Technical Appendix

Lecture 7: Average value-at-risk

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The material is based on the text-book:

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Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures

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Characteristics of conditional loss distributions

- AVaR represents the average of the losses larger than the VaR at tail probability ϵ, which is only one characteristic of the distribution of extreme losses.
- If the distribution function is continuous, then AVaR coincides with ETL which is the mathematical expectation of the conditional loss distribution.
- Moreover, AVaR does not provide any information about how dispersed the conditional losses are around the AVaR value.

Characteristics of conditional loss distributions

- Consider the following tail moment of order n at tail probability ϵ,

$$m_\epsilon^n(X) = \frac{1}{\epsilon}\int_0^\epsilon (F_X^{-1}(t))^n dt, \quad (1)$$

where $n = 1, 2, \ldots$, $F_X^{-1}(t)$ is the inverse c.d.f. of the r.v. X.

- If the distribution function of X is continuous, then the tail moment of order n can be represented through the following conditional expectation,

$$m_\epsilon^n(X) = E(X^n | X < VaR_\epsilon(X)), \quad (2)$$

where $n = 1, 2, \ldots$

- In the general case, if the c.d.f. has a jump at $VaR_\epsilon(X)$, a link exists between the conditional expectation and equation (1), which is similar to formula (13) for AVaR. In fact, AVaR appears as the negative of the tail moment of order one, $AVaR_\epsilon(X) = -m_\epsilon^1(X)$.

Characteristics of conditional loss distributions

- The higher-order tail moments provide additional information about the conditional distribution of the extreme losses.
- In addition to the moments $m_\epsilon^n(X)$, we introduce the central tail moments of order n at tail probability ϵ,

$$M_\epsilon^n(X) = \frac{1}{\epsilon}\int_0^\epsilon (F_X^{-1}(t) - m_\epsilon^1(X))^n dt, \tag{3}$$

  where $m_\epsilon^1(X)$ is the tail moment of order one.
- If the distribution function is continuous, then the central moments can be expressed in terms of the conditional expectation,

$$M_\epsilon^n(X) = E((X - m_\epsilon^1(X))^n | X < VaR_\epsilon(X)).$$

Characteristics of conditional loss distributions

- The tail variance of the conditional distribution appears as $M_\epsilon^2(X)$ and the tail standard deviation equals

$$(M_\epsilon^2(X))^{1/2} = \left(\frac{1}{\epsilon}\int_0^\epsilon (F_X^{-1}(t) - m_\epsilon^1(X))^2 dt\right)^{1/2}.$$

- There is a formula expressing the tail variance in terms of the tail moments introduced in (2),

$$\begin{aligned} M_\epsilon^2(X) &= m_\epsilon^2(X) - (m_\epsilon^1(X))^2 \\ &= m_\epsilon^2(X) - (AVaR_\epsilon(X))^2. \end{aligned}$$

- This formula is similar to the representation of variance in terms of the first two moments,

$$\sigma_X^2 = EX^2 - (EX)^2.$$

Characteristics of conditional loss distributions

- The tail standard deviation can be used to describe the dispersion of conditional losses around AVaR as it satisfies the general properties of dispersion measures.
- If there are two portfolios with equal AVaRs of their return distributions but different tail standard deviations, the portfolio with the smaller standard deviation is preferable.

Characteristics of conditional loss distributions

- Another central tail moment is $M_{\epsilon}^{3}(X)$.
- After proper normalization, it can be employed to measure the skewness of the conditional loss distribution. If the tail probability is sufficiently small, the tail skewness will be quite significant.
- By normalizing the central tail moment of order 4, we obtain a measure of kurtosis of the conditional loss distribution.

Characteristics of conditional loss distributions

- In a similar way, we introduce the absolute central tail moments of order n at tail probability ϵ,

$$\mu_\epsilon^n(X) = \frac{1}{\epsilon}\int_0^\epsilon |F_X^{-1}(t) - m_\epsilon^1(X)|^n dt. \quad (4)$$

- The tail moments $\mu_\epsilon^n(X)$ raised to the power of $1/n$, $(\mu_\epsilon^n(X))^{1/n}$, can be applied as measures of dispersion of the conditional loss distribution if the distribution is such that they are finite.

Characteristics of conditional loss distributions

- The tail of the random variable can be so heavy that AVaR becomes infinite. Even if it is theoretically finite, it can be hard to estimate because the heavy tail will result in the AVaR estimator having a large variability.
- The median tail loss (MTL) defined as the median of the conditional loss distribution, is a robust alternative to AVaR. It has the advantage of always being finite no matter the tail behavior of the random variable. Formally, it is defined as

$$MTL_\epsilon(X) = -F_X^{-1}(1/2|X < -VaR_\epsilon(X)), \tag{5}$$

where $F_X^{-1}(p|X < -VaR_\epsilon(X))$ stands for the inverse distribution function of the c.d.f. of the conditional loss distribution

$$\begin{aligned} F_X(x|X < -VaR_\epsilon(X)) &= P(X \leq x|X < -VaR_\epsilon(X)) \\ &= \begin{cases} P(X \leq x)/\epsilon, & x < -VaR_\epsilon(X) \\ 1, & x \geq -VaR_\epsilon(X). \end{cases} \end{aligned}$$

Characteristics of conditional loss distributions

- MTL, as well as any other quantile of the conditional loss distribution, can be directly calculated as a quantile of the distribution of X,

$$\begin{aligned} MTL_\epsilon(X) &= -F_X^{-1}(\epsilon/2) \\ &= VaR_{\epsilon/2}(X), \end{aligned} \tag{6}$$

  where $F_X^{-1}(p)$ is the inverse c.d.f. of X and ϵ is the tail probability of the corresponding VaR in equation (5).

- MTL shares the properties of VaR. Equation (6) shows that MTL is not a coherent risk measure even though it is a robust alternative to AVaR which is a coherent risk measure.

Higher-order AVaR

AVaR is the average of VaRs larger than the VaR at tail probability ϵ. *What happens if we average all AVaRs larger than the AVaR at tail probability* ϵ*?*

- This quantity is an average of coherent risk measures and, therefore, is a coherent risk measure itself since it satisfies all defining properties of coherent risk measures.
- We call it AVaR of order one and denote it by $AVaR_{\epsilon}^{(1)}(X)$ because it is a derived quantity from AVaR.
- We will consider similar derived quantities from AVaR which we call higher-order AVaRs.

Higher-order AVaR

- The AVaR of order one is represented in the following way,

$$AVaR_{\epsilon}^{(1)}(X) = \frac{1}{\epsilon}\int_0^{\epsilon} AVaR_p(X)dp$$

where $AVaR_p(X)$ is the AVaR at tail probability p.

- Replacing AVaR, we obtain

$$\begin{aligned} AVaR_{\epsilon}^{(1)}(X) &= -\frac{1}{\epsilon}\int_0^{\epsilon}\left(\int_0^1 F_X^{-1}(y)g_p(y)dy\right)dp \\ &= -\frac{1}{\epsilon}\int_0^1 F_X^{-1}(y)\left(\int_0^{\epsilon} g_p(y)dp\right)dy \end{aligned}$$

where

$$g_p(y) = \begin{cases} 1/p, & y \in [0,p] \\ 0, & y > p. \end{cases}$$

Higher-order AVaR

- After certain algebraic manipulations, we get the expression

$$\begin{aligned} AVaR_\epsilon^{(1)}(X) &= -\frac{1}{\epsilon}\int_0^\epsilon F_X^{-1}(y)\log\frac{\epsilon}{y}dy \\ &= \int_0^\epsilon VaR_y(X)\phi_\epsilon(y)dy. \end{aligned} \tag{7}$$

- The AVaR of order one can be expressed as a weighted average of VaRs larger than the VaR at tail probability ϵ with a weighting function $\phi_\epsilon(y)$ equal to

$$\phi_\epsilon(y) = \begin{cases} \frac{1}{\epsilon}\log\frac{\epsilon}{y}, & 0 \leq y \leq \epsilon \\ 0, & \epsilon < y \leq 1. \end{cases}$$

- The AVaR of order one can be viewed as a spectral risk measure with $\phi_\epsilon(y)$ being the risk aversion function.

Higher-order AVaR

- Similarly, we define the higher-order AVaR through the recursive equation
$$AVaR_{\epsilon}^{(n)}(X) = \frac{1}{\epsilon}\int_0^{\epsilon} AVaR_p^{(n-1)}(X)dp \qquad (8)$$
where $AVaR_p^{(0)}(X) = AVaR_p(X)$ and $n = 1, 2\ldots$
- The AVaR of order two equals the average of AVaRs of order one which are larger than the AVaR of order one at tail probability ϵ.
- The AVaR of order n appears as an average of AVaRs of order $n-1$.

Higher-order AVaR

- The quantity $AVaR_\epsilon^{(n)}(X)$ is a coherent risk measure because it is an average of coherent risk measures. This is a consequence of the recursive definition in (8).
- AVaR of order n admits the representation

$$AVaR_\epsilon^{(n)}(X) = \frac{1}{\epsilon}\int_0^\epsilon VaR_y(X)\frac{1}{n!}\left(\log\frac{\epsilon}{y}\right)^n dy \tag{9}$$

  and $AVaR_\epsilon^{(n)}(X)$ can be viewed as a spectral risk measure with a risk aversion function equal to

$$\phi_\epsilon^{(n)}(y) = \begin{cases} \frac{1}{\epsilon n!}\left(\log\frac{\epsilon}{y}\right)^n, & 0 \le y \le \epsilon \\ 0, & \epsilon < y \le 1. \end{cases}$$

- As a simple consequence of the definition, the sequence of higher-order AVaRs is monotonic,

$$AVaR_\epsilon(X) \le AVaR_\epsilon^{(1)}(X) \le \ldots \le AVaR_\epsilon^{(n)}(X) \le \ldots$$

Higher-order AVaR

- We remarked that if the r.v. X has a finite mean, $E|X| < \infty$, then AVaR is also finite.
- This is not true for spectral risk measures and the higher-order AVaR in particular.
- $AVaR_{\epsilon}^{(n)}(X)$ is finite if all moments of X exist. For example, if the random variable X has an exponential tail, then $AVaR_{\epsilon}^{(n)}(X) < \infty$ for any $n < \infty$.

The minimization formula for AVaR

We provide a geometric interpretation of the minimization formula (2) for AVaR.

- We restate equation (2) in the following equivalent form,

$$AVaR_{\epsilon}(X) = \frac{1}{\epsilon} \min_{\theta \in \mathbb{R}} \left(\epsilon\theta + E(-X - \theta)_{+} \right) \quad (10)$$

  where $(x)_{+} = \max(x, 0)$.
- Instead of the integral of the quantile function in the definition of AVaR, a minimization formula appears in (10).

The minimization formula for AVaR

- We interpreted the integral of the inverse c.d.f. as the shaded area in *Figure 4* given in the lecture.
- We will find the area corresponding to the objective function in the minimization formula and we will demonstrate that as θ changes, there is a minimal area which coincides with the area corresponding to the shaded area in *Figure 4*.
- Moreover, the minimal area is attained for $\theta = VaR_\epsilon(X)$ when the c.d.f. of X is continuous at $VaR_\epsilon(X)$.

The minimization formula for AVaR

- Consider first the expectation in equation (10). Assuming that X has a continuous c.d.f., we obtain an expression for the expectation involving the inverse c.d.f.,

$$\begin{aligned} E(-X-\theta)_+ &= \int_{\mathbb{R}} \max(-x-\theta,0)dF_X(x) \\ &= \int_0^1 \max(-F_X^{-1}(t)-\theta,0)dt \\ &= -\int_0^1 \min(F_X^{-1}(t)+\theta,0)dt. \end{aligned}$$

- This representation implies that the expectation $E(-X-\theta)_+$ equals the area closed between the graph of the inverse c.d.f. and a line parallel to the horizontal axis passing through the point $(0,-\theta)$ (See the illustration on the next slide).

The minimization formula for AVaR

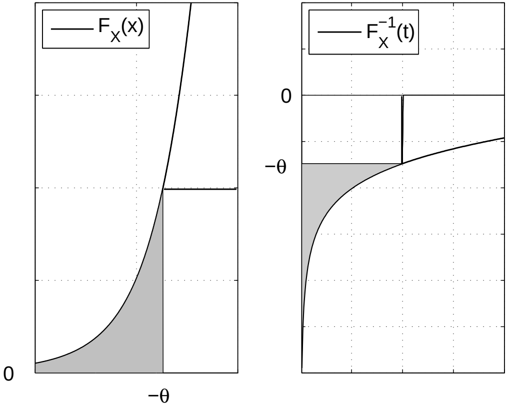

Figure 1. The shaded area is equal to the expectation $E(-X-\theta)_+$ in which X has a continuous distribution function. The same area is represented in terms of the c.d.f on the left plot.

The minimization formula for AVaR

- In equation (10), the tail probability ϵ is fixed. The product $\epsilon \times \theta$ equals the area of a rectangle with sides equal to ϵ and θ. This area is added to $E(-X - \theta)_+$.
- *Figures 2,3* show the two areas together. Comparing the plot to *Figure 1*, we find out that adding the marked area to the shaded area we obtain the total area corresponding to the objective in the minimization formula, $\epsilon\theta + E(-X - \theta)_+$.
- If $-\theta > -VaR_\epsilon(X)$, then we obtain a similar case shown on the *Figure 3*. Again, adding the marked area to the shaded area we obtain the the total area computed by the objective in the minimization formula.
- By varying θ, the total area changes but it always remains larger than the shaded area unless $\theta = VaR_\epsilon(X)$.

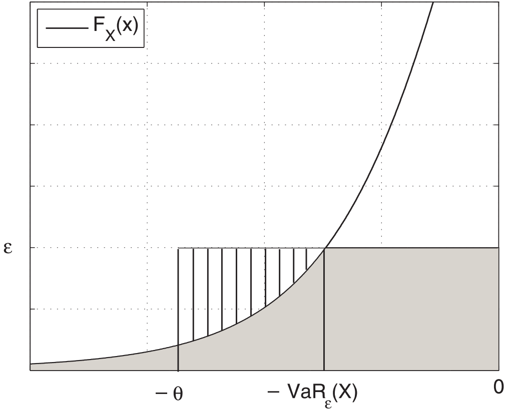

Figure 2. The marked area is in addition to the shaded one. The marked area is equal to zero if $\theta = VaR_\epsilon(X)$. The shaded areas equal $\epsilon \times AVaR_\epsilon(X)$. This is the case in which $-\theta < -VaR_\epsilon(X)$.

The minimization formula for AVaR

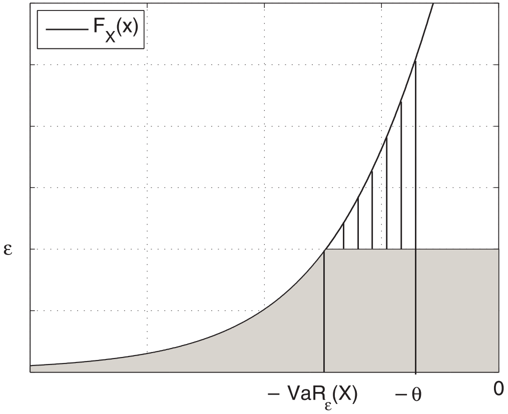

Figure 3. The marked area is in addition to the shaded one. The marked area is equal to zero if $\theta = VaR_\epsilon(X)$. The shaded areas equal $\epsilon \times AVaR_\epsilon(X)$.

The minimization formula for AVaR

- When $\theta = VaR_\epsilon(X)$ the minimum area is attained which equals exactly $\epsilon \times AVaR_\epsilon(X)$.
- According to equation (10), we have to divide the minimal area by ϵ in order to obtain the AVaR.

$\Rightarrow$ We have demonstrated that the minimization formula in equation (2) given in the lecture calculates the AVaR.

AVaR for stable distributions

- Working with the class of stable distributions in practice is difficult because there are no closed-form expressions for their densities and distribution functions.
- Stoyanov et al. (2006) give an account of the approaches to estimating AVaR of stable distributions. There is a formula which is not exactly a closed-form expressions, but is suitable for numerical work.
- It involves numerical integration but the integrand is nicely behaved and the integration range is a bounded interval. Since the formula involves numerical integration, we call it a semi-analytic expression.

AVaR for stable distributions

- Suppose that the r.v. X has a stable distribution with tail exponent α, skewness parameter β, scale parameter σ, and location parameter μ, $X \in S_\alpha(\sigma, \beta, \mu)$.
- If $\alpha \leq 1$, then $AVaR_\epsilon(X) = \infty$. The reason is that stable distributions with $\alpha \leq 1$ have infinite mathematical expectation and the AVaR is unbounded.
- If $\alpha > 1$ and $VaR_\epsilon(X) \neq 0$, then the AVaR can be represented as

$$AVaR_\epsilon(X) = \sigma A_{\epsilon,\alpha,\beta} - \mu \tag{11}$$

  where the term $A_{\epsilon,\alpha,\beta}$ does not depend on the scale and the location parameters.
- The representation (11) is a consequence of the positive homogeneity and the invariance property of AVaR.

AVaR for stable distributions

- Concerning the term $A_{\epsilon,\alpha,\beta}$,

$$A_{\epsilon,\alpha,\beta} = \frac{\alpha}{1-\alpha}\frac{|VaR_\epsilon(X)|}{\pi\epsilon}\int_{-\overline{\theta}_0}^{\pi/2} g(\theta)\exp\left(-|VaR_\epsilon(X)|^{\frac{\alpha}{\alpha-1}}v(\theta)\right)d\theta$$

where

$$g(\theta) = \frac{\sin(\alpha(\overline{\theta}_0+\theta)-2\theta)}{\sin\alpha(\overline{\theta}_0+\theta)} - \frac{\alpha\cos^2\theta}{\sin^2\alpha(\overline{\theta}_0+\theta)},$$

$$v(\theta) = (\cos\alpha\overline{\theta}_0)^{\frac{1}{\alpha-1}}\left(\frac{\cos\theta}{\sin\alpha(\overline{\theta}_0+\theta)}\right)^{\frac{\alpha}{\alpha-1}}\frac{\cos(\alpha\overline{\theta}_0+(\alpha-1)\theta)}{\cos\theta},$$

in which $\overline{\theta}_0 = \frac{1}{\alpha}\arctan\left(\overline{\beta}\tan\frac{\pi\alpha}{2}\right)$, $\overline{\beta} = -\mathrm{sign}(VaR_\epsilon(X))\beta$, and $VaR_\epsilon(X)$ is the VaR of the stable distribution at tail probability ϵ.

AVaR for stable distributions

- If $VaR_\epsilon(X) = 0$, then the AVaR admits a very simple expression,

$$AVaR_\epsilon(X) = \frac{2\Gamma\left(\frac{\alpha-1}{\alpha}\right)}{(\pi - 2\theta_0)} \frac{\cos\theta_0}{(\cos\alpha\theta_0)^{1/\alpha}}.$$

in which $\Gamma(x)$ is the gamma function and $\theta_0 = \frac{1}{\alpha}\arctan(\beta\tan\frac{\pi\alpha}{2})$.

ETL vs AVaR

- The expected tail loss and the average value-at-risk are two related concepts.
- We remarked that ETL and AVaR coincide if the portfolio return distribution is continuous at the corresponding VaR level.
- However, if there is a discontinuity, or a point mass, then the two notions diverge. Still, the AVaR can be expressed through the ETL and the VaR at the same tail probability.

ETL vs AVaR

- The ETL at tail probability ϵ is defined as the average loss provided that the loss exceeds the VaR at tail probability ϵ,

$$ETL_\epsilon(X) = -E(X|X < -VaR_\epsilon(X)). \tag{12}$$

- As a consequence of the definition, the ETL can be expressed in terms of the c.d.f. and the inverse c.d.f.
- Suppose additionally, that the c.d.f. of X has a jump at $-VaR_\epsilon(X)$. Then the loss $VaR_\epsilon(X)$ occurs with probability equal to the size of the jump and, because of the strict inequality in (12), it will not be included in the average.

ETL vs AVaR

Figure below shows the graphs of the c.d.f. and the inverse c.d.f. of a random variable with a point mass at $-VaR_\epsilon(X)$.

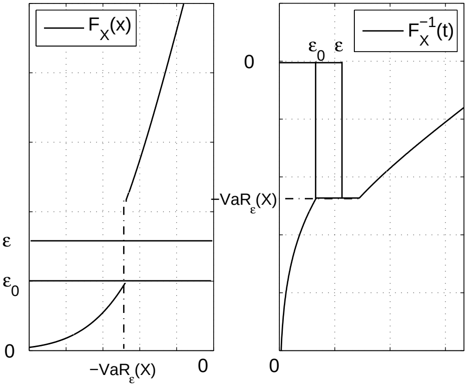

Figure 4. The c.d.f. and the inverse c.d.f. of a random variable X with a point mass at $-VaR_\epsilon(X)$. The tail probability ϵ splits the jump of the c.d.f.

ETL vs AVaR

- If ϵ splits the jump of the c.d.f. as on the left plot in *Figure 4*, then the ETL at tail probability ϵ equals,

$$\begin{aligned} ETL_{\epsilon}(X) &= -E(X|X < -VaR_{\epsilon}(X)) \\ &= -E(X|X < -VaR_{\epsilon_0}(X)) \\ &= ETL_{\epsilon_0}(X). \end{aligned}$$

- In terms of the inverse c.d.f., the quantity $ETL_{\epsilon_0}(X)$ can be represented as

$$ETL_{\epsilon_0}(X) = -\frac{1}{\epsilon_0}\int_0^{\epsilon_0} F_X^{-1}(t)dt.$$

ETL vs AVaR

The relationship between AVaR and ETL follows from the definition of AVaR.

- Suppose that the c.d.f. of the random variable X is as on the left plot in *Figure 4*. Then,

$$\begin{aligned} AVaR_\epsilon(X) &= -\frac{1}{\epsilon}\int_0^\epsilon F_X^{-1}(t)dt \\ &= -\frac{1}{\epsilon}\left(\int_0^{\epsilon_0} F_X^{-1}(t)dt + \int_{\epsilon_0}^{\epsilon} F_X^{-1}(t)dt\right) \\ &= -\frac{1}{\epsilon}\int_0^{\epsilon_0} F_X^{-1}(t)dt + \frac{\epsilon-\epsilon_0}{\epsilon}VaR_\epsilon(X). \end{aligned}$$

  where the last inequality holds because the inverse c.d.f. is flat in the interval $[\epsilon_0, \epsilon]$ and the integral is merely the surface of the rectangle shown on the right plot in *Figure 4*.

- The integral in the first summand can be related to the ETL at tail probability ϵ and, finally, we arrive at the expression

$$AVaR_\epsilon(X) = \frac{\epsilon_0}{\epsilon}ETL_\epsilon(X) + \frac{\epsilon-\epsilon_0}{\epsilon}VaR_\epsilon(X). \qquad (13)$$

ETL vs AVaR

- Equation (13) shows that $AVaR_\epsilon(X)$ can be represented as a weighted average between the ETL and the VaR at the same tail probability as the coefficients in front of the two summands are positive and sum up to one.
- In the special case in which there is no jump, or if $\epsilon = \epsilon_1$, then AVaR equals ETL.

ETL vs AVaR

Why is equation (13) important if in all statistical models we assume that the random variables describing return or payoff distribution have densities?

- Under this assumption, not only are the corresponding c.d.f.s continuous but they are also smooth.
- Equation (13) is important because if the estimate of AVaR is based on the Monte Carlo method, then we use a sample of scenarios which approximate the nicely behaved hypothesized distribution.
- Even though we are approximating a smooth distribution function, the sample c.d.f. of the scenarios is completely discrete, with jumps at the scenarios the size of which equals the $1/n$, where n stands for the number of scenarios.

ETL vs AVaR

- Equation (6) given in the lecture is actually equation (13) restated for a discrete random variable.
- The outcomes are the available scenarios which are equally probable.
- Consider a sample of observations or scenarios $r_1, \ldots, r_n$ and denote by $r_{(1)} \leq r_{(2)} \leq \ldots \leq r_{(n)}$ the ordered sample.
- The natural estimator of the ETL at tail probability ϵ is

$$\widehat{ETL}_\epsilon(r) = -\frac{1}{\lceil n\epsilon \rceil - 1} \sum_{k=1}^{\lceil n\epsilon \rceil - 1} r_{(k)} \tag{14}$$

where $\lceil x \rceil$ is the smallest integer larger than x.

- Formula (14) means that we average $\lceil n\epsilon \rceil - 1$ of the $\lceil n\epsilon \rceil$ smallest observations which is, in fact, the definition of the conditional expectation in (12) for a discrete distribution.
- The VaR at tail probability ϵ is equal to the negative of the empirical quantile,

$$\widehat{VaR}_{\epsilon}(r) = -r_{(\lceil n\epsilon \rceil)}. \tag{15}$$

- It remains to determine the coefficients in (13). Having in mind that the observations in the sample are equally probable, we calculate that

$$\epsilon_0 = \frac{\lceil n\epsilon \rceil - 1}{n}.$$

- Plugging ϵ_0, (15), and (14) into equation (13), we obtain (6) from the lecture which is the sample AVaR.

ETL vs AVaR

Similarly, equation (10) from the lecture also arises from (13).

- The assumption is that the underlying random variable has a discrete distribution but the outcomes are not equally probable.
- The corresponding equation for the average loss on condition that the loss is larger than the VaR at tail probability ϵ is given by

$$\widehat{ETL}_{\epsilon}(r) = -\frac{1}{\epsilon_0}\sum_{j=1}^{k_\epsilon} p_j r_{(j)} \tag{16}$$

where $\epsilon_0 = \sum_{j=1}^{k_\epsilon} p_j$ and k_ϵ is the integer satisfying the inequalities,

$$\sum_{j=1}^{k_\epsilon} p_j \leq \epsilon < \sum_{j=1}^{k_\epsilon+1} p_j.$$

- The sum $\sum_{j=1}^{k_\epsilon} p_j$ stands for the cumulative probability of the losses larger than the the VaR at tail probability ϵ.
- Note that equation (16) turns into equation (14) when the outcomes are equally probable.

ETL is not a coherent risk measure. The sample ETL in (14) is not a smooth function of the tail probability while the sample AVaR is smooth. See the illustrations below.

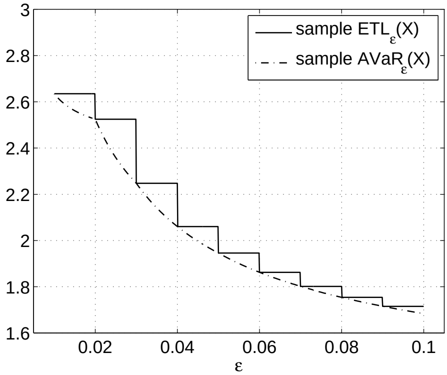

Figure 5. The graphs of the sample ETL and AVaR with tail probability varying between 1% and 10%. The plot is produced from a sample of 100 observations. $X \in N(0,1)$.

ETL vs AVaR

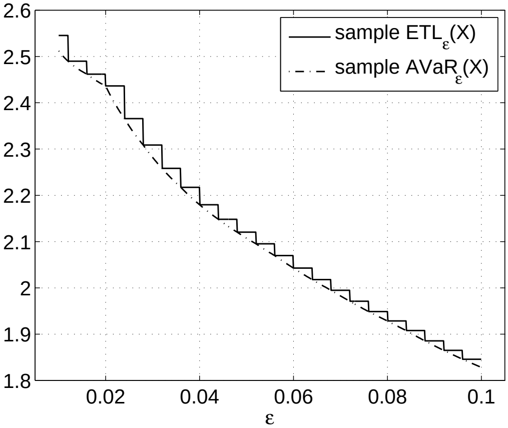

Figure 6. The graphs of the sample ETL and AVaR with tail probability varying between 1% and 10%. The plot is produced from a sample of 250 observations. $X \in N(0,1)$.

ETL vs AVaR

- Both plots demonstrate that the sample ETL is a step function of the tail probability while the AVaR is a smooth function of it.
- This is not surprising because, as ϵ increases, new observations appear in the sum in (14) producing the jumps in the graph of the sample ETL.
- In contrast, the AVaR changes gradually as it is a weighted average of the ETL and the VaR at the same tail probability.
- Note that, as the sample size increases, the jumps in the graph of the sample ETL diminish. In a sample of 5,000 scenarios, both quantities almost overlap. This is because the standard normal distribution has a smooth c.d.f. and the sample c.d.f. constructed from a larger sample better approximates the theoretical c.d.f.
- In this case, as the sample size approaches infinity, the AVaR becomes indistinguishable from the ETL at the same tail probability.

Remarks on spectral risk measures

- By selecting a particular risk-aversion function, we can obtain an infinite risk measure for some return distributions.
- The AVaR can also become infinite but all distributions for which this happens are not reasonable as a model for financial assets returns because they have infinite mathematical expectation.
- This is not the case with the spectral risk measures. There are plausible statistical models which, if combined with an inappropriate risk-aversion function, result in an infinite spectral risk measure.

Remarks on spectral risk measures

- We provide conditions which guarantee that if a risk-aversion function satisfies them, then it generates a finite spectral risk measure.
- These conditions can be divided into two groups depending on what kind of information about the random variable is used:
  1. The first group of conditions is based on information about existence of certain moments;
  2. The second group contains more precise conditions based on the tail behavior of the random variable.

Moment-based conditions

- Moment-based conditions are related to the existence of a certain norm of the risk-aversion function.
- We take advantage of the norms behind the classical Lebesgue spaces of functions denoted by

$$L^p([0,1]) := \left\{ f : ||f||_p = \int_0^1 |f(t)|^p dt < \infty \right\}$$

  where $||\cdot||_p$ denotes the corresponding norm. If $p = \infty$, then the norm is the essential supremum, $||f||_\infty = \operatorname{ess\,sup}_{t\in[0,1]} |f(t)|$.
- If the function f is continuous and bounded, then $||f||_\infty$ is simply the maximum of the absolute value of the function.

Moment-based conditions

- The sufficient conditions for the finiteness of the spectral risk measure involve the quantity

$$I_{\phi}(X) = \int_0^1 |F_X^{-1}(p)\phi(p)|dp \tag{17}$$

  which is, essentially, the definition of the spectral risk measure but the integrand is taken in absolute value.

- Therefore,

$$|\rho_{\phi}(X)| \leq I_{\phi}(X)$$

  and, as a consequence, if the quantity $I_{\phi}(X)$ is finite, so is the spectral risk measure $\rho_{\phi}(X)$.

- This is a sufficient condition for the absolute convergence of the integral behind the definition of spectral risk measures.

- Moment-based conditions are summarized by the following inequalities,

$$C \cdot E|X| \leq I_\phi(X) \leq (E|X|^s)^{1/s} \, ||\phi||_r \tag{18}$$

where $0 \leq C < \infty$ is a constant and $1/s + 1/r = 1$ with $r, s > 1$.

- If $r = 1$ or $s = 1$, the second inequality in (18) changes to

$$\begin{aligned} I_\phi(X) &\leq \sup_{u \in [0,1]} |F_X^{-1}(u)|, \quad \text{if} \quad r = 1 \\ I_\phi(X) &\leq E|X| \cdot ||\phi||_\infty, \quad \text{if} \quad s = 1. \end{aligned} \tag{19}$$

Moment-based conditions

- As a consequence of equation (18), it follows that if the absolute moment of order s exists, $E|X|^s < \infty$, $s > 1$, then $\phi \in L^r([0,1])$ is a sufficient condition for $\rho_\phi(X) < \infty$.
- The $AVaR_\epsilon(X)$ has a special place among $\rho_\phi(X)$ because if $AVaR_\epsilon(X) = \infty$, then $E|X| = \infty$ and $\rho_\phi(X)$ is not absolutely convergent for any choice of ϕ.
- In the reverse direction, if there exists $\phi \in L^1([0,1])$ such that $I_\phi(X) < \infty$, then $AVaR_\epsilon(X) < \infty$.
- The limit cases in inequalities (19) show that if X has a bounded support, then all possible risk spectra are meaningful.
- In addition, if we consider the space of all essentially bounded risk spectra, then the existence of $E|X|$ is a necessary and sufficient condition for the absolute convergence of $\rho_\phi(X)$.

Conditions based on the tail behavior of X

- More precise sufficient conditions can be derived assuming a particular tail behavior of the distribution function of X.
- A fairly general assumption for the tail behavior is regular variation.
- A monotonic function $f(x)$ is said to be *regularly varying* at infinity with index α, $f \in \mathcal{RV}_\alpha$, if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\alpha. \tag{20}$$

- Examples of random variables with regularly varying distribution functions include stable distributions, Student's t distribution, and Parreto distribution.
- Thus, it is natural to look for sufficient conditions for the convergence of $\rho_\phi(X)$ in the general setting of regularly varying tails. A set of such conditions is provided below.

Conditions based on the tail behavior of X

- Suppose that $\rho_\phi(X)$ is the spectral measure of risk of a random variable X such that $E|X| < \infty$ and $P(-X > u) \in \mathcal{RV}_{-\alpha}$.
- Let the inverse of the risk spectrum $\phi^{-1} \in \mathcal{RV}_{-\delta}$, if existing. Then

$$\rho_\phi(X) = \infty, \quad \text{if} \quad 1 < \delta \leq \alpha/(\alpha - 1)$$
$$\rho_\phi(X) < \infty, \quad \text{if} \quad \delta > \alpha/(\alpha - 1)$$

- The inverse of the risk-aversion function ϕ^{-1} exists if we assume that ϕ is smooth because by assumption ϕ is a monotonic function.

Conditions based on the tail behavior of X

- In some cases, we may not know explicitly the inverse of the risk-aversion function, or the inverse may not be regularly varying. Then, the next sufficient condition can be adopted. It is based on comparing the risk-aversion function to a power function.
- Suppose that the same condition as above holds, the random variable X is such that $E|X| < \infty$ and $P(-X > u) \in \mathcal{RV}_{-\alpha}$. If the condition

$$\lim_{x \to 0} \phi(x)x^{\beta} = C$$

is satisfied with $0 < \beta < \frac{\alpha-1}{\alpha}$ and $0 \leq C < \infty$, then $\rho_{\phi}(X) < \infty$. If $\frac{\alpha-1}{\alpha} \leq \beta < 1$ and $0 < C < \infty$, then $\rho_{\phi}(X) = \infty$.

Conditions based on the tail behavior of X

- This condition emphasizes that it is the behavior of the risk-aversion function $\phi(t)$ close to $t = 0$ that matters.
- In this range, the risk aversion function defines the weights of the very extreme losses and if the weights increase very quickly as $t \to 0$, then the risk measure may explode.
- These conditions are more specific than assuming that a certain norm of the risk-aversion function is finite. It is possible to derive them because of the hypothesized tail behavior of the distribution function of X which is a stronger assumption than the existence of certain moments.

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Chapter 7.