which is a measure of dispersion and the other is the mathematical expectation.
- In the case of AVaR, this means that the first term in the decomposition

$$AVaR_\epsilon(r_p) = AVaR_\epsilon(r_p - Er_p) - Er_p$$

  is always non-negative.

$\Rightarrow$ If the expected portfolio return is sufficiently high, then portfolio AVaR can turn negative at any tail probability.

The STARR

- In practice, the empirical AVaR at tail probability $\epsilon \leq 0.5$ is very rarely negative if it is calculated with daily returns. One reason is that the expected portfolio daily return is very close to zero.
- However, negative portfolio AVaRs at tail probability $\epsilon \leq 0.5$ can be observed with monthly returns. Then the portfolios performance cannot be directly compared by ranking with respect to STARRs because a negative AVaR will result in a negative STARR.
- The portfolio with a negative STARR will be among portfolios with very poor performance even though a negative AVaR signifies an exceptional performance.

The STARR

- As a consequence, if there are portfolios with negative empirical AVaRs, then all portfolios should be divided into two groups and a different ordering should be applied to each group.
- The first group contains the portfolios with non-positive AVaRs and the second group contains the portfolios with strictly positive AVaRs.
- We can argue that the portfolios in the first group have a better performance than the portfolios in the second group on the grounds that a negative risk implies that no reserve capital should be allocated.
- Even thought their risk is negative, the portfolios in the first group can be ranked. The smaller the risk is, the more attractive the investment. Thus, *smaller* STARRs indicate better performance.

- Note that STARRs of the portfolios in the first group are necessarily negative because of the inequality,

$$AVaR_{\epsilon}(r_p) \geq -Er_p,$$

valid at any tail probability. This inequality implies

$$0 \leq -AVaR_{\epsilon}(r_p) \leq Er_p$$

meaning that if the portfolio AVaR is negative, then the portfolio expected return is positive.

- If the portfolio AVaR is negative, then the portfolio STARR is negative as well. Thus, smaller STARRs in this case means larger STARRs in absolute value.
- In contrast, the portfolios in the second group should be ranked in the usual way. Larger STARRs imply better performance.

The STARR

- STARR is not defined when AVaR is equal to zero. We noted that this difficulty can be avoided by adopting a linarized form of the ratio.
- According to (7), the linearized STARR is defined as

$$LSTARR(w) = E(r_p) - \lambda AVaR_{\epsilon}(r_p) \tag{10}$$

  where $\lambda \geq 0$ is the risk-aversion parameter.
- The linearized STARR does not have a singularity at AVaR equal to zero and one and the same ordering can be used across all portfolios.

$\Rightarrow$ Higher LSTARR indicates better performance.

The STARR

- In the ex-ante analysis, the problem of finding the portfolio with the best future performance in terms of STARR is

$$\begin{array}{rl} \max\limits_{w} & \dfrac{E(r_p - r_b)}{AVaR_{\epsilon}(r_p - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \tag{11}$$

- According to (6), this problem can be reduced to a simpler optimization problem provided that all feasible portfolios have a positive AVaR of their active return and that there is a feasible portfolio with a positive expected active return.

- Under the additional assumption that the benchmark return is a constant, the simpler optimization problem becomes

$$\begin{array}{rl} \min\limits_{v,t} & AVaR_{\epsilon}(v'X) + tr_b \\ \text{subject to} & v'e = t \\ & v'\mu - tr_b = 1 \\ & v \geq 0, t \geq 0. \end{array} \tag{12}$$

- This optimization problem can be solved by any of the methods discussed in section Mean-risk analysis of *Lecture 8*.

- For example, if there are available scenarios for the assets returns, then AVaR can be linearized and we can formulate a linear programming problem solving (12).
- Combining equation **(15)** in *Lecture 8* with problem (12) we derive the linear programming problem

$$
\begin{array}{rl}
\min\limits_{v,\theta,d,t} & \theta + \dfrac{1}{k\epsilon} d'e + tr_b \\
\text{subject to} & -Hv - \theta e \leq d \\
& v'e = t \\
& v'\mu - tr_b = 1 \\
& w \geq 0,\ d \geq 0,\ t \geq 0,\ \theta \in \mathbb{R},
\end{array}
\tag{13}
$$

The Sortino ratio

- The Sortino ratio is defined as the ratio between the expected active portfolio return and the semi-standard deviation of the underperformance of a fixed target level s.
- If r_b is a constant return target, the ratio is defined as

$$SoR_s(w) = \frac{w'\mu - r_b}{(E(s - r_p)_+^2)^{1/2}} \tag{14}$$

  where the function $(x)_+^2 = (\max(x, 0))^2$.
- The fixed target s is also called the minimum acceptable return level. For example, it can be set to be equal to r_b, $s = r_b$.

The Sortino ratio

- In the ex-post analysis, the Sortino ratio can be calculated as the ratio between the average realized active return and the sample semi-standard deviation,

$$\hat{\sigma}^{-}(s) = \sqrt{\frac{1}{k}\sum_{i=1}^{k}\max(s - r_i, 0)^2},$$

where $r_1, r_2, \ldots, r_k$ is the sample of observed portfolio returns.

- As a result, the empirical Sortino ratio equals

$$\widehat{SoR}_s(w) = \frac{\bar{r} - r_b}{\hat{\sigma}^{-}(s)},$$

where $\bar{r} = \frac{1}{k}\sum_{i=1}^{k} r_i$ is the average realized portfolio return and the "hat" denotes that the formula is an estimator.

The Sortino ratio

- In the ex-ante analysis, the optimal Sortino ratio problem is given by

$$\begin{array}{ll} \max\limits_{w} & \dfrac{w'\mu - r_b}{(E(s-r_p)_+^2)^{1/2}} \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \tag{15}$$

- If there are available scenarios for the assets returns, the simpler problems take the following form[1],

$$\begin{array}{ll} \min\limits_{v,t,d} & d'Id \\ \text{subject to} & tse - Hv \leq d \\ & v'e = t \\ & v'\mu - tr_b = 1 \\ & v \geq 0, t \geq 0, d \geq 0. \end{array} \tag{16}$$

where I denotes the identity matrix and the other notation is consistent with the notation in problem (13).

[1] Under certain technical conditions discussed in the appendix to this lecture.

The Sortino ratio

- We only remark that the matrix H contains the scenarios for the assets returns, e is a vector composed of ones, $e = (1, \ldots, 1)$, and d is a set of additional variables, one for each observation.
- e and d are vectors, the dimension of which equals the number of available observations.
- The simpler problem (16) is a quadratic programming problem because the objective function is a quadratic function of the variables and the constraint set is composed of linear equalities and inequalities.

The Sortino-Satchell ratio

- The Sortino-Satchell ratio is a generalization of the Sortino ratio in which a lower partial moment of order $q \geq 1$ is used as a proxy for risk.
- If r_b is a constant benchmark return, the Sortino-Satchell ratio is defined as

$$SSR_s(w) = \frac{w'\mu - r_b}{(E(s - r_p)_+^q)^{1/q}} \tag{17}$$

where $(x)_+^q = (\max(x, 0))^q$, and q denotes the order of the lower partial moment and the other notation is the same as in the Sortino ratio.

$\Rightarrow$ The Sortino ratio arises from the Sortino-Satchell ratio if $q = 2$.

The Sortino-Satchell ratio

- In the ex-post analysis, the Sortino-Satchell ratio is estimated as the ratio of the sample estimates of the numerator and the denominator,

$$\widehat{SSR}_s(w) = \frac{\bar{r} - r_b}{\hat{\sigma}_q^-(s)},$$

where $\hat{\sigma}_q^-(s)$ denotes the estimate of the denominator,

$$\hat{\sigma}_q^-(s) = \left(\frac{1}{k} \sum_{i=1}^{k} \max(s - r_i, 0)^q \right)^{1/q}. \tag{18}$$

- In the ex-ante analysis, the optimal Sortino-Satchell ratio problem is given by

$$\begin{array}{rl} \max\limits_{w} & \dfrac{w'\mu - r_b}{(E(s - r_p)_+^q)^{1/q}} \\ \text{subject to} & w'e = 1 \\ & w \geq 0. \end{array} \tag{19}$$

which, following the same reasoning as in the Sortino ratio, can be reduced to a simpler form under the same conditions as in the Sortino ratio.

The Sortino-Satchell ratio

- The simpler problem is

$$\begin{array}{rl} \min\limits_{v,t,d} & \sum\limits_{i=1}^{k} d_i^q \\ \text{subject to} & tse - Hv \leq d \\ & v'e = t \\ & v'\mu - tr_b = 1 \\ & v \geq 0, t \geq 0, d \geq 0. \end{array} \tag{20}$$

  where the notation is the same as in the Sortino ratio, and $d = (d_1, \ldots, d_k)$ are the additional variables. Thus, the objective function contains the sum of the additional variables raised to the power q.
- If $q = 1$, then problem (20) is a linear programming problem since the objective function is a linear function of the variables and the constraint set is composed of linear equalities and inequalities.
- If $q = 2$, then (20) is a quadratic programming problem.

A one-sided variability ratio

- Farinelli and Tibiletti (2002) propose a one-sided variability ratio which is based on two partial moments.
- It is different from the Sortino-Satchell ratio because portfolio reward is not measured by the mathematical expectation but by an upper partial moment. The ratio is defined as

$$\Phi_{r_b}^{p,q}(w) = \frac{(E(r_p - r_b)_+^p)^{1/p}}{(E(r_b - r_p)_+^q)^{1/q}}, \tag{21}$$

  where $p \geq 1$, $q \geq 1$ are the orders of the corresponding partial moments and r_b denotes the benchmark return.
- If the portfolio return is above r_b, it is registered as reward and if it is below r_b, it is registered as loss.

A one-sided variability ratio

- In the ex-post analysis, the ratio defined in (21) is computed by replacing the numerator and the denominator by the estimates of the mathematical expectation.
- The estimators can be based on (18).
- Concerning the ex-ante analysis, the optimal $\Phi_{r_b}^{p,q}(w)$ ratio problem does not have nice properties such as the optimal portfolio problems based on STARR or the Sortino-Satchell ratio.
- The reason is that the ratio is a fraction of two convex functions of portfolio weights and, as a result, the optimization problem involving the performance measure given in (21) may have multiple local extrema.

The Rachev ratio

- The Rachev ratio[2] is a performance measure constructed on the basis of AVaR.
- The reward measure is defined as the average of the quantiles of the portfolio return distribution which are above a certain target quantile level. The risk measure is AVaR at a given tail probability.
- Formally, the definition is

$$RaR_{\epsilon_1,\epsilon_2}(w) = \frac{AVaR_{\epsilon_1}(r_b - r_p)}{AVaR_{\epsilon_2}(r_p - r_b)} \tag{22}$$

where the tail probability ϵ_1 defines the quantile level of the reward measure and ϵ_2 is the tail probability of AVaR.

[2] Similar to the performance measure defined in (21) in that it uses a reward measure which is not the mathematical expectation of active portfolio returns.

The Rachev ratio

- Even though AVaR is used in the numerator which is a risk measure, the numerator represents a measure of reward. This is demonstrated by

$$\begin{aligned} AVaR_{\epsilon_1}(X) &= -\frac{1}{\epsilon_1}\int_0^{\epsilon_1} F_X^{-1}(p)dp \\ &= \frac{1}{\epsilon_1}\int_{1-\epsilon_1}^{1} F_{-X}^{-1}(p)dp \end{aligned}$$

where $X = r_b - r_p$ is a r.v. which can be interpreted as benchmark underperformance and $-X$ stands for the active portfolio return.

The Rachev ratio

- The numerator in the Rachev ratio can be interpreted as the average outperformance of the benchmark provided that the ourperformance is larger than the quantile at $1 - \epsilon_1$ probability of the active return distribution.
- Thus, there are two performance levels in the Rachev ratio. The quantile at ϵ_2 probability in the AVaR in the denominator, and the quantile at $1 - \epsilon_1$ probability in the numerator.
- If the active return is below the former, it is counted as loss and if it is above the latter, then it is registered as reward.
- The probability ϵ_2 is often called lower tail probability and ϵ_1 is known as upper tail probability. A possible choice for the lower tail probability is $\epsilon_2 = 0.05$ and for the upper tail probability, $\epsilon_1 = 0.1$.

The Rachev ratio

- In the ex-post analysis, the Rachev ratio is computed by dividing the corresponding two sample AVaRs.
- Since the performance levels in the Rachev ratio are quantiles of the active return distribution, they are relative levels as they adjust according to the distribution.
- For example, if the scale is small, then the two performance levels will be closer to each other. As a consequence, the Rachev ratio is always well-defined.

- In the ex-ante analysis, optimal portfolio problems based on the Rachev ratio are, generally, numerically hard to solve because the Rachev ratio is a fraction of two AVaRs which are convex functions of portfolio weights.
- In effect, the Rachev ratio, if viewed as a function of portfolio weights, may have many local extrema.

Reward-to-variability ratios

- Another general type of performance measures are the reward-to-variability (RV) ratios.
- They are defined as the ratio between the expected active portfolio return and a dispersion measure of the active portfolio return,

$$RV(r_p) = \frac{E(r_p - r_b)}{D(r_p - r_b)}, \tag{23}$$

where

$r_p - r_b$	is the active portfolio return
r_p	denotes the portfolio return
r_b	denotes the return of the benchmark portfolio
$D(r_p)$	is a dispersion measure of the random portfolio return r_p

- The benchmark return r_b can either be a fixed target, or the return of another portfolio, or a reference interest rate.

Reward-to-variability ratios

- In the ex-post analysis, equation (23) is calculated using the available historical returns in a certain period back in time.
- In this case, the numerator is the average of the realized active return and the denominator equals the sample dispersion.
- For example, if $D(X)$ is the standard deviation, then the denominator equals the sample standard deviation of the active return.

- In the ex-ante analysis, the joint distribution of the portfolio return and the benchmark return is hypothesized.
- The parameters of the assumed distribution are estimated from the historical data and the RV ratio is calculated from the fitted distribution.
- In this setting, the portfolio manager is interested in finding a feasible portfolio with highest RV ratio as this portfolio is expected to have the highest return for a unit of variability in its future performance. Formally,

$$\begin{array}{ll} \max\limits_{w} & \dfrac{E(r_p - r_b)}{D(r_p - r_b)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0, \end{array} \tag{24}$$

- On condition that the dispersion measure is a convex function of portfolio weights, the objective function has nice mathematical properties which guarantee that the solution to (3) is unique.

Reward-to-variability ratios

We can consider a simpler version of the optimization problem (24) which arises in the same fashion as the simpler version (6) of the optimal RR ratio problem.

- Since the dispersion measure is non-negative for any r.v. by definition, the only necessary assumption for the RV ratio to be well-defined is that it does not turn into zero for a feasible portfolio.
- This can happen, for example, if the benchmark portfolio itself is a feasible portfolio and can be replicated. In this case, the dispersion measure equals zero because the active portfolio return is zero in all states of the world.

Reward-to-variability ratios

- Suppose the dispersion measure is strictly positive for any feasible portfolio, it satisfies the positive homogeneity property, and there is a feasible portfolio with positive active return.
- Under these assumptions, we can consider the simpler optimization problem

$$\begin{array}{rl} \min\limits_{v,t} & D(v'X - tr_b) \\ \text{subject to} & v'e = t \\ & E(v'X - tr_b) = 1 \\ & v \geq 0, t \geq 0, \end{array} \tag{25}$$

  in which we use the same notation as in problem (6).
- If $(\bar{v}, \bar{t})$ is a solution to (25), then $\bar{v}/\bar{t}$ is a vector of weights solving (24) and, in this sense, the two problems are equivalent.

RV ratios and the efficient portfolios

- Suppose that the risk measure ρ in the M-R analysis is a coherent risk measure satisfying the additional property $\rho(r_p) > -Er_p$.
- In *Lecture 6*, we discussed that in this case the risk measure can be decomposed into

$$\rho(r_p) = D(r_p) - Er_p$$

where $D(r_p) = \rho(r_p - Er_p)$ is a measure of dispersion called a deviation measure.

RV ratios and the efficient portfolios

In *Lecture 8*, we demonstrated that, all optimal portfolios generated by problem **(24)** can be divided into three groups:

- The mean-risk efficient portfolios generated, for example, by problem **(12)**. These efficient portfolios can also be obtained by varying the constant benchmark return in the optimal RR ratio problem (5).
- The middle group contains the mean-deviation efficient portfolios generated by problem **(23)** in which the deviation measure is the dispersion measure underlying the risk measure ρ. They contain the mean-risk efficient portfolios and can be visualized in the mean-deviation plane as in the example in *Figures 10, 11* of *Lecture 8*.

RV ratios and the efficient portfolios

- The mean-deviation efficient portfolios can be obtained from the corresponding optimal RV ratio problem by varying the constant benchmark return.
- Assume that $r_b \neq 0$. Since deviation measures are by definition translation invariant, that is, they satisfy the property

$$D(X + C) = D(X)$$

for any constant C, the optimal RV ratio problem can be formulated as

$$\begin{array}{rl} \max\limits_{w} & \dfrac{w'\mu - r_b}{D(r_p)} \\ \text{subject to} & w'e = 1 \\ & w \geq 0 \end{array} \tag{26}$$

when r_b is a constant benchmark.

RV ratios and the efficient portfolios

- Therefore, the portfolio yielding the maximal RV ratio is positioned on the mean-deviation efficient frontier where a straight line passing through the point with expected return equal to r_b and deviation equal to zero is tangent to it.
- This is illustrated in *Figure 3*. The slope of any straight line passing through the point $(0, r_b)$ on the vertical axis is equal to the RV ratios of the portfolios lying on it. The tangent line has the largest slope among all such straight lines with feasible portfolios lying on them.

RV ratios and the efficient portfolios

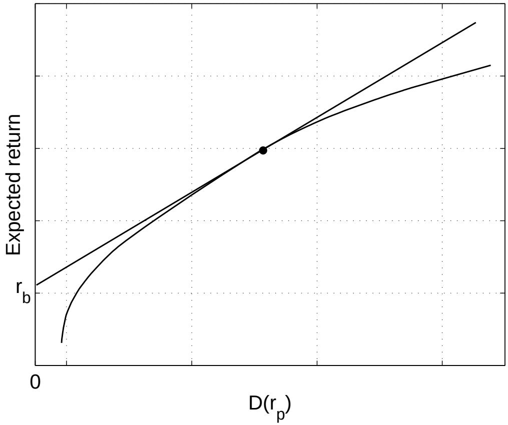

Figure 3. The mean-deviation efficient frontier and the tangent portfolio. Reducing the benchmark return, we obtain a new tangent portfolio without changing the efficient frontier.

RV ratios and the efficient portfolios

- In contrast to the geometric reasoning in the optimal RR ratio problem, changing the benchmark return does not affect the position of the mean-deviation efficient frontier because the deviation measure does not depend on it.
- By increasing or decreasing continuously r_b, we only change the position of the reference point on the vertical axis through which the straight line passes.
- For instance, decreasing the benchmark return to $r_b^1 < r_b$, we obtain a new tangent line and a new tangent portfolio shown on the *Figure 4*.

RV ratios and the efficient portfolios

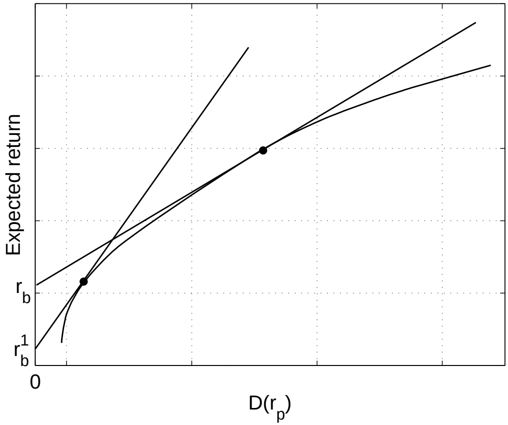

Figure 4. The mean-deviation efficient frontier and the tangent portfolio. Reducing the benchmark return, we obtain a new tangent portfolio without changing the efficient frontier.

RV ratios and the efficient portfolios

- The geometric intuition suggests that decreasing further r_b, we obtain portfolios closer and closer to the global minimum deviation portfolio.
- As a result, with the only exception of the global minimum deviation portfolio, any mean-deviation efficient portfolio can be obtained as a solution to the optimal RV ratio problem when the benchmark return varies $r_b \in (-\infty, r_b^{max}]$ in which r_b^{max} denotes the expected return of the global maximum expected return portfolio.

RV ratios and the efficient portfolios

Since the mean-risk efficient portfolios are only a part of the mean-deviation efficient portfolios, then, as a corollary of the geometric reasoning, we obtain the following relationship between the optimal RR ratio and RV ratio problems.

- The solution to problem (5) coincides with the solution to problem (26) on condition that $r_b \in [-\rho^{min}, r_b^{max}]$ where $\rho^{min} > 0$ denotes the risk of the global minimum risk portfolio.
- The condition $\rho^{min} > 0$ guarantees that the risk of all feasible portfolios is strictly positive and, therefore, the RR ratio is bounded.
- If $r_b < -\rho^{min}$, then the optimal RV ratio portfolio does not belong to the mean-risk efficient frontier but belongs to the mean-deviation efficient frontier.

The Sharpe ratio

- The celebrated Sharpe ratio arises as a RV ratio in which the dispersion measure is represented by the standard deviation, $D(r_p - r_b) = \sigma(r_p - r_b)$. Formally, it is defined as

$$IR(w) = \frac{E(r_p - r_b)}{\sigma(r_p - r_b)} \tag{27}$$

  when the benchmark return is a r.v.

- In this case, the Sharpe ratio equals the mean active return divided by the tracking error and is also known as the information ratio (IR).
- If the benchmark return is a constant, then the Sharpe ratio equals

$$SR(w) = \frac{w'\mu - r_b}{\sigma_{r_p}}. \tag{28}$$

The Sharpe ratio

- The Sharpe ratio was introduced by Sharpe (1966) as a way to compute the performance of mutual funds.
- We provide an example illustrating how the Sharpe ratio is applied in the ex-post analysis.
- Table below contains observed monthly returns of a portfolio.

	Jan	Feb	Mar	Apr
Realized return (%)	1.2	-0.1	1.4	0.3

Table: Realized monthly returns of a hypothetical portfolio.

The Sharpe ratio

- Assume that the monthly target return is a constant and equals 0.5%. In order to compute the Sharpe ratio, we have to calculate the average realized monthly active return and divide it by the sample standard deviation of the portfolio return.
- The average active return can be calculated by subtracting the target return of 0.5% from the average portfolio return,

$$\frac{1}{4}(1.2 - 0.1 + 1.4 + 0.3) - 0.5 = 0.2.$$

- The sample standard deviation $\hat{\sigma}$ is calculated according to the formula

$$\hat{\sigma} = \sqrt{\frac{1}{k-1}\sum_{i=1}^{k}(r_i - \bar{r})^2} = \sqrt{\frac{1}{k-1}\sum_{i=1}^{k} r_i^2 - \frac{k}{k-1}\bar{r}^2} \qquad (29)$$

  where $r_1, r_2, \ldots, r_k$ denote the observed portfolio returns and $\bar{r}$ stands for the average portfolio return.
- In statistics, $\hat{\sigma}$ is called an unbiased estimator of the standard deviation.

The Sharpe ratio

- Making use of equation (29) for $\hat{\sigma}$, we calculate

$$\hat{\sigma} = \sqrt{\frac{1}{3}\left(1.2^2 + 0.1^2 + 1.4^2 + 0.3^2\right) - \frac{4}{3}0.7^2} = 0.716$$

  where the $\bar{r} = 0.7$ is the average monthly return.

- Finally, the ex-post Sharpe ratio of the portfolio equals

$$\hat{SR} = \frac{\bar{r} - 0.5}{\hat{\sigma}} = \frac{0.2}{0.716} = 0.2631.$$

- In the ex-ante analysis, the portfolio manager is looking for the portfolio with the best future performance in terms of the Sharpe ratio. The corresponding optimization problem is

$$\begin{aligned} \max_{w} \quad & \frac{w'\mu - r_b}{\sigma_{r_p}} \\ \text{subject to} \quad & w'e = 1 \\ & w \geq 0, \end{aligned} \tag{30}$$

- (30) according to the general reasoning behind optimal RV ratio problems, can be reduced to the following simpler problem

$$\begin{array}{rl} \min\limits_{v,t} & \sigma(v'X) \\ \text{subject to} & v'e = t \\ & v'\mu - tr_b = 1 \\ & v \geq 0, t \geq 0, \end{array} \tag{31}$$

where the objective function $\sigma(v'X)$ is the standard deviation of the portfolio with scaled weights v and t is an additional variable.

The Sharpe ratio

- In *Lecture 8*, we remarked that it makes no difference whether the standard deviation or the variance of portfolio returns is minimized as far as the optimal solution is concerned.
- This holds because variance is a non-decreasing function of standard deviation and, therefore, the portfolio yielding the minimal standard deviation subject to the constraints also yields the minimal variance.
- Problem (31) can be formulated in terms of minimizing portfolio variance,

$$\begin{array}{rl} \min\limits_{v,t} & v'\Sigma v \\ \text{subject to} & v'e = t \\ & v'\mu - tr_b = 1 \\ & v \geq 0, t \geq 0, \end{array} \tag{32}$$

where Σ is the covariance matrix of the portfolio assets returns.

- The optimization problem (32) is a quadratic programming problem because the objective function is a quadratic function of the scaled portfolio weights and all functions in the constraint set are linear.
- As far as the structure of the optimization problem is concerned, (32) is not more difficult to solve than the traditional quadratic mean-variance problem.
- In fact, the only difference between the two is the additional variable t in (32) but this does not increase significantly the computational complexity.

The capital market line and the Sharpe ratio

- In *Lecture 8*, we discussed the mean-variance analysis when there is a risk-free asset added to the investment universe. In this case, the mean-variance efficient frontier is a straight line in the mean-standard deviation plane, which is called the capital market line.
- The mean-variance efficient portfolios are a combination of the risk-free asset and a portfolio composed of the risky assets known as the market portfolio.
- This is a fundamental result on the structure of the mean-variance efficient portfolios known as the two-fund separation theorem, which is also at the heart of the Capital Asset Pricing Model (CAPM).
- We'll demonstrate that the market portfolio is the portfolio yielding the maximal Sharpe ratio in the universe of the risky assets with the benchmark return equal to the risk-free return, and we provide an interpretation of the optimal value of the additional variable t in problem (32).

The capital market line and the Sharpe ratio

- Consider problem **(9)** of *Lecture 8*, which represents the optimal portfolio problem behind the mean-variance analysis with a risk-free asset.
- In order to make a parallel with (32), we restate problem **(9)** but with an equality constraint on the expected return rather than an inequality constraint,

$$\begin{array}{rl} \min\limits_{\omega,\omega_f} & \omega'\Sigma\omega \\ \text{subject to} & \omega'e + \omega_f = 1 \\ & \omega'\mu + \omega_f r_f = R_* \\ & \omega \geq 0,\ \omega_f \leq 1, \end{array} \tag{33}$$

where

ω_f	stands for the weight of the risk-free asset r_f
ω	denotes the weights of the risky assets
R_*	denotes the bound on the expected portfolio return
Σ	stands for the covariance matrix between the risky assets

- Changing the inequality constraint to equality does not change the optimal solution if the target expected return is not below the risk-free rate, $R_* \geq r_f$.
- Conversely, if the target expected return is below the risk-free rate, then (33) is an infeasible problem.
- Our assumption is $R_* > r_f$ because in the case of equality, the optimal portfolio consists of the risk-free asset only.

The capital market line and the Sharpe ratio

- The weight ω_f of the risk-free asset in the portfolio can be a positive or a negative number.
- If ω_f is negative, this means that borrowing at the risk-free rate is allowed and the borrowed money is invested in the market portfolio.
- In this case, it is said that we have a leveraged portfolio.
- Leveraged portfolios are positioned on the capital market line, illustrated in *Figure 3* in *Lecture 8*, to the right of the tangency portfolio. The efficient portfolios to the left of the tangency portfolio have a positive weight for the risk-free asset.

We can express the weight of the risky assets in the whole portfolio by means of the weight of the risk-free asset.

- The weight of the risky assets equals $1 - \omega_f$ which is a consequence of the requirement that all weights should sum up to 1. We introduce a new variable in (33) computing the weight of the risky assets, $s = 1 - \omega_f$, which we substitute for ω_f. Thus, problem (33) becomes

$$\begin{array}{rl} \min\limits_{\omega,s} & \omega'\Sigma\omega \\ \text{subject to} & \omega' e = s \\ & \omega'\mu - s r_f = R_* - r_f \\ & \omega \geq 0, \ s \geq 0. \end{array} \tag{34}$$

The capital market line and the Sharpe ratio

- There are many similarities between the optimal Sharpe ratio problem (32) and (34). In fact, if $r_b = r_f$, then the only difference is in the expected return constraint. Not only do these two problems look similar but their solutions are also tightly connected.
- Denote by $(\bar{\omega}, \bar{s})$ the optimal solution to (34). Since by assumption $R_* - r_f > 0$, it follows that

$$\bar{v} = \frac{\bar{\omega}}{(R_* - r_f)} \quad \text{and} \quad \bar{t} = \frac{\bar{s}}{(R_* - r_f)} \tag{35}$$

represent the optimal solution to problem (32).

The capital market line and the Sharpe ratio

- According to the analysis made for the generic optimal RV ratio problem (25), we obtain that the weights $\bar{w}$ of the portfolio yielding the maximal Sharpe ratio are computed by

$$\bar{w} = \bar{v}/\bar{t} = \bar{\omega}/\bar{s} = \bar{\omega}/(1 - \bar{\omega}_f). \tag{36}$$

- On the other hand, if $(\bar{\omega}, \bar{s}) = (\bar{\omega}, 1 - \bar{\omega}_f)$ is the optimal solution to (33), then the weights of the market portfolio w_M are calculated by

$$w_M = \bar{\omega}/(1 - \bar{\omega}_f).$$

- As a result, the market portfolio is a portfolio solving the optimal Sharpe ratio problem (30) with $r_b = r_f$.

The capital market line and the Sharpe ratio

- From a geometric viewpoint, the link between the two problems becomes apparent by comparing the plot in *Figure 3* and *Figure 3* in *Lecture 8*.
- The tangent line on the first plot coincides with the capital market line on the second if the benchmark return is equal to the risk-free rate.
- Finally, formula (35) provides a way of interpreting the optimal value $\bar{t}$ of the additional variable t used to simplify the optimal Sharpe ratio problem in (32).
- The optimal value $\bar{t}$ equals the weight of the risky assets in an efficient portfolio obtained with a risk-free rate $r_b = r_f$ and a limit on the expected return R_*, divided by the positive difference $R_* - r_f$.
- Note that this ratio remains one and the same irrespective of the value of the limit on the expected return R_* and, therefore, is a characteristic of the efficient portfolios.

The capital market line and the Sharpe ratio

- In the appendix to *Lecture 8*, we gave a closed-form expression of the solution to a type of mean-variance optimization problems.
- It is possible to derive a closed-form solution to the mean-variance problem with a risk-free asset (33) and also to the optimal Sharpe ratio problem (30) through the simplified problem (32) by removing the inequality constraints on the weights of the assets. Thus, the optimal solution to problem

$$\begin{array}{rl} \min\limits_{\omega,\omega_f} & \omega'\Sigma\omega \\ \text{subject to} & \omega' e + \omega_f = 1 \\ & \omega'\mu + \omega_f r_f = R_* \end{array}$$

is given by

$$\begin{aligned} \bar{\omega} &= \frac{R_* - r_f}{(\mu - r_f e)'\Sigma^{-1}(\mu - r_f e)}\Sigma^{-1}(\mu - r_f e) \\ \bar{\omega}_f &= 1 - \frac{R_* - r_f}{(\mu - r_f e)'\Sigma^{-1}(\mu - r_f e)}(\mu - r_f e)'\Sigma^{-1} e \end{aligned} \qquad (37)$$

where Σ^{-1} denotes the inverse of the covariance matrix Σ.

The capital market line and the Sharpe ratio

- In a similar way, the optimal solution to

$$\begin{array}{rl} \max\limits_{w} & \dfrac{w'\mu - r_b}{\sigma_{r_p}} \\ \text{subject to} & w'e = 1 \end{array}$$

  is given by

$$\bar{w} = \frac{\Sigma^{-1}(\mu - r_b e)}{(\mu - r_b e)'\Sigma^{-1} e} \tag{38}$$

- In this simple case, the relationship in formula (36) between the solution to the optimal Sharpe ratio problem and the mean-variance problem with a risk-free asset is straightforward to check using formula (37) and formula (38).

Summary

- We discussed performance measures from the point of view of the ex-post and ex-ante analysis.
- We distinguish between reward-to-risk and reward-to-variability ratios depending on whether a risk measure or a dispersion measure is adopted in the denominator of the ratio.
- The appendix to this chapter considers a general approach to classifying performance measures in a structural way. We describe the general optimal quasi-concave ratio problem and the arising simpler optimization problems on condition that certain technical properties are met.
- Finally, we give an account of non-quasi-concave ratios and demonstrate that the two-fund separation theorem holds for the general reward-risk analysis when a risk-free asset is added to the investment universe.

Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi
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Chapter 10.