Technical Appendix

Lecture 4: Ideal probability metrics

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The CLT conditions

- The two sets of conditions mentioned in the lecture are sufficient conditions. That is, if any of them holds, then the CLT is valid.
- In the literature, usually the condition of Lindeberg-Feller is given as a general sufficient condition for the CLT. However, the Lindeberg-Feller condition is equivalent to the asymptotic negligibility condition.
- We explained that asymptotic negligibility holds if the summands become negligible with respect to the total sum as their number increases. That is, none of the summands dominates and dictates the behavior of the total sum. We give a more precise formulation of this statement below.

The asymptotic negligibility condition

- Consider a sequence of independent random variables $X_1, X_2, \ldots, X_n, \ldots$ and denote by S_n the sum

$$S_n = X_1 + \ldots + X_n.$$

- We do not assume that the distribution of the random variables is the same, meaning that the means and the variances of the random variables may differ. Denote by μ_n the mean of the sum S_n and by σ_n^2 the variance of the sum,

$$ES_n = EX_1 + \ldots + EX_n = \mu_n$$
$$DS_n = DX_1 + \ldots + DX_n = \sigma_n^2.$$

- The asymptotic negligibility condition holds if

$$\max_{1 \leq j \leq n} P\left(\frac{|X_j - \mu_n|}{\sigma_n} > \delta\right) \longrightarrow 0, \quad n \to \infty \text{ for each } \delta > 0. \qquad (1)$$

The asymptotic negligibility condition

The asymptotic relation (1) can be interpreted in the following way.

- The standard deviation σ_n describes the variability of the total sum. The ratio $|X_j - \mu_n|/\sigma_n$ compares each of the terms in S_n to the variability of the total sum and, thus, the probability in (1) measures the variability of each summand relative to the variability of the sum.
- Therefore, the asymptotic negligibility condition states that as the number of summands increases indefinitely, the most variable term in S_n is responsible for a negligible amount of the variability of the total sum.

The asymptotic negligibility condition

- In the Generalized CLT, the condition (1) does not hold. In fact, the Lévy stable distributions, which are the limit distributions in the Generalized CLT, satisfy a property which is converse to the asymptotic negligibility condition.
- It states that the large deviations of a sum of i.i.d. Lévy stable random variables are due to, basically, one summand,

$$P(Y_1 + \ldots + Y_n > x) \sim P\left(\max_{1 \le k \le n} Y_k > x\right)$$

  where $Y_1, \ldots, Y_n$ are i.i.d. Lévy stable random variables.
- That is, the probability that the sum is large is approximately equal to the probability that one of the summands is large. This fact is a manifestation of the fundamental difference between the Lévy stable distributions and the normal distribution.

The necessary and sufficient condition

While the asymptotic negligibility condition is very general, it is not a necessary and sufficient condition. The CLT may hold even if it is violated. Next, we formulate the necessary and sufficient condition.

- Denote by $\widetilde{S}_n$ the centered and normalized sum,

$$\widetilde{S}_n = Y_1 + \ldots + Y_n = \frac{S_n - \mu_n}{\sigma_n}$$

  where the summands $Y_j = (X_j - EX_j)/\sigma_n$.
- The CLT holds for the centered and normalized sum, $\widetilde{S}_n \xrightarrow{d} Z \in N(0,1)$, if and only if for every $\epsilon > 0$,

$$\sum_{j=1}^{n} \int_{|x|>\epsilon} |F_{Y_j}(x) - F_{Z_j}(x)||x|dx \longrightarrow 0, \qquad n \to \infty \tag{2}$$

  where Z_j has a normal distribution with variance equal to the variance of Y_j,

$$Z_j = \sigma_{Y_j} Z, \quad Z_j \in N\left(0, \sigma^2_{Y_j}\right).$$

The necessary and sufficient condition

- Thus, the absolute difference $|F_{Y_j}(x) - F_{Z_j}(x)|$ is between two distribution functions of random variables with equal scales. The expression in (2) sums up the deviations between the c.d.f.s of the summands Y_j and the scaled normal distributions $F_{Z_j}(x)$.
- The necessary and sufficient condition (2) has a more simple form if the random variables $X_1, \ldots, X_n, \ldots$ have equal distribution. Under this assumption, their means and variances are the same, $EX_j = \mu$ and $DX_j = \sigma^2$. Then the sum in (2) disappears and we obtain that for every $\epsilon > 0$,

$$\int_{|x|>\epsilon\sqrt{n}} |F_{\widetilde{X}_1}(x) - F_Z(x)||x|dx \to 0, \qquad n \to \infty \tag{3}$$

in which $\widetilde{X}_1 = (X_1 - \mu)/\sigma$. Note that as n increases, it is only the integration range that changes in (3).

Remarks on ideal metrics

Here we briefly mention a few general conditions which need to be satisfied in order for the ideal metrics considered to be finite.

- Suppose that the probability metric $\mu(X, Y)$ is a simple ideal metric of order r. The finiteness of $\mu(X, Y)$ guarantees equality of all moments up to order r,

$$\mu(X, Y) < \infty \qquad \Longrightarrow \qquad E(X^k - Y^k) = 0, \quad k = 1, 2, \dots, n < r.$$

- Conversely, if all moments $k = 1, 2, \dots, n < r$ agree and, in addition to this, the absolute moments of order r are finite, then metric $\mu(X, Y)$ is finite,

$$\left.\begin{array}{l} EX^k = EY^k \\ E|X|^r < \infty \\ E|Y|^r < \infty \end{array}\right. \qquad \Longrightarrow \qquad \mu(X, Y) < \infty$$

where $k = 1, 2, \dots, n < r$.

Remarks on ideal metrics

- The conditions which guarantee finiteness of the ideal metric μ are very important when investigating the problem of convergence in distribution of random variables in the context of the metric μ.
- Consider a sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ and a random variable X which satisfy the conditions,

  $$EX_n^k = EX^k, \ \forall n, \ k = 1, 2, \ldots, n < r$$

  and

  $$E|X|^r < \infty, \ E|X_n|^r < \infty, \ \forall n.$$

- For all known ideal metrics $\mu(X, Y)$ of order $r > 0$, given the above moment assumptions, the following holds: $\mu(X_n, X) \to 0$ if and only if X_n converges to X in distribution and the absolute moment of order r converge,

$$\mu(X_n, X) \to 0 \quad \text{if and only if} \quad X_n \xrightarrow{d} X \quad \text{and} \quad E|X_n|^r \to E|X^r|.$$

Remarks on ideal metrics

This abstract result has the following interpretation.

- Suppose that X and Y describe the returns of two portfolios. Choose an ideal metric μ of order $3 < r < 4$, for example. The convergence result above means that if $\mu(X, Y) \approx 0$, then both portfolios have very similar distribution functions and also they have very similar means, volatilities and skewness.
- Note that, generally, the c.d.f.s of two portfolios being "close" to each other does not necessarily mean that their moments will be approximately the same.
- It is of crucial importance which metric is chosen to measure the distance between the distribution functions. The ideal metrics have this nice property that they guarantee convergence of certain moments. Rachev(1991) provides an extensive review of the properties of ideal metrics and their application.

Remarks on ideal metrics

1. The Zolotarev ideal metric
   Only a special case of the Zolotarev ideal metric was given in the lecture. The general form of the Zolotarev ideal metric is

$$\zeta_s(X,Y) = \int_{-\infty}^{\infty} \left| F_{s,X}(x) - F_{s,Y}(x) \right| dx \tag{4}$$

   where $s = 1, 2, \ldots$ and

$$F_{s,X}(x) = \int_{-\infty}^{x} \frac{(x-t)^{s-1}}{(s-1)!} dF_X(t) \tag{5}$$

- The Zolotarev metric $\zeta_s(X,Y)$ is ideal of order $r = s$. Zolotarev(1997) provides more information.

Remarks on ideal metrics

2. The Rachev metric

The general form of the Rachev metric is

$$\zeta_{s,p,\alpha}(X,Y) = \left(\int_{-\infty}^{\infty} |F_{s,X}(x) - F_{s,Y}(x)|^p \, |x|^{\alpha p'} dx\right)^{1/p'} \qquad (6)$$

where $p' = \max(1,p)$, $\alpha \geq 0$, $p \in [0,\infty]$, and $F_{s,X}(x)$ is defined in equation (5). If $\alpha = 0$, then the Rachev metric $\zeta_{s,p,0}(X,Y)$ is ideal of order $r = (s-1)p/p' + 1/p'$.

- Note that $\zeta_{s,p,\alpha}(X,Y)$ can be represented in terms of lower partial moments,

$$\zeta_{s,p,\alpha}(X,Y) = \frac{1}{(s-1)!}\left(\int_{-\infty}^{\infty} \left|E(t-X)_+^s - E(t-X)_+^s\right|^p |t|^{\alpha p'} dt\right)^{1/p'}.$$

- The metric defined in equation (14) in the lecture arises from the metric in (6) when $\alpha = 0$,

$$\zeta_{s,p}(X,Y) = \zeta_{s,p,0}(X,Y).$$

3. The Kolmogorov-Rachev metrics

   The Kolmogorov-Rachev metrics arise from other ideal metrics by a process known as *smoothing*. Suppose the metric μ is ideal of order $0 \leq r \leq 1$.

- Consider the metric defined as

$$\mu_s(X, Y) = \sup_{h \in \mathbb{R}} |h|^s \mu(X + hZ, X + hZ) \tag{7}$$

  where Z is independent of X and Y and is a symmetric random variable $Z \overset{d}{=} -Z$.

- The metric $\mu_s(X, Y)$ defined in this way is ideal of order $r = s$. Note that while (7) always defines an ideal metric of order s, this does not mean that the metric is finite. The finiteness of μ_s should be studied for every choice of the metric μ.

Remarks on ideal metrics

- For example, suppose that $\mu(X, X)$ is the total variation metric $\boldsymbol{\sigma}(X, Y)$ defined in (17) in Lecture 3 "Probability metrics" and Z has the standard normal distribution, $Z \in N(0, 1)$. We calculate that

$$\begin{aligned} \boldsymbol{\sigma}_s(X, Y) &= \sup_{h \in \mathbb{R}} |h|^s \boldsymbol{\sigma}(X + hZ, X + hZ) \\ &= \sup_{h \in \mathbb{R}} |h|^s \frac{1}{2} \int_{\mathbb{R}} |f_X(x) - f_Y(x)| \frac{f_Z(x/h)}{h} dx \\ &= \sup_{h \in \mathbb{R}} |h|^s \frac{1}{2} \int_{\mathbb{R}} |f_X(x) - f_Y(x)| \frac{1}{\sqrt{2\pi h^2}} e^{-\frac{x^2}{2h^2}} dx \end{aligned} \tag{8}$$

  in which we use the explicit form of the standard normal density, $f_Z(u) = \exp(-u^2/2)/\sqrt{2\pi}$, $u \in \mathbb{R}$.

- Note that the absolute difference between the two densities of X and Y in (8) is averaged with respect to the standard normal density. This is why the Kolmogorov-Rachev metrics are also called smoothing metrics.

Remarks on ideal metrics

- The Kolmogorov-Rachev metrics are applied in estimating the convergence rate in the Generalized CLT and other limit theorems.
- *Rachev* and *Rüschendorf* (1998) and *Rachev* (1991) provide more background and further details on the application in limit theorems.

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Chapter 4.