A New Tempered Stable Distribution and Its Application to Finance

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Abstract

In this paper, we will discuss a parametric approach to risk-neutral density extraction from option prices based on the knowledge of the estimated historical density. A flexible distribution is needed in order to find an equivalent change of measure and, at the same time, take into account the historical estimates. To this end, we introduce a new tempered stable distribution we refer to as the KR distribution.

Some properties of this distribution will be discussed in this paper, along with the advantages in applying it to financial modeling. Since the KR distribution is infinitely divisible, a Lévy process can be induced from it. Furthermore, we can develop an exponential Lévy model, called the exponential KR model, and prove that it is an extension of the Carr, Geman, Madan, and Yor (CGMY) model.

The risk-neutral process is fitted by matching model prices to market prices of options using nonlinear least squares. The easy form of the characteristic function of the KR distribution allows one to obtain a suitable solution to the calibration problem. To demonstrate the advantages of the exponential KR model, we will present the results of the parameter estimation for the S&P 500 Index and option prices.

1 Introduction

Since Mandelbrot introduced the Lévy stable (or α-stable) distribution to model the empirical distribution of asset prices in [20], the α-stable distribution became the most popular alternative to the normal distribution which has been rejected by numerous empirical studies that have found financial return series to be heavy-tailed and possibly skewed. Rachev and Mittnik [25] and Rachev et al. [26] have developed financial models with α-stable distributions and applied them to market and credit risk management, option pricing, and portfolio selection. They also discuss the major attack on the α-stable models in the 1970s and 1980s. That is, while the empirical evidence does not support the normal distribution, it is also not consistent with an α-stable distribution. The distribution of returns for assets has heavier tails relative to the normal distribution and thinner tails than the α-stable distribution. Partly in response to those empirical inconsistencies, various alternatives to the α-stable distribution were proposed in the literature. Two examples are the "CGMY" distribution (Carr et al. [7]) and the "Modified Tempered Stable" distribution (Kim, Rachev, and Chung [15]). These two distributions, sometimes called the tempered stable distributions, have not only heavier tails than the normal distribution and thinner than the α-stable distribution, but also have finite moments for all orders. Recently, Rosiński [27] generalized the tempered stable distributions and classified them using the "spectral" (or Rosiński) measure.

In this paper, we will introduce an extension of the CGMY distribution named the "KR tempered stable" (or simply "KR") distribution. The KR distribution is characterized by a new spectral measure. We believe that the simple form of the characteristic function, the exponential decayed tails, and other desirable properties of the KR distribution will result in its use in theoretical and empirical finance, such as modeling asset return processes, portfolio analysis, risk management, derivative pricing, and econometrics in the presence of heavy-tailed innovations.

In the Black-Scholes model [5], the stock price process was described by the exponential of Brownian motion with drift : $S_t = S_0 e^{X_t}$ where $X_t = \mu t + \sigma B_t$ and the process B_t is Brownian motion. Replacing the driving process X_t by a Lévy process we obtain the class of exponential Lévy models. For example, if X_t is replaced by the CGMY process then one can obtain the exponential CGMY model (Carr et al. [7]). In the exponential Lévy model, the equivalent martingale measure (EMM) of a given market measure is not unique in general. For this reason, we have to find a method to select one of them.

One classical method to choose an EMM is the Esscher transform; another reasonable method is finding the "minimal entropy martingale measure", as presented by Fujiwara and Miyahara [23]. However, while these methods are mathematically elegant and have a financial meaning in a utility maximization problem, the model prices obtained from the EMM did not match the market prices observed for options. The other method for handling the problem is to estimate the risk-neutral measure by using current option price data independent of the historical underlying distribution. This method can fit model prices to market prices directly, but it has a problem: the historical market measure and the risk-neutral measure need not to be equivalent and it conflicts with the the no-arbitrage property for option prices. To overcome these drawbacks, one must estimate the market measure and the risk-neutral measure simultaneously, and preserve the equivalent property between two measures. One method for doing so is suggested by Cont and Tankov [10]. Basically, the method finds an EMM of the market measure such that minimizes the least squares error of the model option prices relative to the market option prices. In this paper, we will discuss the last method to find an EMM. We will consider the exponential Lévy model, replacing the driving process X_t by the KR process. Since the change of measure between two KR processes has more freedom than that of the CGMY, we can find the parameters of the EMM such that the least squares error of the KR model prices can be smaller than the error of the CGMY model prices.

The remainder of this paper is organized as follows. Section 2 reviews the tempered stable distribution introduced by Rosiński. The definition and properties of the KR distribution and the change of measure between two KR processes are given in Section 3. Section 4 explains the advantage of the exponential KR model in the calibration problem. In that section, we will show the estimation results for the market parameters for the historical distribution of the log-returns of the S&P 500 index, and compare the performance of the calibration of the risk-neutral distribution for the CGMY model and the KR model.

2 Tempered Stable Distributions

In this section we will review the definition and properties of the tempered stable distributions introduced by Rosiński [27]. The polar coordinates representation of a measure $\nu = \nu(dx)$ on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$ is the measure $\nu = \nu(dv, du)$ on $(0,\infty) \times S^{d-1}$ obtained by the bijection $x \mapsto \left(||x||, \frac{x}{||x||}\right)$. Let the Lévy measure M_0 of an α-stable distribution on $\mathbb{R}^d$ in polar coordinates be of the form

$$M_0(dv, du) = v^{-\alpha-1} dv \sigma(du) \tag{2.1}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. A tempered α-stable distribution is defined by tempering the radial term of M_0 as follows:

Definition 2.1 (Definition 2.1. [27])**.** *Let $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. A probability measure on $\mathbb{R}^d$ is called* tempered α-stable (denoted as TαS) *if is infinitely divisible without Gaussian part and whose Lévy measure M can be written in polar coordinates as*

$$M(dv, du) = v^{-\alpha-1} q(v, u) dv \, \sigma(du). \tag{2.2}$$

where $q : (p, \infty) \times S^{d-1} \mapsto (0, \infty)$ is a Borel function such that $q(\cdot, u)$ is completely monotone with $q(\infty, u) = 0$ for each $u \in S^{d-1}$. A TαS distribution is called a proper TαS *distribution if $\lim_{v\to 0^+} q(v, u) = 1$ for each $u \in S^{d-1}$.*

The completely monotonicity of $q(\cdot, u)$ means that $(-1)^n \frac{d}{dv} q(v, u) > 0$ for all $v > 0$, $u \in S^{d-1}$, and $n = 0, 1, 2, \cdots$. The tempering function q can be represented as the Laplace transform

$$q(v, u) = \int_0^\infty e^{-vs} Q(ds|u) \tag{2.3}$$

where $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ is a measurable family of Borel measures on $(0, \infty)$. Define a measure Q on $\mathbb{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty I_A(vu) Q(dv|u) \sigma(du), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.4}$$

We also define a measure R by

$$R(A) := \int_{\mathbb{R}^d} I_A\left(\frac{x}{||x||^2}\right) ||x||^\alpha Q(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

Clearly $R(\{0\}) = 0$ and $Q(\{0\}) = 0$ and Q can be expressed in terms of the measure R as follows:

$$Q(A) = \int_{\mathbb{R}_0^d} \left(\frac{x}{||x||^2}\right) ||x||^\alpha R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.6}$$

Theorem 2.2 (Theorem 2.3. [27]). *Lévy measure M of $T\alpha S$ distribution can be written in the form*

$$M(A) = \int_{\mathbb{R}_0^d} \int_0^\infty I_A(tx)\alpha t^{-\alpha-1} e^{-t} dt R(dx), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.7}$$

where R is a unique measure on $\mathbb{R}^d$ such that

$$R(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (||x||^2 \wedge ||x||^\alpha) R(dx) < \infty. \tag{2.8}$$

If M is as in (2.2) then R is given by (2.5).

Conversely, if R is a measure satisfying (2.8), then (2.7) defines the Lévy measure of a $T\alpha S$ distribution. M corresponds to a proper $T\alpha S$ distribution if and only if

$$\int_{\mathbb{R}^d} ||x||^\alpha R(dx) < \infty. \tag{2.9}$$

The measure R is called a "spectral measure" of the corresponding TαS distribution. By Theorem 2.9 in [27], the following definition is well defined.

Definition 2.3. *Let X be a random vector having a $T\alpha S$ distribution with the spectral measure R.*
(i) If $\alpha \in (0, 2)$ and $E[||X||] < \infty$, then we will write $X \sim TS_\alpha(R, b)$ to indicate that characteristic function ϕ of X is given by

$$\phi(u) = \exp\left(\int_{\mathbb{R}_0^d} \psi_\alpha(\langle u, x\rangle) R(dx) + i\langle u, b\rangle \right) \tag{2.10}$$

where

$$\psi_\alpha(y) = \begin{cases} \Gamma(-\alpha)((1-iy)^\alpha - 1 + i\alpha y), & \text{if} \quad \alpha \neq 1 \\ (1-iy)\log(1-iy) + iy, & \text{if} \quad \alpha = 1 \end{cases} \tag{2.11}$$

and $b = E[X]$.
(ii) If $\alpha \in (0, 1)$ and

$$\int_{||x|| \leq 1} ||x|| R(dx) < \infty, \tag{2.12}$$

holds, then $X \sim TS_\alpha^0(R, b_0)$ means that the characteristic function ϕ^0 of X is of the form

$$\phi^0(u) = \exp\left(\int_{\mathbb{R}_0^d} \psi_\alpha^0(\langle u, x\rangle) R(dx) + i\langle u, b_0\rangle \right) \tag{2.13}$$

where

$$\psi_\alpha^0(y) = \Gamma(-\alpha)((1-iy)^\alpha - 1) \tag{2.14}$$

and $b_0 \in \mathbb{R}^d$ is the drift vector (i.e. $b_0 = \int_{||x|| \leq 1} ||x|| M(dx)$).

Remark 2.4. *Let X be a $T\alpha S$ distributed random vector with the spectral measure R. By Proposition 2.7 in [27], we can say the following:*

1. *In the above definition, $E[||X||] < \infty$ if and only if $\alpha \in (1,2)$ or*

$$\alpha = 1 \text{ and } \int_{||x||>1} ||x|| \log ||x|| R(dx) < \infty, \tag{2.15}$$

or

$$\alpha \in (0,1) \text{ and } \int_{||x||>1} ||x|| R(dx) < \infty. \tag{2.16}$$

2. *If $\alpha \in (0,1)$ and $\int_{\mathbb{R}^d} ||x|| R(dx) < \infty$, then both form (2.10) and (2.13) are valid for X. Therefore $X \sim TS^0_\alpha(R, b_0)$ and $X \sim TS^0_\alpha(R, b)$, where $b = b_0 + \Gamma(1-\alpha) \int_{\mathbb{R}^d} xR(dx)$.*

The following Lemma shows some relations between the spectral measure R of the TαS distribution and the Lévy measure of the α-stable distribution given by (2.1).

Lemma 2.5 (Lemma 2.14. [27])**.** *Let M be a Lévy measure of a proper $T\alpha S$ distribution, as in (2.2), with the spectral measure R. Let M_0 be the Lévy measure of α-stable distribution given by (2.1). Then*

$$M_0(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.17}$$

Furthermore,

$$\sigma(B) = \int_{\mathbb{R}^d} I_B\left(\frac{x}{||x||}\right) ||x||^\alpha R(dx), \quad B \in \mathcal{B}(S^{d-1}). \tag{2.18}$$

Let X be a α-stable random vector with Lévy measure M_0 given by (2.17). We have

$$E[e^{iuX}] = \exp\left(\int_{S^{d-1}} \bar{\psi}_\alpha(\langle u, x\rangle) \sigma(dx) + i\langle u, a\rangle \right)$$

where some suitable $a \in \mathbb{R}^d$ and

$$\bar{\psi}_\alpha(y) = \begin{cases} \Gamma(-\alpha) \cos\left(\frac{\alpha\pi}{2}\right) |y|^\alpha (1 - i \tan\left(\frac{\alpha\pi}{2}\right) \mathrm{sgn}(y)), & \text{if} \quad \alpha \neq 1 \\ -\frac{\pi}{2}(|y| + i\frac{2}{\pi} y \log(y)), & \text{if} \quad \alpha = 1 \end{cases}$$

(See [29, Theorem 14.10]). In this case, we will write $X \sim S_\alpha(\sigma, a)$.

Since TαS is infinitely divisible, there is a Lévy process $(X_t)_{t\geq 0}$ in $\mathbb{R}^d$ such that X_1 has a TαS (proper TαS) distribution. The process $(X_t)_{t\geq 0}$ will be called a *TαS (proper TαS) Lévy process.*

Let Ω to be the set of all cadlag function on $[0,\infty)$ into $\mathbb{R}^d$, and $(X_t)_{t\geq 0}$ is a canonical process on Ω (i.e, $X_t(\omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$). Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ where

$$\mathcal{F} = \sigma\{X_s; s \geq 0\}$$
$$\mathcal{F}_t = \cap_{s\geq 0} \sigma\{X_u : u \leq s\}, \quad t \geq 0.$$

$(\mathcal{F}_t)_{t\geq 0}$ is the right continuous natural filtration. The canonical process $(X_t)_{t\geq 0}$ is characterized by a probability measure $\mathbb{P}$ on $(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0})$.

Theorem 2.6 (Theorem 4.1. [27])**.** *In the above setting, consider two probability measures $\mathbb{P}_0$ and $\mathbb{P}$ on $(\Omega,\, \mathcal{F})$ such that the canonical process $(X_t)_{t\geq 0}$ under $\mathbb{P}_0$ is an α-stable process while under $\mathbb{P}$ it is a proper $T\alpha S$ Lévy process. Specifically, assume that under $\mathbb{P}_0$, $X_1 \sim S_\alpha(\sigma, a)$, where σ is related to R by (2.18) and $\alpha \in (0,2)$, while under $\mathbb{P}$, $X_1 \sim TS^0_\alpha(R, b)$ when $\alpha \in (0,1)$ and $X_1 \sim TS_\alpha(R, b)$ when $\alpha \in [1,2)$. Let M, the Lévy measure corresponding to R, be as in (2.2), where $q(0^+, u) = 1$ for all $u \in S^{d-1}$. Then $\mathbb{P}_0|_{\mathcal{F}_t}$ and $\mathbb{P}|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{S^{d-1}} \int_0^1 (1 - q(v,u))^2 v^{-\alpha-1} dv\, \sigma(du) < \infty \tag{2.19}$$

and

$$b - a = \begin{cases} 0, & \text{if} \quad \alpha \in (0,1) \\ \int_{\mathbb{R}^d} x(\log ||x|| - 1) R(dx), & \text{if} \quad \alpha = 1 \\ \Gamma(1-\alpha) \int_{\mathbb{R}^d} x R(dx), & \text{if} \quad \alpha \in (1,2). \end{cases} \tag{2.20}$$

Condition (2.19) implies that the integral in (2.20) exists. Furthermore, if either (2.19) or (2.20) fails, then $\mathbb{P}_0|_{\mathcal{F}_t}$ and $\mathbb{P}|_{\mathcal{F}_t}$ are singular for all $t > 0$.

3 KR Tempered Stable Distribution

Consider the proper TαS distribution on $\mathbb{R}$ whose Lévy measure M in polar coordinate is

$$M(ds, du) = s^{-\alpha-1} q(s,u) ds\, \sigma(du) \tag{3.1}$$

where

$$\sigma(A) = \frac{k_+ r_+^\alpha}{\alpha + p_+} I_A(1) + \frac{k_- r_-^\alpha}{\alpha + p_-} I_A(-1), \quad A \subset S^0,$$

and

$$q(v,1) = (\alpha + p_+) r_+^{-\alpha - p_+} \int_0^{r_+} e^{-v/s} s^{\alpha + p_+ - 1} ds$$
$$q(v,-1) = (\alpha + p_-) r_-^{-\alpha - p_-} \int_0^{r_-} e^{-v/s} s^{\alpha + p_- - 1} ds,$$

with $\alpha \in (0,2)$, $k_+, k_-, r_+, r_- > 0$ and $p_+, p_- > -\alpha$. Then the spectral measure R corresponding to the Lévy measure M can be deduced as

$$R(dx) = (k_+ r_+^{-p_+} I_{(0,r_+)}(x) |x|^{p_+ - 1} + k_- r_-^{-p_-} I_{(-r_-,0)}(x) |x|^{p_- - 1})\, dx. \tag{3.2}$$

Lemma 3.1. *If M and R are given by (3.1) and (3.2), respectively, we have*
i) $R(\{0\}) = 0$, $\int_{\mathbb{R}} |x|^\alpha R(dx) < \infty$ *and* $\int_{|x|>1} |x| R(dx) < \infty$ *for all* $\alpha \in (0, 2)$.
ii) By Theorem 2.2, M can be written in the form

$$M(A) = k_+ r_+^{-p_+} \int_0^{r_+} \int_0^\infty I_A(tx) t^{-\alpha-1} e^{-t} dt\, x^{p_+-1} dx + k_- r_-^{-p_-} \int_0^{r_-} \int_0^\infty I_A(-tx) t^{-\alpha-1} e^{-t} dt\, x^{p_--1} dx, \quad A \in \mathcal{B}(\mathbb{R}_0). \tag{3.3}$$

iii) If $\alpha = 1$ *then*

$$\int_{|x|>1} x \log |x| R(dx) < \infty,$$

and if $\alpha \in (0, 1)$,

$$\int_{|x|<1} |x| R(dx) < \infty.$$

Hence, if X is a $T\alpha S$ distributed random variable with Lévy measure M, $E[|X|] < \infty$.

Proof. iii) It can be proved by Proposition 2.7. in [27]. □

Proposition 3.2 (Exponential Moments)**.** *Let X be a random variable with the proper $T\alpha S$ distribution corresponding to the spectral measure R defined in (3.2). Then* $E[e^{\theta X}] < \infty$ *if and only if* $-r_-^{-1} \leq \theta \leq r_+^{-1}$.

Proof. Note that $E[e^{\theta X}] < \infty$ if and only if $\int_{|x|>1} e^{\theta x} M(dx) < \infty$. We have

$$\begin{aligned}\int_{|x|>1} e^{\theta x} M(dx) &= k_+ r_+^{-p_+} \int_0^{r_+} \int_0^\infty e^{\theta tx} I_{(1,\infty)}(tx) t^{-\alpha-1} e^{-t} dt\, x^{p_+-1} dx \\ &\quad + k_- r_-^{-p_-} \int_0^{r_-} \int_0^\infty e^{-\theta tx} I_{(-\infty,-1)}(-tx) t^{-\alpha-1} e^{-t} dt\, x^{p_--1} dx \\ &= k_+ r_+^{-p_+} \int_0^{r_+} \int_{1/x}^\infty e^{t(\theta x-1)} t^{-\alpha-1} dt\, x^{p_+-1} dx \\ &\quad + k_- r_-^{-p_-} \int_0^{r_-} \int_{1/x}^\infty e^{t(-\theta x-1)} t^{-\alpha-1} dt\, x^{p_--1} dx\end{aligned}$$

If $\theta \leq r_+^{-1}$ then $\theta x - 1 \leq 0$ where $x \in (0, r_+)$, and hence

$$\begin{aligned}k_+ r_+^{-p_+} \int_0^{r_+} \int_{1/x}^\infty e^{t(\theta x-1)} t^{-\alpha-1} dt\, x^{p_+-1} dx &\leq k_+ r_+^{-p_+} \int_0^{r_+} \int_{1/x}^\infty t^{-\alpha-1} dt\, x^{p_+-1} dx \\ &= k_+ r_+^{-p_+} \int_0^{r_+} \frac{x^{\alpha+p_+-1}}{\alpha} dx = \frac{k_+ r_+^\alpha}{\alpha(\alpha+p_+)},\end{aligned}$$

Similarly if $-r_-^{-1} \le \theta$ then $-\theta x - 1 \le 0$ where $x \in (0, r_-)$, and hence

$$k_- r_-^{-p_-} \int_0^{r_-} \int_{1/x}^{\infty} e^{t(-\theta x - 1)} t^{-\alpha-1} dt\, x^{p_- - 1} dx \le k_- r_-^{-p_-} \int_0^{r_-} \int_{1/x}^{\infty} t^{-\alpha-1} dt\, x^{p_- -1} dx$$
$$= k_- r_-^{-p_-} \int_0^{r_-} \frac{x^{\alpha+p_- -1}}{\alpha} dx = \frac{k_- r_-^{\alpha}}{\alpha(\alpha+p_-)},$$

Thus, if $-r_-^{-1} \le \theta \le r_+^{-1}$ then $\int_{|x|>1} e^{\theta x} M(dx) < \infty$.

Conversely, if $\theta > r_+^{-1}$ then $\theta x - 1 > r_+^{-1} x - 1 > 0$ for all $x \in (0, r_+)$, so there is ϵ such that $0 < \epsilon < r_+^{-1} x - 1$ for all $h \in (0, r_+)$. Hence

$$k_+ r_+^{-p_+} \int_0^{r_+} \int_{1/x}^{\infty} e^{t(\theta x - 1)} t^{-\alpha-1} dt\, x^{p_+ -1} dx > k_+ r_+^{-p_+} \int_0^{r_+} \int_{1/x}^{\infty} e^{\epsilon t} t^{-\alpha-1} dt\, x^{p_+ -1} dx$$
$$= \infty.$$

Similarly, we can prove that, if $\theta < -r_-^{-1}$ then

$$k_- r_-^{-p_-} \int_0^{r_-} \int_{1/x}^{\infty} e^{t(-\theta x - 1)} t^{-\alpha-1} dt\, x^{p_- -1} dx = \infty.$$

□

Lemma 3.3. *Let* $\alpha \in (0,2)$, $p \in (-\alpha, \infty) \setminus \{-1, 0\}$, $h > 0$, *and* $u \in \mathbb{R}$. *Then we have , if* $\alpha \neq 1$,

$$\int_0^h x^{p-1} (1 - iux)^{\alpha} dx = \frac{h^p}{p} F(p, -\alpha; 1+p; iuh) \tag{3.4}$$

and , if $\alpha = 1$,

(3.5)
$$\int_0^h ((1-iux)\log(1-iux) + iux)\, x^{p-1} dx$$
$$= h^p \Big(\frac{ihu}{1+p} + \frac{hu}{2+3p+p^2} (huF(2+p,1;3+p;ihu) - i(2+p)\log(1-ihu))$$
$$+ \frac{(ihu)^{-p}}{p} ((p-ihu)F_{3,2}(1,1,1-p;2,2;1-ihu) - (1-(ihu)^p)\log(1-ihu)) \Big),$$

where the hypergeometric function $F(a,b;c;x)$ *and the generalized hypergeometric function* $F_{p,q}(a_1, \cdots, a_p; b_1, \cdots, b_q; x)$.

Proof. Suppose $|iux| < 1$ and $\alpha \neq 1$. Since

$$\frac{d}{du} F(a,b;c;x) = \frac{ab}{c} F(a+1, b+1; c+1; x),$$
$$\sum_{n=0}^{\infty} \frac{(p)_n (-\alpha)_n}{(p+1)_n} \frac{(iux)^n}{n!} = \sum_{n=0}^{\infty} \frac{p}{p+n} (-\alpha)_n \frac{(iux)^n}{n!}$$

and

$$\sum_{n=0}^{\infty} \frac{(p+1)_n(-\alpha)_{n+1}}{(p+1)_{n+1}} \frac{(iux)^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{(p+1)_{n-1}(-\alpha)_n}{(p+1)_n} \frac{(iux)^n}{(n-1)!}$$
$$= \sum_{n=1}^{\infty} \frac{n}{p+n}(-\alpha)_n \frac{(iux)^n}{n!}.$$

we have

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{x^p}{p}F(p,-\alpha,1+p;iux)\right) \\
&= x^{p-1}F(p,-\alpha;1+p;iux) + \frac{x^p}{p}\frac{p(-\alpha)}{1+p}F(p+1,1-\alpha;p+2;iux)iu \\
&= x^{p-1}\left(\sum_{n=0}^{\infty}\frac{(p)_n(-\alpha)_n}{(p+1)_n}\frac{(iux)^n}{n!} + \sum_{n=0}^{\infty}\frac{(p+1)_n(-\alpha)_{n+1}}{(p+1)_{n+1}}\frac{(iux)^{n+1}}{n!}\right). \\
&= x^{p-1}\left(1 + \sum_{n=1}^{\infty}(-\alpha)_n\frac{(iux)^n}{n!}\left(\frac{p}{p+n} + \frac{n}{p+n}\right)\right) \\
&= x^{p-1}\left(1 + \sum_{n=1}^{\infty}(-\alpha)_n\frac{(iux)^n}{n!}\right) \\
&= x^{p-1}(1-iux)^{\alpha}.
\end{aligned}$$

Hence, (3.4) is proved if $|iux| < 1$ and this result can be extended analytically if $-1 < \mathrm{Re}(iux) < 1$, so (3.4) is true for all real u. Equation (3.5) can be proved by the same method. □

Theorem 3.4. *Let X be a random variable with the proper $T\alpha S$ distribution corresponding to the spectral measure R defined in (3.2) with conditions $p \neq 0$ and $p \neq -1$, and let $m = E[X]$. Then the characteristic function $E[e^{iuX}]$, $u \in \mathbb{R}$, is given as follows:*
i) if $\alpha \neq 1$,

$$(3.6) \qquad E[e^{iuX}] = \exp\Bigg[H_\alpha(u;k_+,r_+,p_+) + H_\alpha(-u;k_-,r_-,p_-) + iu\left(m + \alpha\Gamma(-\alpha)\left(\frac{k_+r_+}{p_++1} - \frac{k_-r_-}{p_-+1}\right)\right)\Bigg],$$

where

$$H_\alpha(u;a,h,p) = \frac{a\Gamma(-\alpha)}{p}\left(F(p,-\alpha;1+p;ihu) - 1\right),$$

ii) if $\alpha = 1$,

$$(3.7) \qquad E[e^{iuX}] = \exp\Bigg[G_\alpha(u;k_+,r_+,p_+) + G_\alpha(-u;k_-,r_-,p_-) + iu\left(m + \left(\frac{k_+r_+}{p_++1} - \frac{k_-r_-}{p_-+1}\right)\right)\Bigg],$$

where

$$G_\alpha(u;a,h,p) = \frac{ahu}{2+3p+p^2}(huF(2+p,1;3+p;ihu) - i(2+p)\log(1-ihu))$$
$$+ \frac{a(ihu)^{-p}}{p}((p-ihu)F_{3,2}(1,1,1-p;2,2;1-ihu) - (1-(ihu)^p)\log(1-ihu).$$

Proof. By Lemma 3.1 (vi), $m \equiv E[X] < \infty$. By Definition 2.3, we have

$$\log E[e^{iuX}] = \begin{cases} \int_{\mathbb{R}} \Gamma(-\alpha)((1-iux)^\alpha - 1 + i\alpha ux)R(dx) + imu & \text{if} \quad \alpha \neq 1 \\ \int_{\mathbb{R}} ((1-iux)\log(1-iux) + iux)R(dx) + imu & \text{if} \quad \alpha = 1 \end{cases}$$

In case $\alpha \neq 1$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \Gamma(-\alpha)((1-iux)^\alpha - 1 + i\alpha ux)R(dx) + imu \\ &= k_+ r_+^{-p_+}\Gamma(-\alpha)\int_0^{r_+}((1-iux)^\alpha - 1 - i\alpha ux)x^{p_+-1}dx \\ &\quad + k_- r_-^{-p_-}\Gamma(-\alpha)\int_0^{r_-}((1+iux)^\alpha - 1 + i\alpha ux)x^{p_--1}dx + imu. \end{aligned}$$

By (3.4), (3.6) is obtained. Similarly, In case $\alpha = 1$, we have

$$\begin{aligned} &\int_{\mathbb{R}} ((1-iux)\log(1-iux) + iux)R(dx) + imu \\ &= k_+ r_+^{-p_+}\int_0^{r_+}((1-iux)\log(1-iux) + iux)x^{p_+-1}dx \\ &\quad + k_- r_-^{-p_-}\int_0^{r_-}((1+iux)\log(1+iux) - iux)x^{p_--1}dx + imu, \end{aligned}$$

and by (3.5), (3.7) is obtained. □

Now, let's define the KR distribution.

Definition 3.5. *Let $\alpha \in (0,2)$, , $k_+, k_-, r_+, r_- > 0$, $p_+, p_- \in (-\alpha, \infty) \setminus \{-1, 0\}$, and $m \in \mathbb{R}$. A tempered stable distribution is said to be the KR Tempered Stable distribution (or KR distribution) with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) if its characteristic function is given by equations (3.6) and (3.7). If a random variable X follows the KR distribution then we denote $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$.*

The cumulants of the KR distribution can be obtained using the following Lemma.

Lemma 3.6. *Let* $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ *and* $\alpha \neq 1$. *Then we have*

$$
\begin{aligned}
(3.8)\qquad & \frac{d^n}{du^n} \log E[e^{iuX}] \\
&= \Gamma(n-\alpha)\Big(\frac{k_+ i^n r_+^n}{p_+ + n} F(p_+ + n, k-\alpha; p_+ + n + 1; iur_+) \\
&\qquad + \frac{k_-(-i)^k r_-^n}{p_- + n} F(p_- + n, k-\alpha; p_- + n + 1; -iur_-)\Big) \\
&\quad + i\left(b + \alpha\Gamma(-\alpha)\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1}\right)\right) I_{\{1\}}(n).
\end{aligned}
$$

Proof. Since

$$\frac{d^n}{du^n} F(a, b; c; x) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; x),$$

and

$$\Gamma(-\alpha)(-\alpha)_n = \Gamma(-\alpha)\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)} = \Gamma(-\alpha+n),$$

we have

$$
\begin{aligned}
& \frac{d^n}{du^n} \frac{\Gamma(-\alpha)k_\pm}{p} F(p_\pm, -\alpha; 1 + p_\pm; iuh_\pm) \\
&= \frac{k_\pm \Gamma(-\alpha) i^n h_\pm^n}{p_\pm} \frac{(p_\pm)_n (-\alpha)_n}{(p_\pm + 1)_n} F(p_\pm + n, n-\alpha; p_\pm + n + 1; iuh_\pm) \\
&= \frac{k_\pm \Gamma(-\alpha)(-\alpha)_n i^k h_\pm^k}{p_\pm + n} F(p_\pm + n, n-\alpha; p_\pm + n + 1; iuh_\pm) \\
&= \frac{k_\pm \Gamma(n-\alpha) i^n h_\pm^n}{p_\pm + n} F(p_\pm + n, n-\alpha; p_\pm + n + 1; iuh_\pm).
\end{aligned}
$$

Thus, (3.8) can be shown. □

Proposition 3.7. *Let* $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ *with* $\alpha \neq 1$. *Then the cumulants* $c_k(X) \equiv \frac{1}{i^k}\frac{d^k}{du^k} \log E[e^{iuX}]\big|_{u=0}$ *is given by* $c_1(X) = b$ *and*

$$c_k(X) = \Gamma(k-\alpha)\left(\frac{k_+ r_+^k}{p_+ + k} + (-1)^k \frac{k_- r_-^k}{p_- + k}\right)$$

where $k \geq 2$.

Remark 3.8. *Let* $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ *with* $\alpha \neq 1$. *By the Corollary 3.7, we obtain the mean, variance, skewness and excess kurtosis of* X *which are given as follows :*

1. $E[X] = c_1(X) = m$

2. $\mathrm{Var}(X) = c_2(X) = \Gamma(2-\alpha)\left(\frac{k_+ r_+^2}{p_+ + 2} + \frac{k_- r_-^2}{p_- + 2}\right)$

3. $s(X) = \frac{c_3(X)}{c_2(X)^{3/2}} = \frac{\Gamma(3-\alpha)\left(\frac{k_+ r_+^3}{p_+ + 3} - \frac{k_- r_-^3}{p_- + 3}\right)}{\Gamma(2-\alpha)^{3/2}\left(\frac{k_+ r_+^2}{p_+ + 2} + \frac{k_- r_-^2}{p_- + 2}\right)^{3/2}}$

4. $k(X) = \frac{c_4(X)}{c_2(X)^2} = \frac{\Gamma(4-\alpha)\left(\frac{k_+ r_+^4}{p_+ + 4} + \frac{k_- r_-^4}{p_- + 4}\right)}{\Gamma(2-\alpha)^2\left(\frac{k_+ r_+^2}{p_+ + 2} + \frac{k_- r_-^2}{p_- + 2}\right)^2}$

The CGMY distribution is a particular case of the KR distribution.

Proposition 3.9. *The KR distribution with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) converges weakly to the CGMY distribution as $p_\pm \to \infty$ provided that $\alpha \neq 1$ and $k_\pm = c(\alpha + p_\pm) r_\pm^{-\alpha}$ for $c > 0$.*

Proof. By the Lévy theorem, it suffices to prove the convergence of the characteristic function. We have

$$\begin{aligned}
&\lim_{p_+ \to \infty} \frac{k_+ \Gamma(-\alpha)}{p_+} \left(F(p_+, -\alpha; 1 + p_+; i r_+ u) - 1\right) \\
&= c\Gamma(-\alpha) r_+^{-\alpha} \lim_{p_+ \to \infty} \frac{\alpha + p_+}{p_+} \sum_{n=1}^{\infty} \frac{(p_+)_n (-\alpha)_n}{(1 + p_+)_n} \frac{(iur_+)^n}{n!} \\
&= c\Gamma(-\alpha) r_+^{-\alpha} \lim_{p_+ \to \infty} \sum_{n=1}^{\infty} \frac{(\alpha + p_+)(-\alpha)_n}{p_+ + n} \frac{(iur_+)^n}{n!} \\
&= c\Gamma(-\alpha) r_+^{-\alpha} \sum_{n=1}^{\infty} (-\alpha)_n \frac{(iur_+)^n}{n!} \\
&= c\Gamma(-\alpha) r_+^{-\alpha} \sum_{n=1}^{\infty} \binom{\alpha}{n} (-iur_+)^n \\
&= c\Gamma(-\alpha) r_+^{-\alpha} \left((1 - iur_+)^\alpha - 1\right) \\
&= c\Gamma(-\alpha) \left((r_+^{-1} - iu)^\alpha - r_+^{-\alpha}\right).
\end{aligned}$$

Similarly, we have

$$\lim_{p_- \to \infty} \frac{k_- \Gamma(-\alpha)}{p_-} \left(F(p_-, -\alpha; 1 + p_-; -i r_- u) - 1\right) = c\Gamma(-\alpha) \left((r_-^{-1} + iu)^\alpha - r_-^{-\alpha}\right).$$

Moreover, we have

$$\begin{aligned}
\mu &\equiv m + \lim_{p_+ \to \infty} \alpha\Gamma(-\alpha) \frac{k_+ r_+}{p_+ + 1} - \lim_{p_- \to \infty} \alpha\Gamma(-\alpha) \frac{k_- r_-}{p_- + 1} \\
&= m + \lim_{p_+ \to \infty} \alpha\Gamma(-\alpha) \frac{c(\alpha + p_+) r_+^{1-\alpha}}{p_+ + 1} - \lim_{p_- \to \infty} \alpha\Gamma(-\alpha) \frac{c(\alpha + p_-) r_-^{1-\alpha}}{p_- + 1} \\
&= m + c\alpha\Gamma(-\alpha)(r_+^{1-\alpha} - r_-^{1-\alpha}).
\end{aligned}$$

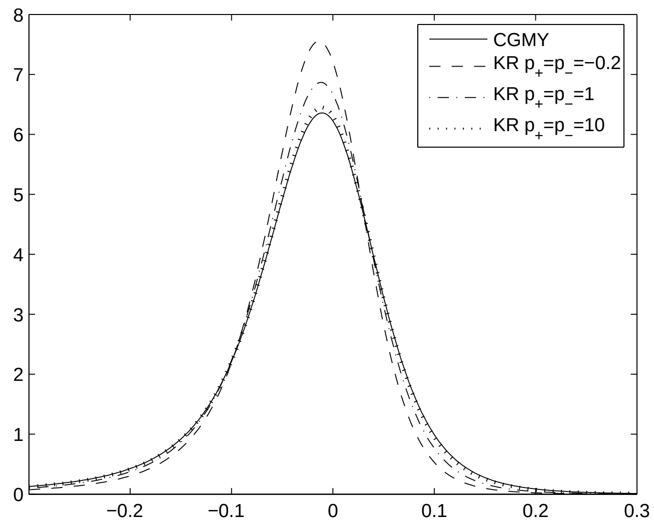

Figure 1: Probability density of the CGMY distribution with parameters $C = 0.01$, $G = 2$, $M = 10$, $Y = 1.25$, and the KR distributions with $\alpha = Y$, $k_\pm = C(Y+p)r_\pm^{-\alpha}$, $r_+ = 1/M$, $r_- = 1/G$, where $p = p_+ = p_- \in \{\ -0.25,\ 1,\ 10\ \}$.

In all, we have

$$\begin{aligned}&\lim_{p_+,p_-\to\infty} E[e^{iuX}]\\ &= \exp\left(i\mu u + c\Gamma(-\alpha)\left(\left((r_+^{-1}-iu)^\alpha - r_+^{-\alpha}\right) + \left((r_-^{-1}+iu)^\alpha - r_-^{-\alpha}\right)\right)\right).\end{aligned}$$

where $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$. That completes the proof. □

Figure 1 shows that the KR distributions converge to the CGMY distribution when parameter $p = p_+ = p_-$ increases.

Definition 3.10. *Let $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ with $\alpha \neq 1$. If the parameters satisfies $m = 0$ and*

$$k_+ = \frac{p_+ + 2}{b\Gamma(2-\alpha)r_+^2}, \qquad k_- = \frac{p_- + 2}{(1-b)\Gamma(2-\alpha)r_-^2},$$

then X is said to be standard KR tempered stable distributed (or standard KR distributed) and denote $X \sim \mathrm{StdKR}(\alpha, r_+, r_-, p_+, p_-, b)$.

Since the KR distribution is infinitely divisible, we can define a Lévy process.

Definition 3.11. *A Lévy process $X = (X_t)_{t\geq 0}$ is said to be a* KR tempered stable process *(or a* KR process*) with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) if $X_1 \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$.*

Proposition 3.12. *The process* $(X_t)_{t\geq 0} \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ *has finite variation if* $\alpha \in (0,1)$ *and infinite variation if* $\alpha \in [1,2)$.

Proof. We have

$$\begin{aligned}\int_{|x|<1} |x| M(dx) &= k_+ r_+^{-p_+} \int_0^{r_+} \int_0^\infty tx I_{(0,1)}(tx) t^{-\alpha-1} e^{-t} dt\, x^{p_+-1} dx \\ &\quad + k_- r_-^{-p_-} \int_0^{r_-} \int_0^\infty (-tx) I_{(-1,0)}(-tx) t^{-\alpha-1} e^{-t} dt\, x^{p_--1} dx \\ &= k_+ r_+^{-p_+} \int_0^{r_+} \int_0^{1/x} t^{-\alpha} e^{-t} dt\, x^{p_+} dx \\ &\quad - k_- r_-^{-p_-} \int_0^{r_-} \int_0^{1/x} t^{-\alpha} e^{-t} dt\, x^{p_-} dx.\end{aligned}$$

For fixed $x > 0$, if $\alpha \in (0,1)$ then

$$\int_0^{1/x} t^{-\alpha} e^{-t} dt \leq \int_0^\infty t^{-\alpha} e^{-t} dt = \Gamma(1-\alpha) < \infty,$$

and if $\alpha \in [1,2)$ then

$$\int_0^{1/x} t^{-\alpha} e^{-t} dt = \infty.$$

Thus

$$\int_{|x|<1} |x| M(dx) \begin{cases} < \infty & \text{if} \quad \alpha \in (0,1) \\ = \infty & \text{if} \quad \alpha \in [1,2) \end{cases}$$

□

3.1 Tail Behavior

In this section, we will discuss the probability tails of the KR distribution. Although the exact asymptotic behavior of its tails is difficult to obtain unlike those of the stable distribution, it is possible to calculate the upper and lower bounds.

In the following, we provide an upper bound for the probability tails by mean of the well-known Chebyshev's Inequality.

Proposition 3.13. *Let be* X *a random variable with KR tempered stable distribution,* $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ *with* $\alpha \neq 1$. *Then the following inequality is fulfilled*

$$\mathbb{P}(|X - m| \geq \lambda) \leq \frac{C}{\lambda^2}$$

where C *does not depend on* λ.

Proof. By Remark 3.8, X has mean and variance, therefore we consider the Chebyshev's Inequality

$$\mathbb{P}(|X-m|\geq\lambda)\leq\frac{1}{\lambda^2}\mathrm{Var}(X).$$

We obtain

$$\mathbb{P}(|X-m|\geq\lambda)\leq\frac{1}{\lambda^2}\Gamma(2-\alpha)\left(\frac{k_+r_+^2}{p_++2}+\frac{k_-r_-^2}{p_-+2}\right)$$

and the result is proved. □

A natural further interest is in a lower bound of the probability tails. By following the approach of [13], below we will give a lower bound. We consider the following result:

Proposition 3.14. *Let X be an infinitely divisible random variable in $\mathbb{R}$, with Lévy triplet $(b,0,M(dx))$. Then we have*

$$\mathbb{P}(|X-m|\geq\lambda)\geq\frac{1}{4}(1-\exp(-M(u\in\mathbb{R}:|u|\geq2\lambda))),\quad\lambda>0. \tag{3.9}$$

for all $m\in\mathbb{R}$.

Proof. See Lemma 5.4 of [4]. □

For further analysis, we need an auxiliary result.

Lemma 3.15. *For $a\in\mathbb{R}_+$, the following equality holds*

$$\int_\beta^\infty s^{-a-1}e^{-s}ds=\beta^{-a-1}e^{-\beta}+o(\beta^{-a-1}e^{-\lambda})$$

as $\beta\to\infty$.

Proof. By integration by parts, if $\beta>0$, we obtain

$$\int_\beta^\infty s^{-a-1}e^{-s}ds=\beta^{-a-1}e^{-\beta}-(a+1)\int_\beta^\infty s^{-a-2}e^{-s}ds\leq\beta^{-a-1}e^{-\beta}$$

and

$$\begin{aligned}\int_\beta^\infty s^{-a-1}e^{-s}ds&=\beta^{-a-1}e^{-\beta}-(a+1)\beta^{-a-2}e^{-\beta}+(a+1)(a+2)\int_\beta^\infty s^{-a-3}e^{-s}ds\\&\geq\beta^{-a-1}e^{-\beta}-(a+1)\beta^{-a-2}e^{-\beta},\end{aligned}$$

when $\beta\to\infty$, the result is proved. □

Taking into account Proposition 3.14 and Lemma 3.15, we can prove the following result.

Proposition 3.16. *Let be X a random variable with KR tempered stable distribution, $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$ with $\alpha \neq 1$. Then the following inequality is fulfilled*

$$\mathbb{P}(|X - m| \geq \lambda) \geq C \frac{e^{-\frac{2\lambda}{\bar{r}}}}{\lambda^{\alpha+2}}$$

as $\lambda \to \infty$, where C does not depend on λ and $\bar{r} = \max(r_+, r_-)$.

Proof. Applying the following elementary fact

$$1 - \exp(-z) \sim z, \quad z \to 0$$

and according to (3.9) and Lemma 3.15, we obtain

$$\mathbb{P}(|X - m| \geq \lambda) \geq \frac{1}{4}\left(1 - \exp\left[-\int_{\mathbb{R}_0}\int_{\frac{2\lambda}{|x|}}^{\infty} s^{-\alpha-1} e^{-s} ds R(dx)\right]\right) \tag{3.10}$$

$$\sim \frac{\lambda^{-\alpha-1}}{2^{\alpha+3}} \int_{\mathbb{R}_0} |x|^{\alpha+1} e^{-\frac{2\lambda}{|x|}} R(dx), \tag{3.11}$$

as $\lambda \to \infty$. By using equality (3.2) and Lemma 3.15, the integral can be written as

$$\begin{aligned}
\int_{\mathbb{R}_0} |x|^{\alpha+1} e^{-\frac{2\lambda}{|x|}} R(dx) &= k_+ r_+^{-p_+} \int_0^{r_+} x^{\alpha+p_+} e^{-\frac{2\lambda}{x}} dx + k_- r_-^{-p_-} \int_0^{r_-} x^{\alpha+p_-} e^{-\frac{2\lambda}{x}} dx \\
&= (2\lambda)^{\alpha+p_++1} k_+ r_+^{-p_+} \int_{\frac{2\lambda}{r_+}}^{\infty} t^{-\alpha-p_+-2} e^{-t} dt \\
&\quad + (2\lambda)^{\alpha+p_-+1} k_- r_-^{-p_-} \int_{\frac{2\lambda}{r_-}}^{\infty} t^{-\alpha-p_--2} e^{-t} dt \\
&\sim (2\lambda)^{-1} \frac{k_+}{r_+^{\alpha+p_++2}} e^{-\frac{2\lambda}{r_+}} + (2\lambda)^{-1} \frac{k_-}{r_-^{\alpha+p_-+2}} e^{-\frac{2\lambda}{r_-}} \\
&\sim \bar{C} (2\lambda)^{-1} e^{-\frac{2\lambda}{\bar{r}}}
\end{aligned}$$

as $\lambda \to 0$, where $\bar{r} = \max(r_+, r_-)$. Combining this with (3.10), we get

$$\mathbb{P}(|X - m| \geq \lambda) \geq C \frac{e^{-\frac{2\lambda}{\bar{r}}}}{\lambda^{\alpha+2}}.$$

□

3.2 Absolute Continuity

Let $(X_t)_{t\geq 0}$ be a canonical process on Ω, the set of all cadlag function on $[0, \infty)$ into $\mathbb{R}$, and consider a space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, where

$$\mathcal{F} = \sigma\{X_s; s \geq 0\}$$
$$\mathcal{F}_t = \cap_{s>t} \sigma\{X_u : u \leq s\}, \quad t \geq 0.$$

Theorem 3.17. *Consider two probability measures $\mathbb{P}_1$, $\mathbb{P}_2$ and the canonical process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ given above. For each $j = 1, 2$, suppose $(X_t)_{t\geq 0}$ is the KR tempered stable process under $\mathbb{P}_j$ with parameters (α_j, $k_{j,+}$, $k_{j,-}$, $r_{j,+}$, $r_{j,-}$, $p_{j,+}$, $p_{j,-}$, m_j) and*

$$\begin{cases} p_{j,\pm} > \frac{1}{2} - \alpha_j, & \alpha_j \in (0,1) \\ p_{j,\pm} > 1 - \alpha_j, & \alpha_j \in [1,2) \end{cases}.$$

Then $\mathbb{P}_1|_{\mathcal{F}_t}$ and $\mathbb{P}_2|_{\mathcal{F}_t}$ are equivalent for every $t > 0$ if and only if

$$\alpha := \alpha_1 = \alpha_2, \tag{3.12}$$

$$\frac{k_{1,+}r_{1,+}^{\alpha}}{\alpha + p_{1,+}} = \frac{k_{2,+}r_{2,+}^{\alpha}}{\alpha + p_{2,+}}, \qquad \frac{k_{1,-}r_{1,-}^{\alpha}}{\alpha + p_{1,+}} = \frac{k_{2,-}r_{2,-}^{\alpha}}{\alpha + p_{2,+}} \tag{3.13}$$

and

$$m_2 - m_1 \tag{3.14}$$
$$= \begin{cases} \displaystyle\sum_{j=1,2} (-1)^j \Bigg(\frac{k_{j,+}r_{j,+}}{p_{j,+}+1}\left(\log r_{j,+} - \frac{p_{j,+}+2}{p_{j,+}+1}\right) \\ \qquad - \frac{k_{j,-}r_{j,-}}{p_{j,-}+1}\left(\log r_{j,-} - \frac{p_{j,-}+2}{p_{j,-}+1}\right)\Bigg) & \text{if } \alpha = 1 \\ \displaystyle\Gamma(1-\alpha)\sum_{j=1,2}(-1)^j\left(\frac{k_{j,+}r_{j,+}}{p_{j,+}+1} - \frac{k_{j,-}r_{j,-}}{p_{j,-}+1}\right) & \text{if } \alpha \neq 1 \end{cases}.$$

Proof. In $\mathrm{KR}(\alpha_j, k_{j,+}, k_{j,-}, r_{j,+}, r_{j,-}, p_{j,+}, p_{j,-}, m_j)$, the spectral measure R_j is equal to

$$R_j(dx) = (k_{j,+}r_{j,+}^{-p_{j,+}} I_{x\in(0,r_{j,+})}|x|^{p_{j,+}-1} + k_{j,-}r_{j,-}^{-p_{j,-}} I_{x\in(0,r_{j,-})}|x|^{p_{j,-}-1})dx$$

and the polar coordinated Lévy measure M_j is equal to

$$M_j(dv, du) = v^{-\alpha_j - 1} q_j(v,u) dv \sigma_j(du)$$

where

$$\sigma_j(A) = \frac{k_{j,+}r_{j,+}^{\alpha_j}}{\alpha_j + p_{j,+}} 1_{1\in A} + \frac{k_{j,-}r_{j,-}^{\alpha_j}}{\alpha_j + p_{j,-}} 1_{-1\in A}, \quad A \subset S^0$$

and

$$q_j(v, \pm 1) = (\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_0^{r_{j,\pm}} e^{-v/s} s^{\alpha_j + p_{j,\pm} - 1} ds$$

By Remark 2.4, we have

$$X_1 \sim \begin{cases} TS_\alpha^0(R_j, b_j), & \alpha_j \in (0,1) \\ TS_\alpha(R_j, b_j), & \alpha_j \in [1,2) \end{cases}$$

where

$$b_j = \begin{cases} m_j - \Gamma(1-\alpha)\int_{\mathbb{R}} x R_j(dx), & \alpha_j \in (0,1) \\ m_j, & \alpha_j \in [1,2) \end{cases}$$

under $\mathbb{P}_j$. Indeed, by Lemma 3.1 iii), $E_{\mathbb{P}_j}[|X_1|] < \infty$ if $\alpha_j \in (0,2)$ and $\int_{|x|<1} |x| R_j(dx) < \infty$ if $\alpha_j \in (0,1)$.

If $p_{j,\pm} > \frac{1}{2} - \alpha_j$ then we have

$$\begin{aligned} \frac{d}{dv} q_j(v,\pm 1) &= -(\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_0^{r_{j,\pm}} e^{-v/s} s^{\alpha_j + p_{j,\pm} - 2} ds \\ &= -(\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_{1/r_{j,\pm}}^{\infty} e^{-vt} t^{-\alpha_j - p_{j,\pm}} dt \\ &\geq -(\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_{1/r_{j,\pm}}^{\infty} \frac{1}{\sqrt{vt}} t^{-\alpha_j - p_{j,\pm}} dt \\ &= -\frac{\alpha_j + p_{j,\pm}}{\sqrt{r_{j,\pm}}(\alpha_j + p_{j,\pm} - \frac{1}{2})} v^{-\frac{1}{2}}. \end{aligned}$$

If $p_{j,\pm} > 1 - \alpha_j$, then we have

$$\begin{aligned} \frac{d}{dv} q_j(v,\pm 1) &= -(\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_0^{r_{j,\pm}} e^{-v/s} s^{\alpha_j + p_{j,\pm} - 2} ds \\ &\geq -(\alpha_j + p_{j,\pm}) r_{j,\pm}^{-\alpha_j - p_{j,\pm}} \int_0^{r_{j,\pm}} s^{\alpha_j + p_{j,\pm} - 2} ds \\ &= -\frac{\alpha_j + p_{j,\pm}}{r_{j,\pm}(\alpha_j + p_{j,\pm} - 1)}. \end{aligned}$$

Let

$$K_j = \begin{cases} \min\left\{ -\frac{\alpha_j + p_{j,+}}{\sqrt{r_{j,+}}(\alpha_j + p_{j,+} - 1/2)}, -\frac{\alpha_j + p_{j,-}}{\sqrt{r_{j,-}}(\alpha_j + p_{j,-} - 1/2)} \right\}, & \alpha_j \in (0,1) \\ \min\left\{ -\frac{\alpha_j + p_{j,+}}{r_{j,+}(\alpha_j + p_{j,+} - 1)}, -\frac{\alpha_j + p_{j,-}}{r_{j,-}(\alpha_j + p_{j,-} - 1)} \right\}, & \alpha_j \in [1,2) \end{cases}$$

then

$$0 > \frac{d}{dv} q_j(v,\pm 1) \geq \begin{cases} K_j v^{-1/2}, & \alpha_j \in (0,1) \\ K_j, & \alpha_j \in [1,2) \end{cases}.$$

By the integration of the last inequality on the interval $(0,v)$, we obtain

$$0 \geq q_j(v,\pm 1) - 1 = q_j(v,\pm 1) - q_j(0,\pm 1) \geq \begin{cases} 2K_j v^{1/2}, & \alpha_j \in (0,1) \\ K_j v, & \alpha_j \in [1,2) \end{cases}.$$

Hence,

$$
\begin{aligned}
&\int_{S^0}\int_0^1 (1-q_j(v,u))^2 v^{-\alpha_j-1} dv\,\sigma(du) \\
&\le \begin{cases} \int_{S^0}\int_0^1 4K_j^2 v^{-\alpha_j} dv\,\sigma(du), & \alpha_j\in(0,1) \\ \int_{S^0}\int_0^1 K_j^2 v^{-\alpha_j+1} dv\,\sigma(du), & \alpha_j\in[1,2) \end{cases} \\
&= \begin{cases} \frac{4K_j^2}{1-\alpha_j}\int_{S^0}\sigma(du), & \alpha_j\in(0,1) \\ \frac{K_j^2}{2-\alpha_j}\int_{S^0}\sigma(du), & \alpha_j\in[1,2) \end{cases} \\
&< \infty.
\end{aligned}
$$

By Theorem 2.6, there is a measure $\mathbb{P}_j^0$ such that $\mathbb{P}_j^0|_{\mathcal{F}_t}$ and $\mathbb{P}_j|_{\mathcal{F}_t}$ are equivalent for every $t>0$ and $(X_t)_{t\ge 0}$ is an α-stable process with $X_1\sim S_{\alpha_j}(\sigma_j,a_j)$ under $\mathbb{P}_j^0$ where

$$
\begin{aligned}
a_j &= \begin{cases} b_j & \text{if } \alpha\in(0,1) \\ b_j-\int_{\mathbb{R}} x(\log|x|-1)R_j(dx) & \text{if } \alpha=1 \\ b_j-\Gamma(1-\alpha)\int_{\mathbb{R}} xR_j(dx) & \text{if } \alpha\in(1,2) \end{cases} \\
&= \begin{cases} m_j-\int_{\mathbb{R}} x(\log|x|-1)R_j(dx) & \text{if } \alpha=1 \\ m_j-\Gamma(1-\alpha)\int_{\mathbb{R}} xR_j(dx) & \text{if } \alpha\ne 1 \end{cases}.
\end{aligned}
$$

Note that, if $p>-1$ and $y>0$,

$$
\int_0^y x^p\log x\,dx = \left[\frac{x^{p+1}}{p+1}\log x\right]_0^y - \frac{1}{p+1}\int_0^y x^p dx = \frac{y^{p+1}}{p+1}\log y - \frac{y^{p+1}}{(p+1)^2},
$$

by the integration by parts. If $\alpha=1$, then $p_{j,\pm}>0$ and

$$
\begin{aligned}
&\int_{\mathbb{R}} x(\log|x|-1)R_j(dx) \\
&= k_{j,+}r_{j,+}^{-p_{j,+}}\int_0^{r_{j,+}}(\log x-1)x^{p_{j,+}}dx - k_{j,-}r_{j,-}^{-p_{j,-}}\int_0^{r_{j,-}}(\log x-1)x^{p_{j,-}}dx \\
&= \left(\frac{k_{j,+}r_{j,+}}{p_{j,+}+1}\left(\log r_{j,+}-\frac{p_{j,+}+2}{p_{j,+}+1}\right) - \frac{k_{j,-}r_{j,-}}{p_{j,-}+1}\left(\log r_{j,-}-\frac{p_{j,-}+2}{p_{j,-}+1}\right)\right),
\end{aligned}
$$

and if $\alpha\ne 1$, then

$$
\begin{aligned}
\int_{\mathbb{R}} xR_j(dx) &= k_{j,+}r_{j,+}^{-p_{j,+}}\int_0^{r_{j,+}} x^{p_{j,+}}dx + k_{j,-}r_{j,-}^{-p_{j,-}}\int_0^{r_{j,-}} x^{p_{j,-}}dx \\
&= \left(\frac{k_{j,+}r_{j,+}}{p_{j,-}+1} - \frac{k_{j,-}r_{j,-}}{p_{j,-}+1}\right)
\end{aligned}
$$

Since $\mathbb{P}_1^0|_{\mathcal{F}_t}$ and $\mathbb{P}_2^0|_{\mathcal{F}_t}$ are equivalent for every $t>0$ if and only if $\alpha_1=\alpha_2$, $\sigma_1=\sigma_2$, and $a_1=a_2$, we obtain the result that $\mathbb{P}_1|_{\mathcal{F}_t}$ and $\mathbb{P}_2|_{\mathcal{F}_t}$ are equivalent for every $t>0$ if and only if the parameters satisfy (3.12), (3.13) and (3.14). □

4 KR Tempered Stable Market Model

In the remainder of this paper, let us denote a time horizon by $T > 0$ and the risk-free rate by $r > 0$. Let Ω to be the set of all cadlag functions on $[0, T]$ into $\mathbb{R}$, and $(X_t)_{t\in[0,T]}$ is a canonical process on Ω (i.e. $X_t(\omega) = \omega(t)$, $t \in [0, T]$, $\omega \in \Omega$). Consider a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]})$ where

$$\mathcal{F}_T = \sigma\{X_s; s \in [0, T]\}$$
$$\mathcal{F}_t = \cap_{s\in(t,T]}\sigma\{X_u : u \leq s\}, t \in [0, T].$$

$(\mathcal{F}_t)_{t\in[0,T]}$ is the right continuous natural filtration. The *continuous-time market* is modeled by a probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, for some measure $\mathbb{P}$ named the *market measure*. In the market, the stock price is given by the random variable $S_t = S_0 e^{X_t}$, $t \in [0, T]$ for some initial value of the stock price $S_0 > 0$, and the discounted stock price $\tilde{S}_t$ of S_t is given by $\tilde{S}_t = e^{-rt}S_t$, $t \in [0, T]$. The processes $(S_t)_{t\in[0,T]}$ and $(\tilde{S}_t)_{t\in[0,T]}$ are called the *stock price process* and the *discounted (stock) price process*, respectively. The process $(X_t)_{t\in[0,T]}$ is called the *driving process* of $(S_t)_{t\in[0,T]}$. The driving process $(X_t)_{t\in[0,T]}$ is completely described by the market measure $\mathbb{P}$. If $(X_t)_{t\in[0,T]}$ is a Lévy process under the measure $\mathbb{P}$, we say that the stock price process follows the *exponential Lévy model*. Assume a stock buyer receives continuous dividend yield d. A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is called an *equivalent martingale measure* (EMM) of $\mathbb{P}$ if the stock price process net of the cost of carry (Lewis [18]) is a $\mathbb{Q}$-martingale; that is $E_{\mathbb{Q}}[S_t] = e^{(r-d)t}S_0$ or $E_{\mathbb{Q}}[e^{X_t}] = 1$.

Now, we intend to define the KR model. For convenience, we exclude the case $\alpha = 1$ and define a function

$$\psi_\alpha(u; k_+, k_-, r_+, r_-, p_+, p_-, m) = H_\alpha(u; k_+, r_+, p_+) + H_\alpha(-u; k_-, r_-, p_-) + iu\left(m + \alpha\Gamma(-\alpha)\left(\frac{k_+r_+}{p_+ + 1} - \frac{k_-r_-}{p_- + 1}\right)\right),$$

on $u \in \{z \in \mathbb{C} \mid -\mathrm{Im}(z) \in (-r_-^{-1}, r_+^{-1})\}$, which is same as the exponent of (3.6).

Definition 4.1. *In the above setting, if $(X_t)_{t\in[0,T]}$ is the KR process with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, m) where*

$$\alpha \in (0, 1) \cup (1, 2),$$
$$k_+, k_-, r_- \in (0, \infty),$$
$$r_+ \in (0, 1),$$
$$p_+, p_- \in (1/2 - \alpha, \infty) \setminus \{0\}, \text{ if } \alpha \in (0, 1),$$
$$p_+, p_- \in (1 - \alpha, \infty) \setminus \{0\}, \text{ if } \alpha \in (1, 2),$$

and $m = \mu - \psi_\alpha(-i; k_+, k_-, r_+, r_-, p_+, p_-, 0)$ for some $\mu \in \mathbb{R}$, then the process $(S_t)_{t\in[0,T]}$ is called the KR price process *with parameters (α, k_+, k_-, r_+, r_-, p_+, p_-, μ) and we say that the stock price process follows the* exponential KR model.

Remark 4.2.

1. *We have the condition* $r_+ \in (0,1)$ *for* $\psi_\alpha(-i; k_+, k_-, r_+, r_- p_+, p_-, 0)$ *and* $E[e^{X_t}]$ *to be well defined.*

2. *By the condition* $\begin{cases} p_+, p_- \in (1/2 - \alpha, \infty) \setminus \{0\}, & \text{if } \alpha \in (0,1) \\ p_+, p_- \in (1 - \alpha, \infty) \setminus \{0\}, & \text{if } \alpha \in (1,2) \end{cases}$, *we are able to use Theorem 3.17 for finding an equivalent measure.*

3. *Since* $m = \mu - \psi_\alpha(-i; k_+, k_-, r_+, r_-, p_+, p_-, 0)$, *we have*

$$E[S_t] = S_0 E[e^{X_t}] = S_0 e^{\mu t}.$$

Theorem 4.3. *Assume that* $(S_t)_{t\in[0,T]}$ *is the the KR price process with parameters* $(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, \mu)$ *under the market measure* $\mathbb{P}$, *and with parameters* $(\tilde{\alpha}, \tilde{a}_+, \tilde{a}_-, \tilde{r}_+, \tilde{r}_-, \tilde{p}_+, \tilde{p}_-, r - d)$ *under a measure* $\mathbb{Q}$. *Then* $\mathbb{Q}$ *is an* EMM *of* $\mathbb{P}$ *if and only if*

$$\alpha = \tilde{\alpha}, \tag{4.1}$$

$$\frac{k_+ r_+^\alpha}{\alpha + p_+} = \frac{\tilde{k}_+ \tilde{r}_+^\alpha}{\alpha + \tilde{p}_+}, \qquad \frac{k_- r_-^\alpha}{\alpha + p_-} = \frac{\tilde{k}_- \tilde{r}_-^\alpha}{\alpha + \tilde{p}_-} \tag{4.2}$$

and

$$\begin{aligned} \mu - (r - d) = {} & H_\alpha(-i; k_+, r_+, p_+) + H_\alpha(i; k_-, r_-, p_-) \\ & - H_\alpha(-i; \tilde{k}_+, \tilde{r}_+, \tilde{p}_+) - H_\alpha(i; \tilde{k}_-, \tilde{r}_-, \tilde{p}_-). \end{aligned} \tag{4.3}$$

Proof. By Definition 4.1 and Corollary 3.17, it can be proved. □

4.1 Estimation of Market Parameters

In this section, we will present the estimation results of the fit of our model to the historical log-returns of the S&P 500 Index. In order to compare the KR model with other well-known models, let us consider the normal, CGMY, and KR density fit. The CGMY process is defined in the Appendix and in [7]. In our empirical study, we focus on two sets of data. We estimated the market parameters from time-series data on the S&P 500 Index over the period January 1, 1992 to April 18, 2002, with $\tilde{n} = 2573$ closing prices (Data1), and over the period January 1, 1984 to January 1, 1994, with $\bar{n} = 2498$ closing prices (Data2). The estimation of market parameters based on Data1 will be used to extract the risk-neutral density by using observed option prices, while the historical series Data2 is selected to demonstrate the benefit of the KR distribution in fitting historical log-returns containing extreme events (*Black Monday*, October 19, 1987).

Our estimation procedure follows the classical maximum likelihood estimation (MLE) method (see Table 1). The discrete Fourier transform (DFT) is used to invert the characteristic function and evaluate the likelihood function in the CGMY and KR cases.

In order to compare how the stock market process can be explained by these different models, Figures 2 and 3 show the results of density fits.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{X_i\}_{1\leq i\leq n}$ a given set of independent and identically distributed real random variables. In the following, let us consider $X_i(\omega) = x_i$, for each $i = 1, \ldots, n$. Let F be the distribution of X_i, and $x_1 \leq x_2 \leq \ldots \leq x_n$. The empirical cumulative distribution function $\hat{F}_n(x)$ is defined by

$$\hat{F}_n(x) = \frac{\text{no. observations} \leq x}{n} = \begin{cases} 0, & x < x_1 \\ \frac{i}{n}, & x_i \leq x \leq x_{i+1},\ i = 1, \ldots, n-1 \\ 1, & x_n \leq x. \end{cases}$$

A statistic measuring the difference between $\hat{F}_n(x)$ and $F(x)$ is called the empirical distribution function (EDF) statistic [11]. These statistics include the Kolmogorov-Smirnov (KS) statistic [11, 21, 31] and Anderson-Darling (AD) statistic [1, 2, 22]. Our goal is to test if the empirical distribution function of an observed data sample belongs to a family of hypothesized distributions, i.e.

$$H_0 : F = F_0 \ \text{ vs } \ H_1 : F \neq F_0 \tag{4.4}$$

Suppose a test statistic D takes the value d, the p-value of the statistic will then be the value

$$p\text{-value} = P(D \geq d).$$

We reject the hypothesis H_0 if the p-value is less than a given level of significance, which we take to be equal to 0.05. Let us consider a test for hypotheses of the type (4.4) concerning continuous cumulative distribution function, the Kolmogorov-Smirnov test. The KS statistic D_n measures the absolute value of the maximum distance between the empirical distribution function $\hat{F}$ and the theoretical distribution function F, putting equal weight on each observation,

$$D_n = \sup_{x_i} |F(x_i) - \hat{F}_n(x_i)| \tag{4.5}$$

where $\{x_i\}_{1\leq i\leq n}$ is a given set of observations. Using the procedure of [21], we can easily evaluate the distribution of D_n and find the p-value for our test.

It might be of interest to test the ability to model to forecast extreme events. To this end, we also provide the AD statistics. We consider two different versions of the AD statistic. In its simplest version, it is a variance-weighted KS statistic

$$AD_n = \sup_{x_i} \frac{|F(x_i) - \hat{F}(x_i)|}{\sqrt{F(x_i)(1 - F(x_i))}} \tag{4.6}$$

Since the distribution of AD_n is not known in closed form, p-values were obtained via 1000 Monte Carlo simulations.

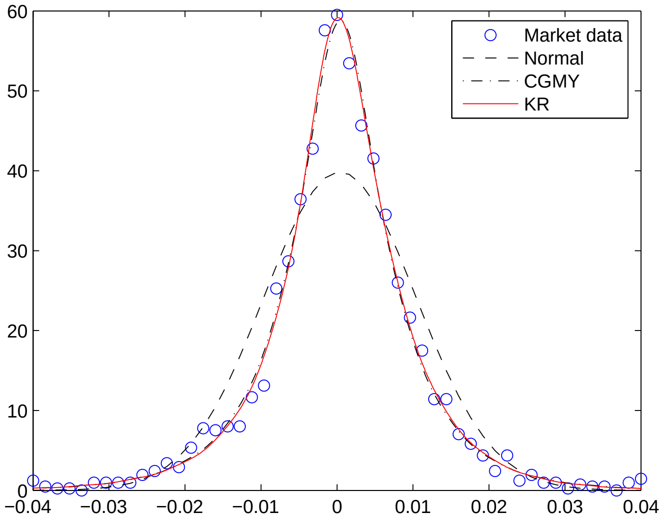

Figure 2: S&P 500 Index (from January 1, 1992 to April 18, 2002) MLE density fit. Circles are density of the market data. The solid curve is the KR fit, the dotted curve is the CGMY fit and the dashed curve is the normal fit

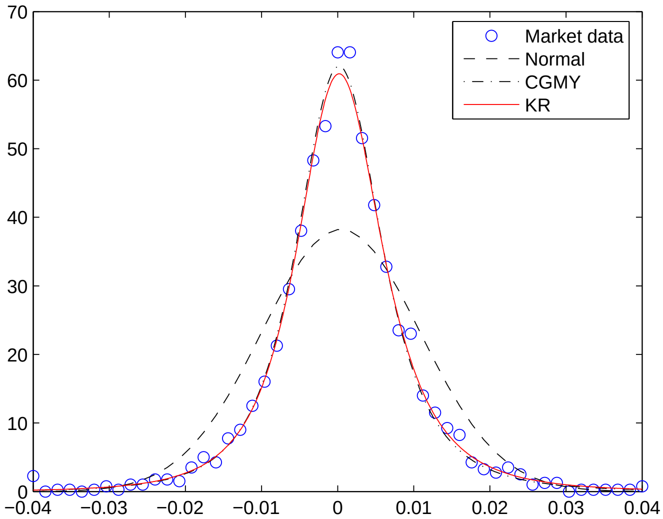

Figure 3: S&P 500 Index (from January 1, 1984 to January 1, 1994) MLE density fit. Circles are density of the market data. The solid curve is the KR fit, the dotted curve is the CGMY fit and the dashed curve is the normal fit

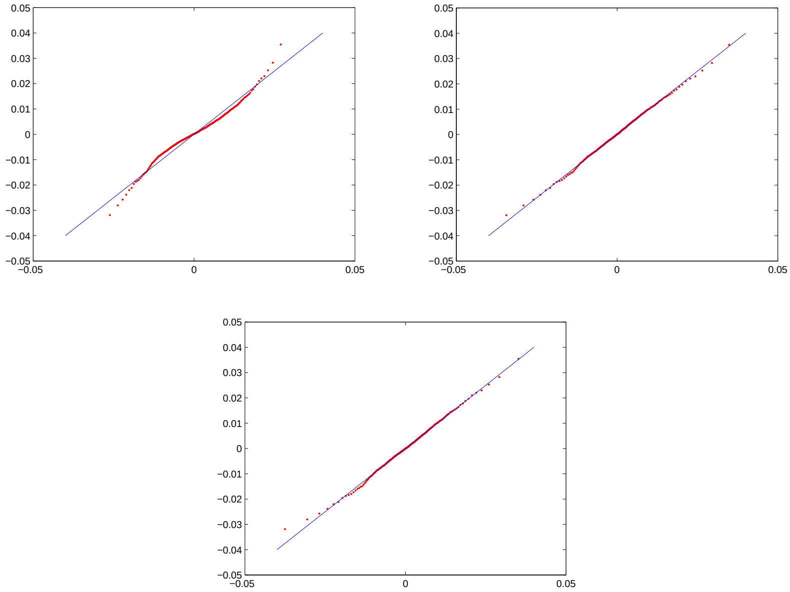

Figure 4: QQ-plots of S&P 500 Index (from January 1, 1992 to April 18, 2002) MLE density fit. Normal model (left), CGMY model (right) and KR model (down).

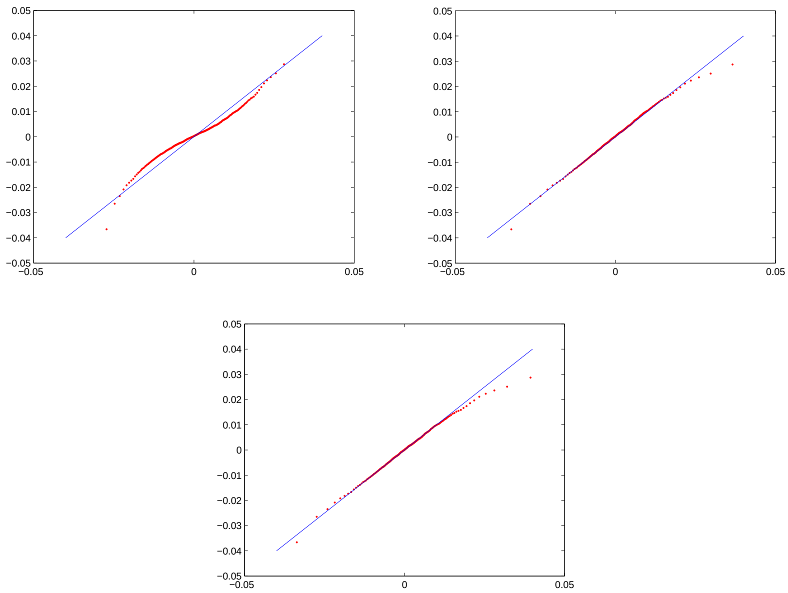

Figure 5: QQ-plots of S&P 500 Index (from January 1, 1984 to January 1, 1994) MLE density fit. Normal model (left), CGMY model (right) and KR model (down).

Table 1: S&P 500 Index MLE density fit

S&P 500 Index from January 1, 1992 to April 18, 2002

	Parameters							
	μ	σ						
Normal	0.096364	0.15756						
	C	G	M	Y	m			
CGMY	10.161	97.455	98.891	0.5634	0.1135			
	k_+	k_-	r_+	r_-	p_+	p_-	α	μ
KR	3286.1	2124.8	0.0090	0.0113	17.736	17.736	0.5103	0.1252

S&P 500 Index from January 1, 1984 to January 1, 1994

	Parameters							
	μ	σ						
Normal	0.11644	0.15008						
	C	G	M	Y	m			
CGMY	0.41077	59.078	49.663	1.0781	0.1274			
	k_+	k_-	r_+	r_-	p_+	p_-	α	μ
KR	598.38	694.71	0.0222	0.0183	20.662	20.662	1.0416	0.1840

A more generally used version of this statistic belongs to the quadratic class defined by the Cramér-von Mises family [11], i.e.

$$AD_n^2 = n \int_{-\infty}^{\infty} \frac{(\hat{F}_n(x) - F(x))^2}{F(x)(1 - F(x))} dF(x) \tag{4.7}$$

and by the Probability Integral Transformation (PIT) formula [11], we obtain the computing formula for the AD_n^2 statistic

$$AD_n^2 = -n + \frac{1}{n}\sum_{i=1}^{n}(1-2i)\log(z_i) - \frac{1}{n}\sum_{i=1}^{n}(1+2(n-i))\log(1-z_i)$$

where z_i is $z_i = F(x_i)$, with $i = 1, \ldots, n$. To evaluate the distribution of the AD_n^2 statistic, we use the procedure described in [22]. As in the KS case, the distribution of AD_n^2 does not depend on F. Results of our tests are shown in Tables 2 and 3. Following the approach of [21, 22], p-values can be obtained with a computational time much less than Monte Carlo simulations.

A parametric procedure for testing the goodness of fit is the χ^2-test. We define the null hypotheses as follows:

H_0^{normal}: The daily returns follow the normal distribution.

H_0^{CGMY}: The daily returns follow the CGMY distribution.

H_0^{KR}: The daily returns follow the KR distribution.

Let us consider a partition $\mathcal{P} = \{A_1, \ldots, A_m\}$ of the support of our distribution. Let N_k, with $k = 1, \ldots, m$, be the number of observations x_i falling into the

Table 2: χ^2, KS, AD and AD^2 statistics (degrees of freedom in round brackets).

S&P 500 Index from January 1, 1992 to April 18, 2002				
Model	χ^2	KS	AD	AD^2
Normal	546.49(288)	0.0663	2180.7	23.762
CGMY	273.4(255)	0.0103	0.2945	0.6130
KR	268.91(252)	0.0109	0.2315	0.3367

p-value							
	Theoretical$^\sharp$			Monte Carlo‡			
Model	χ^2	KS	AD^2	χ^2	KS	AD	AD^2
Normal	0	0	0	0	0	0	0
CGMY	0.2045	0.9450	0.6356	0.43	0.908	0.098	0.656
KR	0.2216	0.9165	0.9082	0.53	0.875	0.242	0.916

$^\sharp$Theoretical p-values were obtained from [21, 22] and χ^2 distribution.

‡Monte Carlo p-values were obtained via 1000 simulations.

interval A_k. We will compare these numbers with the theoretical frequency distribution π_k, defined by

$$\pi_k = P(X \in A_k) \;\; k = 1, \dots, m$$

through the Pearson statistic

$$\hat{\chi}^2 = \sum_{k=1}^{m} \frac{N_k - n\pi_k}{n\pi_k}.$$

If necessary, we collapse outer cells A_k, so that the expected value $n\pi_k$ of the observations always becomes greater than 5 [30].

From the results reported in Tables 2 and 3, we conclude that H_0^{normal} is rejected but H_0^{CGMY} and H_0^{KR} are not rejected. QQ-plots (see Figures 4 and 5) show that the empirical density strongly deviated from the theoretical density for the normal model, but this deviation almost disappears in both the CGMY and KR cases.

4.2 Estimation of Risk Neutral Parameters

In this section, we will discuss a parametric approach to risk-neutral density extraction from option prices based on knowledge of the estimated historical density. Therefore, taking into account the estimation results of Section 4.1 under the market probability measure, we want to estimate parameters under a risk-neutral measure.

Let us consider a given market model and observed prices $\hat{C}_i$ of call options with maturities T_i and strikes K_i, $i \in \{1, \dots, N\}$, where N is the number of options on a fixed day. The risk-neutral process is fitted by matching model prices to market prices using nonlinear least squares. Hence, to obtain a practical solution to the calibration problem, our purpose is to find a parameter set

Table 3: χ^2, KS, AD and AD^2 statistics (degrees of freedom in round brackets).

S&P 500 Index from January 1, 1984 to January 1, 1994

Model	χ^2	KS	AD	AD^2
Normal	482.39(202)	0.0699	3.9e+6	33.654
CGMY	191.68(179)	0.0191	0.1527	2.0475
KR	180.07(181)	0.0107	0.1302	0.9719

p-value

	Theoretical$^\sharp$			Monte Carlo‡			
Model	χ^2	KS	AD^2	χ^2	KS	AD	AD^2
Normal	0	0	0	0	0	0	0
CGMY	0.2451	0.3180	0.0865	0.893	0.305	0.696	0.086
KR	0.5055	0.9343	0.3723	0.974	0.875	0.872	0.361

$^\sharp$Theoretical p-values were obtained from [21, 22] and χ^2 distribution.

‡Monte Carlo p-values were obtained via 1000 simulations.

$\tilde{\theta}$, such that the optimization problem

$$\min_{\tilde{\theta}} \sum_{i=1}^{N} (\hat{C}_i - C^{\tilde{\theta}}(T_i, K_i))^2 \tag{4.8}$$

is solved, where by $\hat{C}_i$ we denote the price of an option as observed in the market and by $C_i^{\tilde{\theta}}$ the price computed according to a pricing formula in a chosen model with a parameter set $\tilde{\theta}$.

By Proposition 3.9, we obtain that the KR model is an extension of the CGMY model. Therefore, to demonstrate the advantages of the KR tempered stable distribution model, we will compare it with the well-known CGMY model. To find an equivalent change of measure in the CGMY model, we consider the result reported in the Appendix.

By Proposition A.2, we can consider the historical estimation for parameters $\tilde{Y}$ and $\tilde{C}$ and find a solution to the minimization problem (4.8) which satisfies condition (A.1). Therefore, we can estimate parameters $\tilde{M}$ and $\tilde{G}$ under a risk-neutral measure. The optimization procedure involves 4 parameters except r and 3 equality constraints. Consequently we have only one free parameter to solve (4.8).

If we consider the KR exponential model, according to Definition 4.1 and Proposition 4.3 , we can find parameters $\tilde{k}_+$, $\tilde{k}_-$, $\tilde{r}_+$ and $\tilde{r}_-$, such that conditions (4.1), (4.2), and (4.3) are satisfied and (4.8) is solved. We have 7 parameters except r and 4 equality constraints, namely 3 free parameters to minimize (4.8), i.e.

$$\alpha = \tilde{\alpha},$$

$$\tilde{p}_+ = \frac{\tilde{k}_+\tilde{r}_+^\alpha}{k_+r_+^\alpha}(\alpha + p_+) - \alpha,$$
$$\tilde{p}_- = \frac{\tilde{k}_-\tilde{r}_-^\alpha}{k_-r_-^\alpha}(\alpha - p_-) - \alpha$$

and

$$\begin{aligned}\mu - r = {} & H_\alpha(-i; k_+, r_+, p_+) + H_\alpha(i; k_-, r_-, p_-) \\ & - H_\alpha(-i; \tilde{k}_+, \tilde{r}_+, \tilde{p}_+) - H_\alpha(i; \tilde{k}_-, \tilde{r}_-, \tilde{p}_-).\end{aligned}$$

In the CGMY case we have only one free parameter but in the KR case we have 3 free parameters to fit model prices to market prices; therefore, we can obtain a better solution to the optimization problem. The KR distribution is more flexible in order to find an equivalent change of measure and, at the same time, takes into account the historical estimates.

The time-series data were for the period January 1, 1992 to April 18, 2002, while the option data were April 18, 2002.

Contrary to the classical Black-Scholes case, in the exponential-Levý models there is no explicit formula for call option prices, since the probability density of a Levý process is typically not known in closed form. Due to the easy form of the characteristic functions of the CGMY and KR distributions, we follow the generally used pricing method for standard vanilla options, which can be applied in general when the characteristic function of the risk-neutral stock-price process is known [8, 30]. Let ρ be a positive constant such that the ρ-th moment of the price exists and ϕ the characteristic function of the random variable $\log S_T$. A value of $\rho = 0.75$ will typically do fine [30]. Carr and Madan [8, 30] then showed that

$$C(K,T) = \frac{\exp(-\rho \log K)}{\pi} \int_0^\infty \exp(-iv \log K)\varrho(v)dv,$$

where

$$\varrho(v) = \frac{\exp(-rT)\phi(v - (\rho+1)i)}{\rho^2 + \rho - v^2 + i(2\rho+1)v}$$

Furthermore, we need to guarantee the analyticity of the integrand function in the horizontal strip of the complex plane, on which the line $L_\rho = \{x + i\rho \in \mathbb{C} | -\infty < x < \infty\}$ lies [18, 19]. If we consider the exponential KR model, we obtain the following additional inequality constraint,

$$r_+^{-1} \geq 1 + \rho,$$

by Proposition 3.2. Since α is less than 1 in the estimated market parameter for the given time-series data, we have to consider an additional condition

$$p_+,\ p_- \in (1/2 - \alpha, \infty),$$

by Remark 4.2.

Table 4: Estimated Risk-Neutral Parameters

	CGMY		KR			
T	$\tilde{M}$	$\tilde{G}$	$\tilde{k}_+$	$\tilde{k}_-$	$\tilde{r}_+$	$\tilde{r}_-$
0.0880	106.5827	96.1341	5325.8	33.727	0.0065	0.0330
0.1840	103.4463	93.3887	9126.3	33.024	0.0066	0.034
0.4360	92.4701	83.7430	4757.3	31.327	0.0074	0.0381
0.6920	89.4576	81.0851	3866.4	30.776	0.0076	0.0395
0.9360	90.0040	81.5675	6655.4	30.78	0.0075	0.03953
1.1920	82.6216	75.0354	9896.7	29.483	0.0079	0.0430
1.7080	77.3594	70.3609	10000	28.468	0.0084	0.046

Table 5: Error Estimators

T	Model	APE	AEE	RMSE	ARPE
0.0880					
	CGMY	0.0149	0.4019	0.4613	0.0175
	KR	0.0030	0.0826	0.1023	0.0035
0.1840					
	CGMY	0.0341	1.0998	1.4270	0.0442
	KR	0.0234	0.7541	0.9937	0.0295
0.4360					
	CGMY	0.0437	3.1727	3.5159	0.0788
	KR	0.0361	2.6249	2.8972	0.0651
0.6920					
	CGMY	0.0577	4.4063	5.0448	0.1093
	KR	0.0503	3.8468	4.4086	0.0953
0.9360					
	CGMY	0.0802	4.4772	5.2826	0.1378
	KR	0.0717	4.0071	4.7401	0.1233
1.1920					
	CGMY	0.0898	6.7185	7.5797	0.2003
	KR	0.0820	6.1366	6.9289	0.1825
1.7080					
	CGMY	0.1238	9.0494	9.8394	0.2588
	KR	0.1156	8.4512	9.1809	0.2409

Each maturity has been calibrated separately (see Table 4). Unfortunately, due to the independence and stationarity of their increments, exponential Lévy models perform poorly when calibrating several maturities at the same time [10]. In Table 5, we resume the error estimator of our option price fits. If we consider the exponential CGMY or KR models, we can estimate simultaneously market and risk-neutral parameters using historical prices and observed option prices. The flexibility of the KR distribution allows one to obtain a suitable solution to the calibration problem (see Table 5).

5 Conclusion

In this paper, we introduce a new tempered stable distribution named the KR distribution. Theoretically, the KR distribution is a proper tempered stable distribution with a simple closed form for the characteristic function. One can easily calculate the moments of the distribution and observe the behavior of the tails. Moreover, it is an extension of the well-known CGMY distribution and the change of measure for the KR distributions has more freedom than that for the CGMY distributions.

Empirically, we find that there are advantages supporting the KR distribution in the fitting of the historical distribution and the calibration of the risk-neutral distribution. In the fitting of S&P 500 index returns, the χ^2 and KS tests do not reject the KR distribution, but they do reject the normal distribution. The p-values of χ^2 and KS statistic for the KR distribution are similar to (sometimes better than) those of the CGMY distribution which is also not rejected. Furthermore, the p-values of AD and AD^2 statistic for the KR distribution fitting exceed those of the CGMY distribution fitting, suggesting that the KR distribution can capture extreme events better than the CGMY distribution. In the calibration of the risk-neutral distribution using the S&P 500 index option prices, the performance of the calibration for the exponential KR model is better than the CGMY model. The relatively flexible change of measure for the KR distribution seems to generate the result.

As mentioned at the outset of this paper, the KR distribution can be applied to other areas within finance. For example, it can be used in risk management because of its tail property. If we apply it to the modeling of innovation processes of the GARCH model, we can obtain an enhanced GARCH model. Since the KR distribution has the exponential moment with proper condition, we can calculate prices for exotic options with the partial integro-differential equation method. Finally, we can study asset pricing models and portfolio analysis with the KR distribution.

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APPENDIX

Exponential CGMY Model

The CGMY process is a pure jump process, introduced by Carr et al. [7].

Definition A.1. *A Lévy process* $(X_t)_{t\geq 0}$ *is called a* CGMY process *with parameters* (C, G, M, Y, m) *if the characteristic function of* X_t *is given by*

$$\begin{aligned}&\phi_{X_t}(u; C, G, M, Y, m)\\&= \exp(iumt + tC\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y)), \quad u \in \mathbb{R}.\end{aligned}$$

where $C, M, G > 0$*,* $Y \in (0, 2)$ *and* $m \in \mathbb{R}$*.*

For convenience, let us denote

$$\Psi_0(u; C, G, M, Y) \equiv C\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y).$$

Now, we focus on a way to find an equivalent measure for CGMY processes.

Proposition A.2. *Let* $(X_t)_{t\in[0,T]}$ *be CGMY processes with parameters* (C, G, M, Y, m) *and* $(\tilde{C}, \tilde{G}, \tilde{M}, \tilde{Y}, \tilde{m})$ *under* $\mathbb{P}$ *and* $\mathbb{Q}$*, respectively. Then* $\mathbb{P}|_{\mathcal{F}_t}$ *and* $\mathbb{Q}|_{\mathcal{F}_t}$ *are equivalent for all* $t > 0$ *if and only if* $C = \tilde{C}$*,* $Y = \tilde{Y}$ *and* $m = \tilde{m}$*.*

Proof. See Corollary 3 in [16]. □

The exponential CGMY model is defined under the continuous-time market as follows.

Definition A.3. *Let* $C > 0$*,* $G > 0$*,* $M > 1$*,* $Y \in (0, 2)$ *and* $\mu > 0$*. In the continuous-time market, if the driving process* $(X_t)_{t\in[0,T]}$ *of* $(S_t)_{t\in[0,T]}$ *is a CGMY process with parameters* (C, G, M, Y, m) *and* $m = \mu - \Psi_0(-i; C, G, M, Y)$*, then* $(S_t)_{t\in[0,T]}$ *is called the* CGMY stock price process *with parameters* (C, G, M, Y, μ) *and we say that the stock price process follows the* exponential CGMY model.

The function $\Psi_0(-i; C, G, M, Y)$ is well defined with the condition $M > 1$, and hence $E[S_t] = S_0 e^{\mu t}$, $t \in [0, T]$.

If we apply Proposition A.2 to the exponential CGMY model, we obtain the following proposition.

Theorem A.4. *Assume that* $(S_t)_{t\in[0,T]}$ *is the CGMY stock price process with parameters* (C, G, M, Y, μ) *under the market measure* $\mathbb{P}$*, and with parameters* $(\tilde{C}, \tilde{G}, \tilde{M}, \tilde{Y}, r - d)$ *under a measure* $\mathbb{Q}$*. Then* $\mathbb{Q}$ *is an* EMM *of* $\mathbb{P}$ *if and only if* $\tilde{C} = C$*,* $\tilde{Y} = Y$*, and*

$$r - d - \Psi_0(-i; C, \tilde{G}, \tilde{M}, Y) = \mu - \Psi_0(-i; C, G, M, Y). \tag{A.1}$$

Proof. See [16]. □