Asymptotic distribution of the sample average value-at-risk in the case of heavy-tailed returns

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Abstract

In this paper, we provide a stable limit theorem for the asymptotic distribution of the sample average value-at-risk when the distribution of the underlying random variable X describing portfolio returns is heavy-tailed. We illustrate the convergence rate in the limit theorem assuming that X has a stable Paretian distribution and Student's t distribution.

Keywords average value-at-risk, risk measures, heavy-tails, asymptotic distribution, Monte Carlo

1 Introduction

The average value-at-risk (AVaR) risk measure has been proposed in the literature as a coherent alternative to the industry standard Value-at-Risk (VaR), see Artzner et al. (1998) and Pflug (2000). It has been demonstrated that it has better properties than VaR for the purposes of risk management and, being a downside risk-measure, it is superior to the classical standard deviation and can be adopted in a portfolio optimization framework, see Rachev et al. (2006), Stoyanov et al. (2007), Biglova et al. (2004), and Rachev et al. (2008).

The AVaR of a random variable X at tail probability ϵ is defined as

$$AVaR_{\epsilon}(X) = -\frac{1}{\epsilon}\int_0^{\epsilon} F^{-1}(p)dp.$$

where $F^{-1}(x)$ is the inverse of the cumulative distribution function (c.d.f.) of the random variable X. The random variable may describe the return of stock, for example. A practical problem of computing portfolio AVaR is that usually we do not know explicitly the c.d.f. of portfolio returns. In order to solve this practical problem, the Monte Carlo method is employed. The returns of the portfolio constituents are simulated and then the returns of the portfolio are calculated. In effect, we have a sample from the portfolio return distribution which we can use to estimate AVaR. The sample AVaR equals,

$$\widehat{AVaR}_{\epsilon}(X) = -\frac{1}{\epsilon}\int_0^{\epsilon} F_n^{-1}(p)dp.$$

where $F_n^{-1}(p)$ denotes the inverse of the sample c.d.f. $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \leq x\}$ in which $I\{A\}$ denotes the indicator function of the event A, and $X_1, \dots, X_n$

is a sample of independent, identically distributed (i.i.d.) copies of a random variable X.

Under a very general regularity condition, the larger the sample, the closer the estimate to the true value. Suppose that $E\max(-X,0) < \infty$. Then, it is easy to demonstrate that the following relation holds,

$$E\max(-X,0) < \infty \qquad \Longleftrightarrow \qquad AVaR_\epsilon(X) < \infty.$$

Thus, by the strong law of large numbers, the condition $E\max(-X,0) < \infty$ is necessary and sufficient for the almost sure convergence of the sample AVaR to the true one,

$$\widehat{AVaR}_\epsilon(X) \xrightarrow{a.s.} AVaR_\epsilon(X) \quad \text{as} \quad n \to \infty. \tag{1}$$

However, with any finite sample, the sample AVaR will fluctuate about the true value and, having only a sample estimate, we have to know the probability distribution of the sample AVaR in order to build a confidence interval for the true value. The problem of computing the distribution of the sample AVaR is a complicated one even if we know the distribution of X. From a practical viewpoint, X describes portfolio return which can be a complicated function of the joint distribution of the risk drivers. Therefore, we can only rely on large sample theory to gain insight into the fluctuations of sample AVaR. That is, for a large n, we can use a limiting distribution to calculate a confidence interval. In this respect, a limit theorem for the distribution of the sample AVaR can be regarded as a way to describe the speed of convergence in (1).

Concerning the finite sample properties, the estimator $\widehat{AVaR}_\epsilon(X)$ has a negative bias,

$$\widehat{AVaR}_\epsilon(X) \leq AVaR_\epsilon(X).$$

The asymptotic bias is of order $O(n^{-1})$ and we consider it negligible for the purposes of our study. For further details, see Trindade et al. (2007).

In this paper, we discuss the asymptotic distribution of the sample AVaR assuming that the random variable X can be heavy-tailed and may have an infinite second moment. In such a case, we cannot take advantage of the classical Central Limit Theorem (CLT) to establish a limit theorem. For this reason, we resort to the Generalized CLT and the characterization of the domains of attraction of stable distributions which appear as limiting distribution in it.

Stable distributions are introduced by their characteristic functions. The random variable Z is said to have a stable distribution if its characteristic function $\varphi(t) = Ee^{itZ}$ has the form

$$\varphi(t) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) + i\mu t\}, & \alpha \neq 1 \\ \exp\{-\sigma |t| (1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \ln(|t|)) + i\mu t\}, & \alpha = 1 \end{cases} \quad (2)$$

and is denoted by $Z \in S_\alpha(\sigma, \beta, \mu)$. The parameter $\alpha \in (0, 2]$ is called the tail index and governs the tail behavior and the kurtosis of the distribution. Smaller α indicates heavier tails and higher kurtosis. If $\alpha < 2$, then Z has infinite variance. If $1 < \alpha \leq 2$, then Z has finite mean and the AVaR of Z can be calculated. The Gaussian distribution appears as a stable distribution with $\alpha = 2$. The stable distributions with $\alpha < 2$ are referred to as *stable Paretian distributions*. The parameter $\beta \in [-1, 1]$ is a skewness parameter. If $\beta = 0$, the distribution is symmetric with respect to μ. Positive β indicates that the distribution is skewed to the right and negative β indicates that the distribution is skewed to the left. The parameter $\sigma > 0$ is a scale parameter and $\mu \in \mathbf{R}$ is a location parameter.

The notion of slowly varying functions is extensively used in the paper. A positive function $L(x)$ is said to be slowly varying at infinity if the following limit relation is satisfied,

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad \forall t > 0. \quad (3)$$

The main result concerning the domains of attraction of stable distributions is given in the following theorem.

Theorem 1. *Let $X_1, \ldots, X_n$ be i.i.d. with c.d.f. $F(x)$. There exist $a_n > 0$, $b_n \in \mathbf{R}$, $n = 1, 2, \ldots$, such that the distribution of*

$$a_n^{-1}[(X_1 + \ldots + X_n) - b_n]$$

converges as $n \to \infty$ to $S_\alpha(1, \beta, 0)$ if and only if both

(i) $x^\alpha[1 - F(x) + F(-x)] = L(x)$ *is slowly varying at infinity.*

(ii) $\dfrac{1 - F(x) - F(-x)}{1 - F(x) + F(-x)} \to \beta$ *as* $x \to \infty$

The a_n must satisfy

$$\lim_{n \to \infty} \frac{nL(a_n)}{a_n^\alpha} = \begin{cases} (\Gamma(1-\alpha)\cos(\pi\alpha/2))^{-1} & \text{if } 0 < \alpha < 1, \\ 2/\pi & \text{if } \alpha = 1, \\ \left(\frac{\Gamma(2-\alpha)}{\alpha-1} |\cos\frac{\pi\alpha}{2}|\right)^{-1} & \text{if } 1 < \alpha < 2. \end{cases} \quad (4)$$

The b_n may be chosen as follows:

$$b_n = \begin{cases} 0 & \textit{if} \quad 0 < \alpha < 1, \\ na_n \int_{-\infty}^{\infty} \sin(x/a_n) dF(x) & \textit{if} \quad \alpha = 1, \\ n \int_{-\infty}^{\infty} x dF(x) & \textit{if} \quad 1 < \alpha < 2. \end{cases} \tag{5}$$

In all cases, $a_n = n^{1/\alpha} L_0(n)$ *where* $L_0(n)$ *is slowly varying at infinity.*

For further information about stable distributions and their properties, see Samorodnitsky and Taqqu (1994).

The result in Theorem 1 characterizes the domains of attraction of stable Paretian laws. If the index α characterizing the tails of the c.d.f. $F(x)$ in condition *(i)* satisfies $\alpha \geq 2$, then the tail index of the limiting distribution equals $\alpha^* = 2$. Thus, the relationship between the tail index of the limiting distribution, which we denote by α^*, and the tail index in condition *(i)* can be generalized as $\alpha^* = \min(\alpha, 2)$. If $\alpha > 2$, then $EX_1^2 < \infty$ and we are in the setting of the classical CLT. The centering and normalization can be done $b_n = nEX_1$ and $a_n = n^{1/2}\sigma_{X_1}$, where σ_{X_1} denotes the standard deviation of X_1. The case $\alpha = 2$ is more special because the variance of X_1 is infinite and a_n cannot be chosen in this fashion. Moreover, the proper normalization cannot be obtained by computing the limit $\alpha \to 2$ in equation (4). Under the more simple assumptions that the function $L(x)$ in condition *(i)* equals a constant A, Zolotarev and Uchaikin (1999) provide the formula $a_n = (n \log n)^{1/2} A^{1/2}$.

The paper is organized in the following way. Section 2 provides a stable limit theorem for the asymptotic distribution of the sample AVaR. In Section 3, we apply the theorem assuming that the random variable X has a stable Paretian distribution and also Student's t distribution. Under these assumptions, we study the effect of skewness and heavy tails on the convergence rate in the limit theorem.

2 A stable limit theorem

In order to develop the limit theorem, we need a few additional facts related to building a linear approximation to AVaR and estimating the rate of improvement of the linear approximation. They are collected in the following proposition.

Proposition 1. *Suppose* X *is a r.v. with c.d.f.* F *which satisfies the condition* $E\max(-X, 0) < \infty$ *and* F *is differentiable at the* ϵ*-quantile of* X*. Denote by* F_n *the sample c.d.f. of* $X_1, \dots, X_n$ *which is a sample of i.i.d.*

copies of X. There exists a linear functional Δ defined on the difference $G-F$ where the functions G and F are c.d.f.s, such that

$$|\phi(F_n)-\phi(F)-\Delta(F_n-F)|=o(\rho(F_n,F)) \tag{6}$$

where $\rho(F_n,F)=\sup_x|F_n(x)-F(x)|$ stands for the Kolmogorov metric and

$$\phi(G)=-\frac{1}{\epsilon}\int_0^{\epsilon}G^{-1}(p)dp$$

in which G^{-1} is the inverse of the c.d.f. G. The linear functional Δ has the form

$$\Delta(F_n-F)=\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}(q_\epsilon-x)d(F_n(x)-F(x)). \tag{7}$$

where q_ϵ is the ϵ-quantile of X.

Proof. The condition $E\max(-X,0)<\infty$ guarantees $\phi(F)<\infty$. Note that $\phi(F_n)$ is convergent with any finite sample.

Consider the difference $\phi(F_n)-\phi(F)$.

$$\begin{aligned}
\phi(F_n)-\phi(F) &= -\frac{1}{\epsilon}\int_0^{\epsilon}F_n^{-1}(p)dp+\frac{1}{\epsilon}\int_0^{\epsilon}F^{-1}(p)dp\\
&= -\frac{1}{\epsilon}\int_{-\infty}^{F_n^{-1}(\epsilon)}pdF_n(p)+\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}pdF(p)\\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}pdF_n(p)-\frac{1}{\epsilon}\int_{q_\epsilon}^{F_n^{-1}(\epsilon)}pdF_n(p)+\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}pdF(p)\\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}pd(F_n(p)-F(p))-\frac{C_n}{\epsilon}(F(q_\epsilon)-F_n(q_\epsilon))\\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon}pd(F_n(p)-F(p))+\frac{C_n}{\epsilon}(F_n(q_\epsilon)-F(q_\epsilon))
\end{aligned}$$

where, by the mean-value theorem, the constant C_n is between q_ϵ and $F_n^{-1}(\epsilon)$. For example if we assume, for the sake of being particular, that $q_\epsilon\le F_n^{-1}(\epsilon)$, then $q_\epsilon\le C_n\le F_n^{-1}(\epsilon)$. Due to the assumption that F is differentiable at q_ϵ, $F_n^{-1}(\epsilon)\to q_\epsilon$ in almost sure sense as n increases indefinitely. As a result, $C_n\to q_\epsilon$ in almost sure sense.

Choose the linear functional $\Delta(F_n-F)$ as in equation (7). The fact that it is linear with respect to the difference of the c.d.f.s is a property of the integral. Consider the left-had side of (7), which we denote by LHS, having in mind the expression derived above. We obtain

$$
\begin{aligned}
LHS &= \left| -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} x d(F_n(x) - F(x)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) - L(F_n - F) \right| \\
&= \left| -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} q_\epsilon d(F_n(x) - F(x)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) \right| \\
&= \left| -\frac{q_\epsilon}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) \right| \\
&= \frac{|C_n - q_\epsilon|}{\epsilon} |F_n(q_\epsilon) - F(q_\epsilon)| \\
&\leq \frac{|C_n - q_\epsilon|}{\epsilon} \sup_x |F_n(x) - F(x)| \\
&= \frac{|C_n - q_\epsilon|}{\epsilon} \rho(F_n, F)
\end{aligned}
$$

As a result,

$$
\frac{|\phi(F_n) - \phi(F) - L(F_n - F)|}{\rho(F_n, F)} \to 0, \text{ as } n \to \infty
$$

in almost sure sense. As a result we obtain the asymptotic relation in equation (6). □

Corollary 1. *Under the assumptions in the proposition,*

$$
|\phi(F_n) - \phi(F) - \Delta(F_n - F)| = o(n^{-1/2}). \tag{8}
$$

Proof. By the Kolmogorov theorem, the metric $\rho(F_n, F)$ approaches zero at a rate equal to $n^{-1/2}$ which indicates the rate of improvement of the linear approximation $\Delta(F_n - F)$. □

The main result is given in the theorem below. The idea is to use the linear approximation $\Delta(F_n - F)$ of the AVaR functional in order to obtain an asymptotic distribution as $n \to \infty$.

Theorem 2. *Suppose that X is random variable with c.d.f. $F(x)$ which satisfies the following conditions*

a) $x^\alpha F(-x) = L(x)$ *is slowly varying at infinity*

b) $\int_{-\infty}^{0} x dF(x) < \infty$

c) $F(x)$ *is differentiable at* $x = q_\epsilon$*, where* q_ϵ *is the* ϵ*-quantile of* X*.*

Then, there exist $c_n > 0$, $n = 1, 2 \ldots$, *such that for any* $0 < \epsilon < 1$,

$$c_n^{-1}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right) \stackrel{w}{\to} S_{\alpha^*}(1,1,0), \tag{9}$$

in which $\stackrel{w}{\to}$ *denotes weak limit,* $1 < \alpha^* = \min(\alpha, 2)$*, and* $c_n = n^{1/\alpha^*-1}L_0(n)/\epsilon$ *where* L_0 *is slowly varying at infinity. Furthermore, the* c_n *are representable as* $c_n = a_n/n\epsilon$ *where* a_n *stands for the normalizing sequence in Theorem 1 and must satisfy the condition in equation (4).*

Proof. By the result in Proposition 1,

$$\phi(F_n) - \phi(F) = \Delta(F_n - F) + o(n^{-1/2}) \tag{10}$$

where ϕ is the AVaR functional and $\Delta(F_n - F)$ is given in (6). Simplifying the expression for $\Delta(F_n - F)$, we obtain

$$\phi(F_n) - \phi(F) = \frac{1}{n\epsilon}\sum_{i=1}^{n}[(q_\epsilon - X_i)_+ - E(q_\epsilon - X_i)_+] + o(n^{-1/2}) \tag{11}$$

It remains to apply the domains of attraction characterization in Theorem 1 to the right-hand side of equation (11). For this purpose, consider the expression

$$\sum_{i=1}^{n} Y_i - nEY_1 \tag{12}$$

where $Y_i = (q_\epsilon - X_i)_+$ are i.i.d. random variables. Denote by $F_Y(x)$ the c.d.f. of Y. The left-tail behavior of X assumed in a) implies $x^\alpha(1-F_Y(x)) = L(x)$ as $x \to \infty$ where $L(x)$ is the slowly varying function assumed in a). This is demonstrated by

$$\begin{aligned} x^\alpha(1 - F_Y(x)) &= x^\alpha P(\max(q_\epsilon - X, 0) > x) \\ &= x^\alpha P(X < q_\epsilon - x) \\ &\sim x^\alpha P(X < -x) \end{aligned} \tag{13}$$

Furthermore, the asymptotic behavior of the left tail of Y is $F_Y(-x) = 0$ which holds for any $x \geq -q_\epsilon$. As a result, condition (i) from Theorem 1 holds.

Condition b) implies that the tail exponent α in a) must satisfy the inequality $\alpha > 1$. Therefore, subtracting nEY_1 in (12) is a proper centering of the sum as suggested in (5) in Theorem 1. Note that if $\alpha \geq 2$, then Y is in the domain of attraction of the normal distribution and the same choice of centering is appropriate. Thus, the tail index of the limiting distribution satisfies $1 < \alpha^* = \min(\alpha, 2)$.

Finally, computing condition (ii) in Theorem 1 from the tail behavior of Y yields $\beta = 1$. Essentially, this follows because $F_Y(-x) = 0$ if $x \geq -q_\epsilon$.

Therefore, all conditions in Theorem 1 are satisfied and as, a result, there exists a sequence of normalizing constants a_n satisfying (4), such that

$$a_n^{-1}\left(\sum_{i=1}^{n} Y_i - nEY_1\right) \xrightarrow{w} S_{\alpha^*}(1,1,0). \tag{14}$$

as $n \to \infty$. In order to apply this result to sample AVaR, we need (14) reformulated for the average rather than the sum of Y_i. Thus, a more suitable form is

$$n\epsilon a_n^{-1}\left(\frac{1}{n\epsilon}\sum_{i=1}^{n}(Y_i - EY_i)\right) \xrightarrow{w} S_{\alpha^*}(1,1,0). \tag{15}$$

as $n \to \infty$.

As a final step, we apply the limit result in (15) to equation (11). Multiplying both sides of (11) by $n\epsilon a_n^{-1}$ yields the limit

$$n\epsilon a_n^{-1}(\phi(F_n) - \phi(F)) \xrightarrow{w} S_{\alpha^*}(1,1,0) \tag{16}$$

as $n \to \infty$. It remains only to verify if the normalization does not lead to explosion of the residual. Indeed,

$$n\epsilon a_n^{-1} o(n^{-1/2}) = \frac{n^{1/2}}{a_n} o(1) = o(1),$$

because the factor $n^{1/2}/a_n$ approaches zero by the asymptotic behavior of a_n given in the domains of attraction characterization in Theorem 1.

□

A number of comments are collected in the following remarks.

Remark 1. By definition, the AVaR is the negative of the average of the quantiles of X beyond a reference quantile q_ϵ. For this reason, it is only the behavior of the left tail of X which matters and the assumptions a) and b) in Theorem 2 concern the left tail only. Condition c) is technical and allows the calculation of the influence function of AVaR.

Remark 2. If $\alpha > 2$ in condition a), then $\int_{-\infty}^{0} x^2 dF(x) < \infty$ and the limiting distribution is the standard normal distribution. In this case, the normalizing sequence c_n should be calculated using $\sigma_\epsilon^2 = D(q_\epsilon - X)_+$,

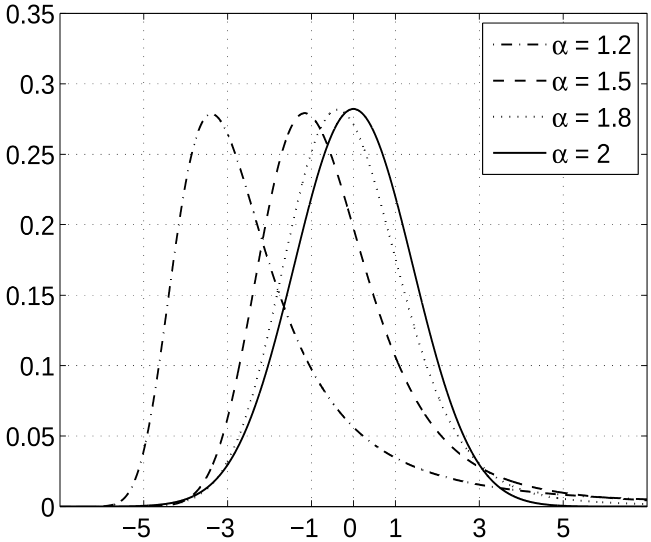

Figure 1: Densities of the limiting stable distribution corresponding to different tail behavior.

$$c_n = n^{-1/2}\sigma_\epsilon/\epsilon.$$

The case $\alpha > 2$ is considered in detail in Stoyanov and Rachev (2007).

Remark 3. The limiting stable distribution is totally skewed to the right, $\beta = 1$. However, the observed skewness in the shape of the distribution decreases as $\alpha \to 2$, see Figure 1. At the limit, when $\alpha = 2$, the limiting distribution is Gaussian and is symmetric irrespective of the value of β. Therefore, the degree of the observed skewness in the limiting distribution is essentially determined by the tail behavior of X, or by the value of α, and is not influenced by any other characteristic.

Remark 4. When $\epsilon \to 1$, then AVaR approaches the mean of X (or the sample average if we consider the sample AVaR),

$$\lim_{\epsilon \to 1} AVaR_\epsilon(X) = EX.$$

Unfortunately, there is no such continuity in equation (9) unless X has finite variance. That is, generally it is not true that the weak limit in equation (9) holds for the sample average letting $\epsilon \to 1$. The reason is that if $\epsilon = 1$, then both tails of the distribution of X matter and the limiting stable distribution can have any $\beta \in [-1, 1]$. The condition $DX < \infty$ is sufficient to guarantee that the limiting distribution is normal for any $\epsilon \in (0, 1]$ and in this case there is continuity in equation (9) as $\epsilon \to 1$.

As an illustration of the singularity at $\epsilon = 1$, consider the following example. Suppose that the right tail of X is heavier than the left tail and as a consequence,

$$\int_{-\infty}^{q_\epsilon} x^2 dF(x) < \infty, \quad \text{for any} \quad \epsilon < 1,$$

but $EX^2 = \infty$. Under this assumption, the limiting distribution of the sample AVaR is normal for any $\epsilon < 1$. If $\epsilon = 1$, then the limiting distribution becomes stable non-Gaussian due to the heavier right tail. Thus, there is a change in the limiting distribution of the sample AVaR with $\epsilon < 1$ and the sample average.

3 Examples

The result in Theorem 2 provides the limiting distribution but does not provide any insight on the rate of convergence. That is, it does not give an answer to the question how many observations are needed in order for the distribution of the left-had side in equation (9) to be sufficiently close to the distribution of the right-hand side in terms of a selected probability metric. In this section, we provide illustrations of the stable limit theorem and the rate of convergence assuming particular distributions of X.

3.1 Stable Paretian Distributions

We remarked that stable Paretian distributions are stable distributions with tail index $\alpha < 2$. This distinction is made since their properties are very different from the properties of the normal distribution which appears as a stable distribution with $\alpha = 2$. For example, in contrast to the normal distribution, stable Paretian distributions have heavy tails exhibiting power decay. In the field of finance, stable Paretian distribution were proposed as a model for stock returns and other financial variables, see Rachev and Mittnik (2000).

Denote by X the random variable describing the return of a given stock. In this section, we assume that $X \in S_\alpha(\sigma, \beta, \mu)$ with $1 < \alpha < 2$, $\beta \neq 1$, and our goal is to apply the result in Theorem 2 which provides a tool of computing the confidence interval of the sample AVaR of X on condition that the Monte Carlo method is used with a large number of scenarios. Since by

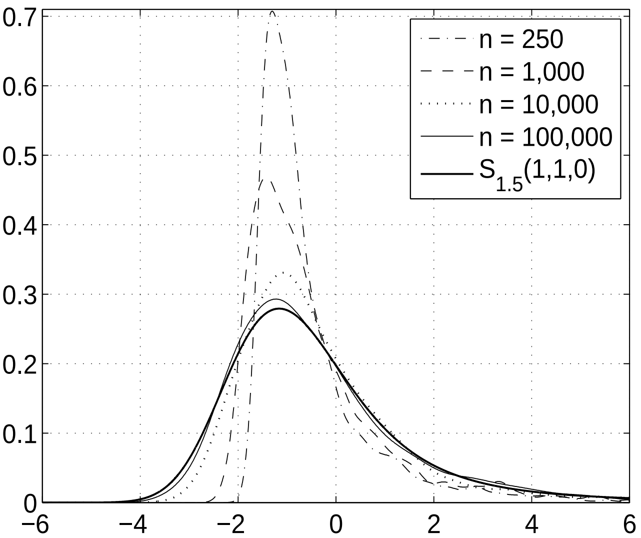

Figure 2: The density of the sample AVaR as n increases with $\beta = 0.7$ and $\epsilon = 0.01$.

assumption $\alpha > 1$, which guarantees convergence of the sample AVaR to the theoretical AVaR in almost sure sense. In the case of stable distributions, the quantity $AVaR_\epsilon(X)$ can be calculated using a semi-analytic expression given in Stoyanov et al. (2006).

In order to apply the result in Theorem 2, first we have to check if the conditions are satisfied and then choose the scaling constants c_n. For this purpose, we use the following property of stable Paretian distributions, see Samorodnitsky and Taqqu (1994).

Property 1. *Let* $X \in S_\alpha(\sigma, \beta, \mu)$ $\;0 < \alpha < 2$. *Then*

$$\lim_{\lambda\to\infty} \lambda^\alpha P(X > \lambda) = C_\alpha \frac{1+\beta}{2}\sigma^\alpha$$

$$\lim_{\lambda\to\infty} \lambda^\alpha P(X < -\lambda) = C_\alpha \frac{1-\beta}{2}\sigma^\alpha$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin(x)dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \alpha \neq 1 \\ 2/\pi, & \alpha = 1 \end{cases}$$

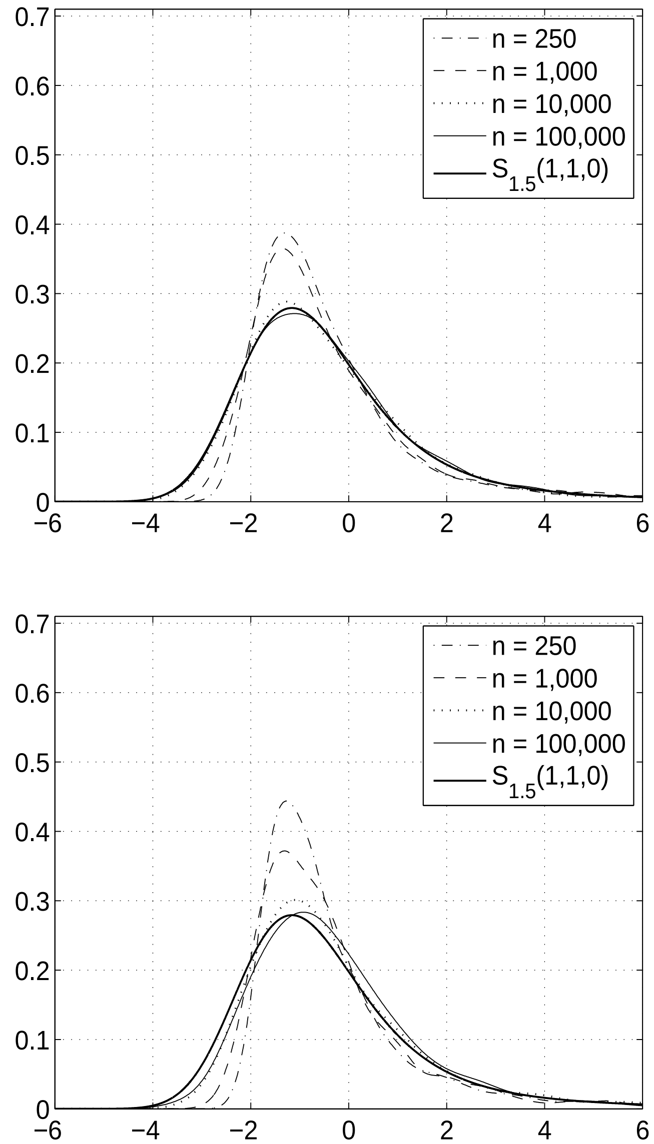

Figure 3: The density of the sample AVaR as n increases with $\beta = 0.7$ (top) and $\beta = -0.7$ (bottom) and $\epsilon = 0.05$.

This property provides the asymptotic behavior of the left tail of the distribution. We further assume that $\beta \neq 1$ since in this case the asymptotic behavior of the left tail is different, see Samorodnitsky and Taqqu (1994). Condition b) is satisfied because of the assumption $1 < \alpha < 2$ and, finally, condition c) is satisfied for any choice of $0 < \epsilon < 1$ since all stable distributions have densities. Therefore, all assumptions are satisfied and the result in Theorem 2 holds with $\alpha^* = \alpha$ and the scaling constants c_n should be chosen in the following way,

$$c_n = n^{1/\alpha - 1} \left(\frac{1-\beta}{2} \right)^{1/\alpha} \frac{\sigma}{\epsilon}.$$

Note that in this case, the skewness in the distribution of X translates into a different scaling of the normalizing constants. If X is negatively skewed ($\beta < 0$), the scaling factor is larger than if X is skewed positively ($\beta > 0$).

We carry out a Monte Carlo study assuming $X \in S_{1.5}(\beta, 1, 0)$ where $\beta = \pm 0.7$ and two choices of the tail probability $\epsilon = 0.01$ and $\epsilon = 0.05$. We generate 2,000 samples from the corresponding distribution the size of which equals $n = 250, 1,000, 10,000,$ and $100,000$.

Figure 2 illustrates the convergence rate for the case $\epsilon = 0.01$ as the number of observations increases. While from the plot it seems that $n = 100,000$ results in a density which is very close to that of the limiting distribution, but the Kolmogorov test fails. The convergence rate is much slower in the heavy-tailed case than in the setting of the classical CLT. Stoyanov and Rachev (2007) suggest that about 5,000 simulations are sufficient for the purposes of confidence bounds estimation when the distribution has bounded support. Apparently, much more observations are needed in this heavy-tailed case.

The plots in Figure 3 indicate that as the tail probability ϵ increases, the behavior of the sample AVaR distribution improves. Furthermore, the the behavior improves when X turns from being negatively to positively skewed.

3.2 Student's t distribution

Student's t distribution is a widely used model for a stock return distribution. X has Student's t distribution, $X \in t(\nu)$, with $\nu > 0$ degrees of freedom if the density of X equals,

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \ x \in \mathbb{R}.$$

A few simple properties of Student's t distribution are collected in the next proposition.

Proposition 2. *Suppose that $X \in t(\nu)$ and denote the c.d.f. of X by $F(x)$. Then, $x^\nu F(-x) = L(x)$ where $L(x)$ is a slowly varying function at infinity and also*

$$\lim_{x\to\infty} x^\nu F(-x) = \nu^{\nu/2-1} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)\sqrt{\pi}}. \tag{17}$$

Proof. The fact that $L(x)$ is a slowly varying function is checked directly applying the definition and the limit in (17) is obtained by applying l'Hospital's rule. □

The result in this proposition and Theorem 2 imply that for $\nu > 2$, the limiting distribution of the sample AVaR is the Gaussian distribution. If $1 < \nu \leq 2$, then the limiting distribution is stable with $\alpha^* = \nu$. If $\nu \leq 1$, then the AVaR of X diverges. The scaling constants c_n should be chosen in a different way depending on the value of ν,

$$c_n = \begin{cases} n^{-1/2}\sigma_\epsilon/\epsilon, & \text{if } \nu > 2 \\ n^{1/\nu-1}A_\nu/\epsilon, & \text{if } 1 < \nu < 2 \end{cases} \tag{18}$$

where $\sigma_\epsilon^2 = D(q_\epsilon - X)_+$ and

$$A_\nu^\nu = \nu^{\nu/2-1} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)\sqrt{\pi}} \frac{\Gamma(2-\nu)}{\nu-1} |\cos(\pi\nu/2)|.$$

The value of the constant A_ν is obtained by taking into account the limit in (17) and the condition in equation (4). Stoyanov and Rachev (2007) consider in detail the case $\nu > 2$ and provide the formula for σ_ϵ. This case is in the classical setting of the CLT as the variance of X is finite.

We carry out a Monte Carlo experiment in order to study the convergence rate of the sample AVaR distribution to the limiting distribution. We fix the degrees of freedom, the number of simulations to 100,000, and $\epsilon = 0.05$. Next we generate 2,000 samples from which the sample AVaR is estimated. Thus we obtain 2,000 estimates of $AVaR_\epsilon(X)$, $X \in t(\nu)$. Finally, we calculate the Kolmogorov distance

$$\rho(G_\nu, G) = \sup_x |G_\nu(x) - G(x)|$$

where G_ν is the c.d.f. of the sample AVaR approximated by the sample c.d.f. obtained with the 2,000 estimates, and G is the c.d.f. of the limiting distribution $S_{\alpha^*}(1,1,0)$ where $\alpha^* = \min(\nu, 2)$.

Figure 4 shows the values of $\rho(G_\nu, G)$ as ν varies from 1.05 to 3. The horizontal line shows the critical value of the Kolmogorov statistic: if the

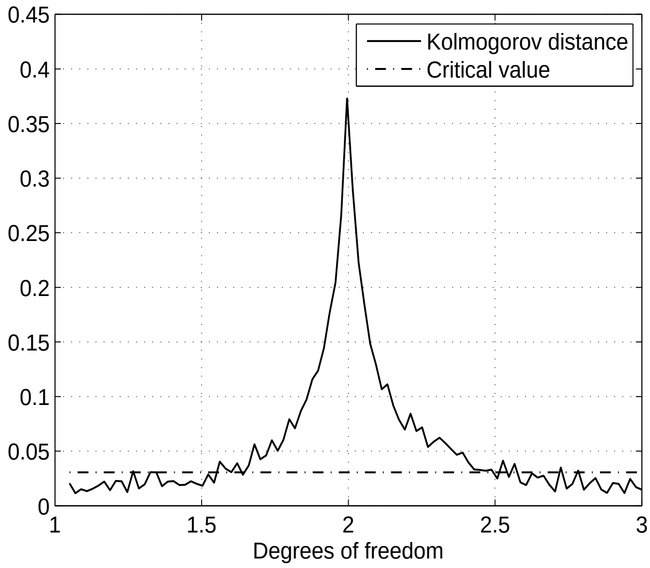

Figure 4: The Kolmogorov distance between the sample AVaR distribution of $X \in t(\nu)$ obtained with 100,000 simulations and the limiting distribution.

calculated $\rho(G_\nu, G)$ is below the critical value, we accept the hypothesis that the sample AVaR distribution is the same as the limiting distribution, otherwise we reject it. Since we use a sample c.d.f. to approximate $G_\nu(x)$, the solid line fluctuates a little but we notice that for $\nu \leq 1.5$ and $\nu \geq 2.5$ it seems that 100,000 observations are enough in order to accept the limiting distribution as a model. For the middle values, larger samples are needed. This observation indicates that the rate of convergence of the sample AVaR distribution to the limiting distribution deteriorates as ν approaches 2 and is slowest for $\nu = 2$. This finding can be summarized in the following way by considering all possible cases for ν:

- $\nu > 2$. As ν decreases from larger values to 2, the tail thickness increases which results in higher absolute moments becoming divergent, $E|X|^\delta = \infty$, $\delta \geq \nu$. The limiting distribution is the Gaussian distribution but the tails becoming thicker results in deterioration of the convergence rate to the Gaussian distribution.

- $\nu = 2$. The limiting distribution is the Gaussian distribution even though the variance of X is infinite. This case is not covered by the limit theory behind the classical CLT.

- $1 < \nu < 2$. We continue decreasing ν and the tails become so thick that they start influencing the limiting distribution which is stable Paretian, $S_\nu(1, 1, 0)$, and depends on ν. However, the convergence rate starts improving.

- $0 < \nu \leq 1$. The tails of X become so heavy that $AVaR_\epsilon(X) = \infty$.

4 Conclusion

In the paper, we study the asymptotic distribution of the sample AVaR. We provide a stable limit theorem describing all possible asymptotic laws depending on the behavior of the left tail of the random variable X. If we assume that X describes the return distribution of a stock, then the left tail describes losses. Intuitively, the asymptotic distribution of the sample AVaR is determined by the behavior of extreme losses.

Furthermore, in order to adopt the asymptotic law and draw conclusions based on it, we need insight on the rate of convergence in the stable limit theorem. We illustrate the rate of convergence by Monte Carlo experiments assuming a stable distribution and Student's t distribution for X. In summary, the convergence rate deteriorates as the tail exponent $\alpha \to 2$ and it improves as the distribution of X becomes more positively skewed. Generally, the skewness of X does not influence the asymptotic law.

References

Artzner, P., F. Delbaen, J.-M. Eber and D. Heath (1998), 'Coherent measures of risk', *Math. Fin.* **6**, 203–228.

Biglova, A., S. Ortobelli, S. Rachev and S. Stoyanov (2004), 'Different approaches to risk estimation in portfolio theory', *The Journal of Portfolio Management* **Fall 2004**, 103–112.

Pflug, G. (2000), 'Some remarks on the value-at-risk and the conditional value-at-risk', *In: Uryasev, S. (Ed.), Probabilistic Constrained Optimization: Methodology and Applications. Kluwer Academic Publishers, Dordrecht* .

Rachev, S., D. Martin, B. Racheva-Iotova and S. Stoyanov (2006), 'Stable etl optimal portfolios and extreme risk management', *forthcoming in Decisions in Banking and Finance, Springer/Physika, 2007* .

Rachev, S. T., Stoyan V. Stoyanov and F. J. Fabozzi (2008), *Advanced stochastic models, risk assessment, and portfolio optimization: The ideal risk, uncertainty, and performance measures*, Wiley, Finance.

Rachev, S.T. and S. Mittnik (2000), *Stable Paretian Models in Finance*, John Wiley & Sons, Series in Financial Economics.

Samorodnitsky, G. and M.S. Taqqu (1994), *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, London.

Stoyanov, S., G. Samorodnitsky, S. Rachev and S. Ortobelli (2006), 'Computing the portfolio conditional value-at-risk in the α-stable case', *Probability and Mathematical Statistics* **26**, 1–22.

Stoyanov, S. and S. Rachev (2007), 'Asymptotic distribution of the sample average value-at-risk', *forthcoming in Journal of Computational Analysis and Applications* .

Stoyanov, S., S. Rachev and F. Fabozzi (2007), 'Optimal financial portfolios', *forthcoming in Applied Mathematical Finance* .

Trindade, A. A., S. Uryasev, A. Shapiro and G. Zrazhevsky (2007), 'Financial prediction with constrained tail risk', *forthcoming in Journal of Banking and Finance* .

Zolotarev, V. M. and V. V. Uchaikin (1999), *Chance and stability, stable distributions and their applications*, Brill Academic Publishers.