Technical Appendix
Lecture 5: Choice under uncertainty

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The material is based on the text-book:
**Svetlozar T. Rachev, Stoyan Stoyanov, and Frank J. Fabozzi**
*Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures*

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The axioms of choice are fundamental assumptions defining a preference order.

- $\mathcal{X}$ stands for the set of the probability distributions of the ventures also known as lotteries, and the notation $P_X \succeq P_Y$ means that the economic agent prefers $P_X$ to $P_Y$ or is indifferent between them.

- The notation $P_X \succ P_Y$ means that $P_X$ is strictly preferred to $P_Y$. 
The axioms of choice

The axioms of choice are the following:

Completeness
For all $P_X, P_Y \in \mathcal{X}$, either $P_X \succeq P_Y$ or $P_Y \succeq P_X$ or both are true, $P_X \sim P_Y$.

Transitivity
If $P_X \succeq P_Y$ and $P_Y \succeq P_Z$, then $P_X \succeq P_Z$, where $P_X, P_Y$ and $P_Z$ are three lotteries.

Archimedean Axiom
If $P_X, P_Y, P_Z \in \mathcal{X}$ are such that $P_X \succ P_Y \succ P_Z$, then there is an $\alpha, \beta \in (0, 1)$ such that $\alpha P_X + (1 - \alpha)P_Z \succ P_Y$ and also $P_Y \succ \beta P_X + (1 - \beta)P_Z$.

Independence Axiom
For all $P_X, P_Y, P_Z \in \mathcal{X}$ and any $\alpha \in [0, 1]$, $P_X \succeq P_Y$ if and only if $\alpha P_X + (1 - \alpha)P_Z \succeq \alpha P_Y + (1 - \alpha)P_Z$. 
The axioms of choice

- The *completeness* axiom states that economic agents should always be able to compare two lotteries, e.g. two portfolios. They either prefer one or the other, or are indifferent.

- The *transitivity* axiom rules out the possibility that an investor may prefer $P_X$ to $P_Y$, $P_Y$ to $P_Z$, and also $P_Z$ to $P_X$. It states that if the first two relations hold, then necessarily the investor should prefer $P_X$ to $P_Z$.

- The *Archimedean* axiom is like a “continuity” condition. It states that given any three distributions strictly preferred to each other, we can combine the most and the least preferred distribution through an $\alpha \in (0, 1)$ such that the resulting distribution is strictly preferred to the middle distribution. Likewise, we can combine the most and the least preferred distribution through a $\beta \in (0, 1)$ so that the middle distribution is strictly preferred to the resulting distribution.

- The *independence* axiom claims that the preference between two lotteries remains unaffected if they are both combined in the same way with a third lottery.
The axioms of choice

The basic result of von Neumann-Morgenstern is that a preference relation satisfies the four axioms of choice if and only if there is a real-valued function, $U : \mathcal{X} \to \mathbb{R}$, such that:

a) $U$ represents the preference order,

$$P_X \succeq P_Y \iff U(P_X) \geq U(P_Y)$$

for all $P_X, P_Y \in \mathcal{X}$.

b) $U$ has the linear property,$^1$

$$U(\alpha P_X + (1 - \alpha) P_Y) = \alpha U(P_X) + (1 - \alpha) U(P_Y)$$

for any $\alpha \in (0, 1)$ and $P_X, P_Y \in \mathcal{X}$.

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$^1$Functions satisfying this property are also called affine.
Moreover, the numerical representation $U$ is unique up to a positive linear transform. That is, if $U_1$ and $U_2$ are two functions representing one and the same preference order, then $U_2 = aU_1 + b$ where $a > 0$ and $b$ are some coefficients.

It turns out that the numerical representation has a very special form under some additional technical continuity conditions:

$$U(P_X) = \int \limits_{\mathbb{R}} u(x) dF_X(x)$$

where the function $u(x)$ is the utility function of the economic agent and $F_X(x)$ is the c.d.f. of the probability distribution $P_X$.

Thus, the numerical representation of the preference order of an economic agent is the expected utility of $X$.

The fact that $U$ is known up to a positive linear transform means that the utility function of the economic agent is not determined uniquely from the preference order but is also unique up to a positive linear transform.
Stochastic dominance relations of order \( n \)

- Including additional characteristics of the investors by imposing conditions on the utility function, we end up with more refined stochastic orders. This method can be generalized in the \( n \)-th order stochastic dominance.

- Denote by \( U_n \) the set of all utility functions, the derivatives of which satisfy the inequalities
  \[
  (-1)^{k+1} u^{(k)}(x) \geq 0, \quad k = 1, 2, \ldots, n
  \]
  where \( u^{(k)}(x) \) denotes the \( k \)-th derivative of \( u(x) \).

- For each \( n \), we have a set of utility functions which is a subset of \( U_{n-1} \),
  \[
  U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots
  \]

  The classes of investors characterized by the first-, second-, and third-order stochastic dominance are \( U_1, U_2, \) and \( U_3 \).
Stochastic dominance relations of order $n$

- Imposing further properties on the derivatives of the utility function requires that we make more assumptions for the moments of the random variables we consider.

- We assume that the absolute moments $E|X|^k$ and $E|Y|^k$, $k = 1, \ldots, n$ of the random variables $X$ and $Y$ are finite.

- We say that the portfolio $X$ dominates the portfolio $Y$ in the sense of the $n$-th order stochastic dominance, $X \succeq_n Y$, if no investor with a utility function in the set $\mathcal{U}_n$ would prefer $Y$ to $X$,

$$X \succeq_n Y \text{ if } Eu(X) \geq Eu(Y), \forall u(x) \in \mathcal{U}_n.$$

- Thus, the first-, second-, and third-order stochastic dominance appear as special cases from the $n$-th order stochastic dominance with $n = 1, 2, 3$. 
Stochastic dominance relations of order $n$

There is an equivalent way of describing the $n$-th order stochastic dominance in terms of the c.d.f.s of the ventures only.

- The condition is the following one,

$$X \succeq_n Y \iff F_X^{(n)}(x) \leq F_Y^{(n)}(x), \forall x \in \mathbb{R} \quad (1)$$

where $F_X^{(n)}(x)$ stands for the $n$-th integral of the c.d.f. of $X$ which can be defined recursively as

$$F_X^{(n)}(x) = \int_{-\infty}^{x} F_X^{(n-1)}(t)dt.$$

- An equivalent form of the condition in (1) can be derived, which is close to the form of TSD condition (8) in the lecture,

$$X \succeq_n Y \iff E(t - X)_{+}^{n-1} \leq E(t - Y)_{+}^{n-1}, \forall t \in \mathbb{R} \quad (2)$$

where $$(t - x)_{+}^{n-1} = \max(t - x, 0)^{n-1}$$. 
Since in the $n$-th order stochastic dominance we furnish the conditions on the utility function as $n$ increases, the following relation holds,

$$X \succeq_1 Y \implies X \succeq_2 Y \implies \ldots \implies X \succeq_n Y,$$

which generalizes the relationship between FSD, SSD, and TSD.

It is possible to extend the $n$-th order stochastic dominance to the $\alpha$-order stochastic dominance in which $\alpha \geq 1$ is a real number and instead of the ordinary integrals of the c.d.f.s, fractional integrals are involved $^2$.

$^2$See Ortobelli et al. (2007) for details
The lotteries in von Neumann-Morgenstern theory are usually interpreted as probability distributions of payoffs. That is, the domain of the utility function $u(x)$ is the positive half-line which is interpreted as the collection of all possible outcomes in terms of dollars from a given venture.

Assume that the payoff distribution is actually the price distribution $P_t$ of a financial asset at a future time $t$. In line with the von Neumann-Morgenstern theory, the expected utility of $P_t$ for an investor with utility function $u(x)$ is given by

$$U(P_t) = \int_0^\infty u(x)dF_{P_t}(x)$$

where $F_{P_t}(x) = P(P_t \leq x)$ is the c.d.f. of the random variable $P_t$. 
Further on, suppose that the price of the common stock at the present time is $P_0$. Consider the substitution $x = P_0 \exp(y)$. Under the new variable, the c.d.f. of $P_t$ changes to

$$F_{P_t}(P_0 \exp(y)) = P(P_t \leq P_0 \exp(y)) = P\left(\log\frac{P_t}{P_0} \leq y\right)$$

which is, in fact, the distribution function of the log-return of the financial asset $r_t = \log(P_t/P_0)$.

The integration range changes from the positive half-line to the entire real line and equation (3) becomes

$$U(P_t) = \int_{-\infty}^{\infty} u(P_0 \exp(y)) dF_{r_t}(y).$$

(4)
On the other hand, the expected utility of the log-return distribution has the form

$$U(r_t) = \int_{-\infty}^{\infty} v(y) dF_{r_t}(y) \tag{5}$$

where $v(y)$ is the utility function of the investor on the space of log-returns which is unique up to a positive linear transform.

Note that $v(y)$ is defined on the entire real line as the log-return can be any real number.
Return versus payoff and stochastic dominance

- Compare equations (4) and (5). From the uniqueness of the expected utility representation, it appears that (4) is the expected utility of the log-return distribution. Therefore, the utility function \( v(y) \) can be computed by means of the utility function \( u \),

\[
v(y) = a.u(P_0 \exp(y)) + b, \quad a > 0
\]  

(6)

in which the constants \( a \) and \( b \) appear because of the uniqueness result.

- Conversely, the utility function \( u(x) \) can be expressed via \( v \),

\[
u(x) = c.v(\log(x/P_0)) + d, \quad c > 0.
\]  

(7)

- Note that the two utilities in equations (4) and (5) are identical (up to a positive linear transform), because the investor is the same. We only change the way we look at the venture.
Because of the relationship between the functions $u$ and $v$, properties imposed on the utility function $u$ may not transfer to the function $v$ and vice versa.

We remark on what happens with the properties connected with the $n$-th order stochastic dominance.

Suppose that the utility function $v(y)$ belongs to the set $\mathcal{U}_n$, i.e. it satisfies the conditions

$$(-1)^{k+1} v^{(k)}(y) \geq 0, \quad k = 1, 2, \ldots, n$$

where $v^{(k)}(y)$ denotes the $k$-th derivative of $v(y)$.

It turns out that the function $u(x)$ given by (7) satisfies the same properties and, therefore, it also belongs to the set $\mathcal{U}_n$. This is verified directly by differentiation.
In the reverse direction, the statement holds only for \( n = 1 \). If \( u \in \mathcal{U}_n, n > 1 \), then the function \( v \) given in (6) may not belong to \( \mathcal{U}_n, n > 1 \), and we obtain a set of functions to which \( \mathcal{U}_n \) is a subset.

The \( n \)-th degree stochastic dominance, \( n > 1 \), on the space of payoffs implies the \( n \)-th degree stochastic dominance, \( n > 1 \), on the space of the corresponding log-returns but not vice versa,

\[
P_t^1 \succeq_n P_t^2 \quad \implies \quad r_t^1 \succeq_n r_t^2.
\]

where \( P_t^1 \) and \( P_t^2 \) are the payoffs of the two common stocks, for example, at time \( t > 0 \), and \( r_t^1 \) and \( r_t^2 \) are the corresponding log-returns for the same period.

Note that this relationship holds if we assume that the prices of the two common stocks at the present time are equal to \( P_0^1 = P_0^2 = P_0 \).
There are ways of obtaining stochastic dominance relations other than the $n$-th order stochastic dominance which is based on certain properties of investors’ utility functions.

- We borrow an example from reliability theory and adapt it for distributions describing payoffs, losses or returns.
- Consider the conditional probability

$$Q_X(t, x) = P(X > t + x | X > t).$$

(8)

where $x \geq 0$ and suppose that $X$ describes a random loss.

Then, equation (8) calculates the probability of losing more than $t + x$ on condition that the loss is larger than $t$. This probability may vary depending on the level $t$ with the additional amount of loss being fixed (x does not depend on t).
Other stochastic dominance relations

- For example, if $t_1 \leq t_2$, then the corresponding conditional probabilities may be related in the following way,

$$Q_X(t_1, x) \geq Q_X(t_2, x).$$  \hfill (9)

- Thus, the deeper we go into the tail, the less likely it is to lose additional $x$ dollars provided that the loss is larger than the selected threshold.

- Conversely, if the inequality is

$$Q_X(t_1, x) \leq Q_X(t_2, x),$$  \hfill (10)

then the further we go into the tail, the more likely it becomes to lose additional $x$ dollars.

- Basically, the inequalities in (9) and (10) describe certain tail properties of the random variable $X$. 
Other stochastic dominance relations

- Denote by $\bar{F}_X(x) = 1 - F_X(x) = P(X > x)$ the tail of the random variable $X$. Then, according to the definition of conditional probability, equation (8) can be stated in terms of $\bar{F}_X(x)$,

$$Q_X(t, x) = \frac{\bar{F}_X(x + t)}{\bar{F}_X(t)}.$$  \hspace{1cm} (11)

- Denote by $Q$ the class of all random variables for which $Q_X(t, x)$ is a *non-increasing* function of $t$ for any $x \geq 0$, and by $Q^*$ the class of all random variables for which $Q_X(t, x)$ is a *non-decreasing* function of $t$ for any $x \geq 0$.

- The random variables belonging to $Q$ satisfy inequality (9) and those belonging to $Q^*$ satisfy inequality (10) for any $x \geq 0$. 
Other stochastic dominance relations

- In case the random variable $X$ has a density $f_X(x)$, then it can be determined whether it belongs to $Q$ or $Q^*$ by the behavior of the function

$$h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)} \quad (12)$$

which is known as the hazard rate function or the failure rate function.

- If $h_X(t)$ is a non-increasing function, then $X \in Q$. If it is a non-decreasing function, then $X \in Q^*$.

- The only distribution which belongs to both classes is the exponential distribution. The hazard rate function of the exponential distribution is constant with respect to $t$. 
Now we introduce a stochastic dominance order assuming that the random variables describe random profits.

- Denote by $\Lambda_X(t)$ the transform
  \[
  \Lambda_X(t) = -\log(\bar{F}_X(t)).
  \]  
  (13)

- A positive random variable $X$ is said to dominate another positive random variable $Y$ with respect to the $\Lambda$ transform, $X \preceq_\Lambda Y$, if the random variable $Z = \Lambda_Y(X)$ is such that $Z \in Q$. 

The rationale behind the $\Lambda$ transform is the following. Consider the special case $Y = X$. The r.v. $Z = \Lambda_Y(X)$ has exactly the exponential distribution because $\bar{F}_Y(X)$ is uniformly distributed. If $Y$ has a heavier tail than $X$, then $Z$ has a tail which increases no more slowly than the tail of the exponential distribution and, therefore, $Z \in Q$.

$\Rightarrow$ The stochastic order $\succeq_{\Lambda}$ emphasizes the tail behavior of $X$ relative to $Y$.

This stochastic order is interesting since it does not arise from a class of utility functions and it has application in finance describing choice under uncertainty. We illustrate this by showing a relationship with SSD.
Other stochastic dominance relations

- Suppose that $X \succeq_{\Lambda} Y$. Then, Kalashnikov and Rachev (1990) show that the following condition holds

$$\int_{x}^{\infty} \bar{F}_X(x)dx \leq \int_{x}^{\infty} \bar{F}_Y(x)dx, \quad \forall x \geq 0. \quad (14)$$

- The converse statement is not true; that is, condition (14) does not ensure $X \succeq_{\Lambda} Y$. Equation (14) can be directly connected with SSD. In fact, if (14) holds and we assume that the expected payoffs of $X$ and $Y$ are equal, then

$$\int_{0}^{x} F_X(x)dx \leq \int_{0}^{x} F_Y(x)dx, \quad \forall x \geq 0.$$

- This inequality means that $X$ dominates $Y$ with respect to RSD and, therefore, with respect to SSD. Thus, we have demonstrated that if $EX = EY$, then

$$X \succeq_{\Lambda} Y \implies X \succeq_{RSD} Y \implies X \succeq_{SSD} Y. \quad (15)$$
Other stochastic dominance relations

- Suppose that the random variables describe losses (applied in operational risk management).

- We modify the stochastic order in the following way. A positive random variable $X$ is said to dominate another positive random variable $Y$ with respect to the $\Lambda$ transform, $X \preceq_{\Lambda^*} Y$, if the random variable $Z = \Lambda_Y(X)$ is such that $Z \in Q^*$. In this case, the tail of $X$ is heavier than the tail of $Y$. 
Other stochastic dominance relations

- If the random variables describe returns, then the left tail describes losses and the right tail describes profits.
- The random variable can be decomposed into two terms,

\[ X = X_+ - X_- , \]

where \( X_+ = \max(X, 0) \) stands for the profit and \( X_- = \max(-X, 0) \) denotes the loss.
- By modifying the stochastic order, we can determine the tail of which of the two components influences the stochastic order.
- Consider two real valued random variables \( X \) and \( Y \) describing random returns. The order \( \succeq^\Lambda \) compares the tails of the profits \( X_+ \) and \( Y_+ \), and \( \succeq^\Lambda^* \) compares the tails of the losses \( X_- \) and \( Y_- \).
The stochastic orders $\succeq_\Lambda$ and $\succeq_{\Lambda^*}$ are constructed without considering first a particular class of investors but by imposing directly a condition on the tail of the random variable.

- There may or may not be a corresponding set of utility functions such that if $Eu(X) \geq Eu(Y)$ for all $u(x)$ in this class, then $X \succeq_\Lambda Y$, for example.
- We have demonstrated that the order $\succeq_\Lambda$ is consistent with SSD and is not implied by it.

The stochastic order can be defined without seeking first a class of investors which can generate it, but we can only search for a consistency relation with an existing stochastic order (Eq. (15)).
Chapter 5.