Credit Risk : Firm Value Model

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Risk

- Market Risk (30% of the total risk)
  - economic factors : current state of the economy
  - government band rate : default free rate
  - FX-rate (Currency)
  - unemployment rate
- Credit Risk (40% of the total risk)
  - loans
  - defaultable bonds (Corporate bonds)
  - defaultable fixed income securities
- Operational Risk (30% of the total risk)
  - business lines
  - event types
  - management
  - internal / external fraud

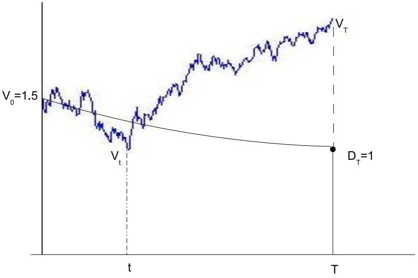
Contents

- Firm value model
  - The defaults are endogenous
  - Option pricing method
  - APT (Arbitrage Pricing Theory) : Stochastic calculus, risk neutral valuation and no-arbitrage markets
- Intensity based model
  - The defaults are exogenous.
  - The model is designed for large portfolios of corporate bonds.
- Rating based model
  - Markov chains
  - Rating agencies - nationally recognized statistical rating organizations (NRSROs)
- Credit derivatives
  - Description
  - Valuation

Merton's Firm Value Model

- The defaultable bond and the stock price are derivatives with underlying the value of the firm.
- The default time is endogeneous for the model.

Example



Firm value: $V_t = \underbrace{D_t}_{\text{debt}} + \underbrace{E_t}_{\text{equity}} = \underbrace{B_t}_{\text{defaultable bond price}} + \underbrace{S_t}_{\text{stock value}}$

Merton:

- What is B_t the value of the corporate bond at time t?
- What is S_t the stock value at time t?

Simplest Model

Suppose V_t is the value of the firm at t.
Geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t \tag{1}$$

where $(W_t)_{t \geq 0}$ is the Brownian motion on the market measure $\mathbb{P}$ (natural world).
Let r_t be the risk free rate at t, and assume $r_t \equiv r$.
The bank account:

$$b_t = b_0 e^{rt}, \quad b_0 = 1. \tag{2}$$

Discount factor:

$$\beta_t = \frac{1}{b_t} = e^{-rt}$$

*(1) and (2) : classical Black-Scholes model for option pricing.

B_t under Black-Scholes model

Let $B_t = \bar{B}(t, T)$.
At maturity $t = T$,

$$B(T,T) = \begin{cases} \bar{D} & , V_T > \bar{D} \\ V_T & , V_T \leq \bar{D} \end{cases}$$
$$= \min(V_T, \bar{D})$$
$$= \bar{D} - \max(\bar{D} - V_T, 0).$$

Hence

$\bar{B}(t, T)$

$=$ the value of European contingent claim with$B(T, T) = \min(V_T, \bar{D})$

$= E_{\tilde{\mathbb{P}}}\left[e^{-r(T-t)}\bar{B}(T,T)|\mathcal{F}_t\right]$

$=$ discounted final payoff

under risk-neutral measure $\tilde{\mathbb{P}}$ given $\mathcal{F}_t = \sigma(w_u, u \leq t)$

B_t under Black-Scholes model

Recall that under $\tilde{\mathbb{P}}$ (risk-neutral world),

$$dV_t = rV_t dt + \sigma V_t d\tilde{W}_t \tag{3}$$

where $(\tilde{W}_t)_{t\geq 0}$ is the Brownian motion on $\tilde{\mathbb{P}}$.
* On the natural world, $\tilde{W}_t = W_t + \theta t$ where $\theta = (\mu - r)/\sigma$ is the market price of risk.
The solution for (3) is

$$V_t = V_0 e^{(r-\frac{\sigma^2}{2})t+\sigma\tilde{W}_t}, \quad t \geq 0. \tag{4}$$

By (4), given information $\mathcal{F}_t$,

$$V_T = V_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)} \quad \text{on } \mathbb{P}.$$

B_t under Black-Scholes model

Therefore,

$$\begin{aligned}\bar{B}(t,T) &= e^{-r(T-t)} E_{\tilde{\mathbb{P}}}\left[\bar{B}(T,T)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)} E_{\tilde{\mathbb{P}}}\left[\bar{D} - \max(\bar{D} - V_T, 0)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\bar{D} - e^{-r(T-t)} E_{\tilde{\mathbb{P}}}\left[\max(\bar{D} - V_T, 0)|\mathcal{F}_t\right].\end{aligned}$$

By the Black-Scholes put option price formula,

$$\begin{aligned}\bar{B}(t,T) &= e^{-r(T-t)}\bar{D} - e^{-r(T-t)}\bar{D}N(-d_2) + V_t N(-d_1) \\ &= e^{-r(T-t)}\bar{D}(1 - N(-d_2)) + V_t N(-d_1) \\ &= e^{-r(T-t)}\bar{D}N(d_2) + V_t N(-d_1) \qquad (5)\end{aligned}$$

where

$$\begin{aligned}d_1 &= \frac{\ln(V_t/\bar{D}) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}\end{aligned}$$

and $N(x)$ is the cumulative density function of the standard normal distribution.

S_t under Black-Scholes model

Answer 1: Having (5) and $V_t = \bar{B}(t,T) + S_t$, we have

$$\begin{aligned} S_t &= V_t - \bar{B}(t,T) \\ &= V_t(1 - N(-d_1)) - e^{-r(T-t)}\bar{D}N(d_2) \\ &= V_t N(d_1) - e^{-r(T-t)}\bar{D}N(d_2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(V_t/\bar{D}) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

S_t under Black-Scholes model

Answer 2: If $\bar{B}(t, T)$ is unknown, we view S_t as European contingent claim on V_t. By Black-Scholes theory, we have

$$S_t = E_{\tilde{\mathbb{P}}}\left[e^{-r(T-t)} S_T | \mathcal{F}_t\right].$$

At the terminal time (i.e. at the maturity T),

$$\begin{aligned} S_T &= \begin{cases} V_T - \bar{D} & , V_T > \bar{D} \\ 0 & , V_T \leq \bar{D} \end{cases} \\ &= \max(V_T - \bar{D}, 0). \end{aligned}$$

Therefore,

$$\begin{aligned} S_t &= E_{\tilde{\mathbb{P}}}\left[e^{-r(T-t)} \max(V_T - \bar{D}, 0) | \mathcal{F}_t\right] \\ &= V_t N(d_1) - e^{-r(T-t)} \bar{D} N(d_2) \end{aligned}$$

* Note : $\bar{B}(t, T) = V_t - S_t$.

Remark

Under Merton's model, regardless how complex a defaultable instrument is, price of a " structural instrument" at time t is given by

$$
\begin{aligned}
\bar{F}_t &= \text{price of the structural instrument} \\
&= \bar{F}(\bar{B}(t,T),t), \\
&\overset{\bar{B}(t,T)=\bar{B}(V_t,t,T)}{=\!=\!=\!=\!=\!=\!=} \bar{F}(V_t,t,T)
\end{aligned}
$$

which is an European contingent claim on V_t.

Hedge

Q:How do the bond holders hedge their risk?
A: The bond holders are "long" in the bond. The only security they can use for hedging is the stock. The stock is the only security available for trade.

- They can buy or sell the stock.
- Let Δ_t ("Delta" position at t) be the number of stock shares bought (or sold) at t.
- The bond holders form a riskless portfolio.

$$\Pi_t = 1 \cdot \bar{B}(t, T) + \Delta_t \cdot S_t$$

(=riskless, like risk free bank account, complete immunization, perfect hedge)

* Bond holders typically (in US) immunize 7% of their holding.

Hedge

The hedge strategy $(a_t, b_t) = (1, \Delta_t)$ should be self-financing

$$\begin{aligned}\Pi_t &= a_t \bar{B}(t,T) + b_t S_t \\ &= \Pi_0 + \underbrace{\int_0^t a_s d\bar{B}(s,T)}_{\text{total gain from keeping the bond}} + \underbrace{\int_0^t b_s dS_s}_{\text{total gain from trading the stock}}\end{aligned}$$

Then

$$d\Pi_t = a_t d\bar{B}(t,T) + b_t dS_t = d\bar{B}(t,T) + \Delta_t dS_t$$

Because the bond holders want full immunization, i.e.

$$\Pi_t = C_t e^{rt} : \text{ like a bank account (no randomness).}$$

So,

$$d\Pi_t = [\ \cdots\ ]dt + \underbrace{0 dW_t}_{\text{no risk}}$$

Hedge

Under the Merton's model

$$\bar{B}(t,T) = \bar{B}(V_t,t,T) = \bar{B}(V_t,t)$$
$$S_t = S(V_t,t) = V_t - \bar{B}(V_t,t)$$

Thus

$$\begin{aligned} d\Pi_t &= 1 \cdot d\bar{B}(V_t,t) + \Delta_t dS(V_t,t) \\ &= [\ \cdots\ ]dt + 0dW_t \end{aligned}$$

(Instantaneously risk free portfolio).

Hedge

By the Ito formula and

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \quad (dV_t)^2 = \sigma^2 V_t^2 dt,$$

we obtain

$$\begin{aligned} d\bar{B}(V_t, t) &= \frac{\partial \bar{B}}{\partial t} dt + \frac{\partial \bar{B}}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 \bar{B}}{\partial V^2} (dV_t)^2 \\ &= \left(\frac{\partial \bar{B}}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 \bar{B}}{\partial V^2} + \mu V_t \frac{\partial \bar{B}}{\partial V} \right) dt + \left(\sigma V_t \frac{\partial \bar{B}}{\partial V} \right) dW_t \end{aligned}$$

and

$$\begin{aligned} dS(V_t, t) &= \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 S}{\partial V^2} (dV_t)^2 \\ &= \left(\frac{\partial S}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 S}{\partial V^2} + \mu V_t \frac{\partial S}{\partial V} \right) dt + \left(\sigma V_t \frac{\partial S}{\partial V} \right) dW_t. \end{aligned}$$

Hedge

Therefore,

$$
\begin{aligned}
& d\Pi_t \\
&= \left[\frac{\partial \bar{B}}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 \bar{B}}{\partial V^2} + \mu V_t \frac{\partial \bar{B}}{\partial V} + \Delta_t \left(\frac{\partial S}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 S}{\partial V^2} + \mu V_t \frac{\partial S}{\partial V}\right)\right] dt \\
&\qquad + \underbrace{\sigma V_t}_{>0} \left(\underbrace{\frac{\partial \bar{B}}{\partial V} + \Delta_t \frac{\partial S}{\partial V}}_{=0}\right) dW_t
\end{aligned}
$$

and hence we obtain

$$\Delta_t = -\frac{\frac{\partial \bar{B}}{\partial V}}{\frac{\partial S}{\partial V}}.$$

Perfect Hedge!!

Generalization of Merton's model

In general $V(t)$ follows Itô process (Continuous diffusion process):

$$dV(t) = \mu(t)V(t)dt + \sigma(t)V(t)dW(t) \text{ on } \mathbb{P} \text{ natural world}$$

$$b(t) = b_0 e^{\int_0^t r(u)du}, \quad r(t) : \text{FRB or ECB rate}, \mathcal{F}(t)\text{adapted}, b_0 = 1.$$

Suppose, there exists unique equivalent martingale measure $\tilde{\mathbb{P}}$. Then every security (portfolio) price $P(t)$ after discounting with $b(t)$ is $\tilde{\mathbb{P}}$-martingale. i.e.

$$\frac{P(t)}{b(t)} = E_{\tilde{\mathbb{P}}}\left[\frac{P(s)}{b(s)}|\mathcal{F}(t)\right], \quad 0 < t < s. \tag{6}$$

then it implies, on $\tilde{\mathbb{P}}$,

$$dP(t) = r(t)P(t)dt + \underbrace{[\cdots\cdots\cdots]}_{\text{historical diffusion coefficient}} d\tilde{W}(t)$$

$$= \text{known from historical data}$$

Generalization of Merton's model

In our case $P(t) = V(t)$. Hence

$$dV(t) = r(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t)$$

where $\tilde{W}(t)$ is a Brownian motion on $\tilde{\mathbb{P}}$. (Note that the process $(\sigma(t))_{t\geq 0}$ is estimated from historical data.)

Generalization of Merton's model

The interest rate on $\tilde{\mathbb{P}}$ will have general form

$$dr(t) = \mu_r(t)dt + \sigma_r(t)d\tilde{W}_r(t)$$

where $\tilde{W}_r(t)$ is a Brownian motion on $\tilde{\mathbb{P}}$.

Example (CIR (Cox Ingersol Ross) model.) In real application, $\mu_r(t)$ and $\sigma_r(t)$ have simple form.

$$dr(t) = (a_r - b_r r(t))dt + \sigma_r\sqrt{r(t)}d\tilde{W}(t) \tag{7}$$

:mean reverting Ornstein-Uhlenbeck process. We estimate $a_r > 0$, $b_r > 0$, and $\sigma_r > 0$ by calibrating the default free term structure interest rate, that is we found the best a^*, b^* and σ^* so that

$$B(0, T_i) = E_{\tilde{\mathbb{P}}}\left[e^{\int_0^t r_{a_r, b_r, \sigma_r}(u)du}\right], i = 1, 2, \cdots, M$$

are as closed as possible to the market prices $B^{\text{market}}(0, T_i)$ at time $t = 0$ (today).

Generalization of Merton's model

Recall

$$dV(t) = r(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t)$$
$$dr(t) = \mu_r(t)dt + \sigma_r(t)d\tilde{W}_r(t).$$

Here $\tilde{W}(t)$ and $\tilde{W}_r(t)$ are Brownian motions on $\tilde{\mathbb{P}}$ and they are correlated

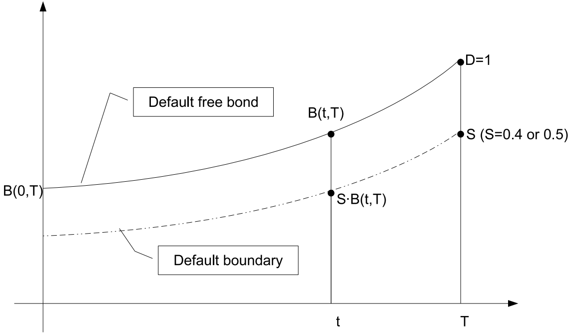
$$d\tilde{W}(t)d\tilde{W}_r(t) = \rho dt,$$

where ρ is correlation coefficient with $-1 \leq \rho \leq 1$. More precisely, $\langle \tilde{W}, \tilde{W}_r \rangle_t = \rho t$

Generalization of Merton's model

- Typically $\rho < 0$: when the interest rate $r(t)$ goes up, the firm cannot easily borrow money and default are more likely, and hence the firm value $V(t)$ goes down.
- ρ must be calibrated from defaultable term structure interest rate.
- In some cases, in practice, ρ is estimated from historical data with the hope that the model is flexible enough to avoid arbitrages.

General Default Boundary



Value of the default free zero with maturity T evaluated at t $(0 \leq t \leq T)$:

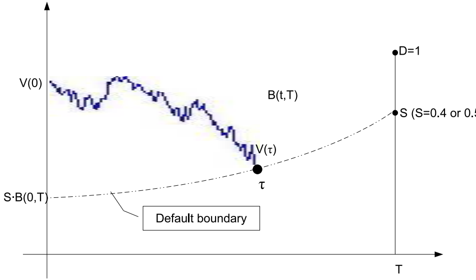
$$B(t,T) = E_{\tilde{\mathbb{P}}}\left[e^{-\int_t^T r(u)du} | \mathcal{F}_t\right]$$

where $r(t)$ is default free Term Structure Interest Rate.

Default Structure

Case 1:

- At time $\tau < T$ (stopping time), $V(\tau)$ hits the boundary, i.e. $V(\tau) = S \cdot B(\tau, T)$.
- Then at τ the bond holders sell the company cost $C > 0$, and get $\bar{B}(\tau, T)$: the value of corporate (defaultable) bond.



$$\bar{B}(\tau, T) = V(\tau) - C$$

Remark:

$$\begin{aligned}\bar{B}(\tau, T) &= V(\tau) - C \\ &= S \cdot B(\tau, T) - C \\ &= (S - \bar{C})B(\tau, T)\end{aligned}$$

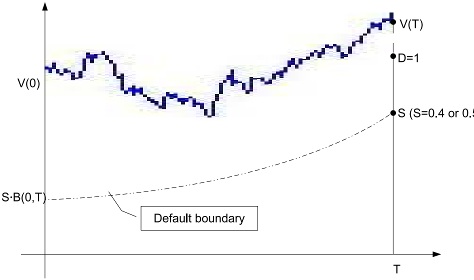
where $\bar{C}$ is relative cost (e.g. 0.05) such that $\bar{C} \cdot B(\tau, T) = C$.

Default Structure

Case 2:

- Then at T, $\bar{B}(T,T)$ is the value of corporate bond at maturity T.

$$B(T,T) = \begin{cases} \bar{D} & , V_T > \bar{D} \\ V_T & , V_T \leq \bar{D} \end{cases}$$
$$= \min(\bar{D}, V_T)$$



Valuation of the Corporate Bond

Since the market is complete with unique equivalent martingale measure $\tilde{\mathbb{P}}$, for any payoff P_s at $s > 0$, we have present value at time $t < s$ as

$$P_t = E_{\tilde{\mathbb{P}}}\left[e^{-\int_t^s r(u)du} P_s | \mathcal{F}_t\right]$$

In our case, $P_t = \bar{B}(t, T)$ is the value of the corporate bond at t. Hence

$$\begin{aligned}
&\bar{B}(t, T) \\
&= E_{\tilde{\mathbb{P}}}\left[1_{\tau < T} e^{-\int_t^\tau r(u)du}(V(\tau) - C) + 1_{\tau \geq T} e^{-\int_t^T r(u)du} min(\bar{D}, V(T)) | \mathcal{F}_t\right]
\end{aligned} \tag{8}$$

Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

We have to do M-C simulation :

$$\begin{cases} dV(t) = r(t)V(t)dt + \sigma(t)V(t)dW_V(t) \\ dr(t) = \mu_r(t)dt + \sigma_r(t)dW_r(t) \\ d\tilde{W}(t)d\tilde{W}_r(t) = \rho dt \end{cases} \tag{9}$$

- We simulate $r(t)$, the default free Term Structure Interest Rate, and obtain the value $B(t, T)$, $0 \leq t \leq T \leq T^*$. (T^* : Time horizon. e.g. 30 years)
- Thus we know the boundary $S \cdot B(t, T)$ for $0 \leq t \leq T$.

Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

Simulate the joint process $(V(t), r(t))$ on $\tilde{\mathbb{P}}$.
Discrete version of (9):

$$\begin{cases} V(t+\Delta t) = V(t) + r(t)V(t)\Delta t + \sigma(t)V(t)(W_V(t+\Delta t) - W_V(t)) \\ r(t+\Delta t) = r(t) + \mu_r(t)\Delta t + \sigma_r(t)(W_r(t+\Delta t) - W_r(t)) \\ corr(W_V(t+\Delta t) - W_V(t), W_r(t+\Delta t) - W_r(t)) = \rho\Delta t \end{cases}, \tag{10}$$

where $t = 0, \Delta t, 2\Delta t, \cdots, (N-1)\Delta t$.
Remark: (10) converges with probability 1 to (9) as $\Delta t \to 0$ if (9) has unique strong solution. The drift and diffusion coefficients must be linear.

Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

For $t = 0$,

$$\begin{cases} V(\Delta t) = V(0) + r(0)V(0)\Delta t + \sigma(0)V(0)\sqrt{\Delta t}\varepsilon_V \\ r(\Delta t) = r(0) + \mu_r(0)\Delta t + \sigma_r(0)\sqrt{\Delta t}\varepsilon_r \end{cases}, \tag{11}$$

Furthermore,

$$\begin{aligned} & corr(\sqrt{\Delta t}\varepsilon_V, \sqrt{\Delta t}\varepsilon_r) = \rho\Delta t \\ \Rightarrow & corr(\varepsilon_V, \varepsilon_r) = \rho \end{aligned}$$

where $\varepsilon_V \sim N(0,1)$ and $\varepsilon_r \sim N(0,1)$.

Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

To simulate the pair $(\varepsilon_V, \varepsilon_r)$, we simulate two independent standard normal random variables (N_1, N_2)
(i.e. $N_1 \sim N(0,1)$, $N_2 \sim N(0,1)$, and $corr(N_1, N_2) = 0$),
we set

$$\begin{aligned} \varepsilon_V &:= N_1 \\ \varepsilon_r &:= \rho N_1 + \sqrt{1-\rho^2} N_2. \end{aligned}$$

Then

$$\begin{aligned} E[\varepsilon_V] = E[\varepsilon_r] &= 0, \\ \mathrm{Var}[\varepsilon_V] = \mathrm{Var}[\varepsilon_r] &= 1, \\ corr(\varepsilon_V, \varepsilon_r) &= \rho. \end{aligned}$$

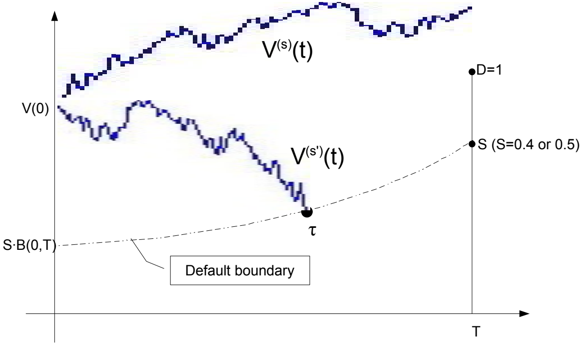
Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

We continue using independent pairs (N_1, N_2) for every step, then we obtain one scenario for

$$(V(t+\Delta t), r(t+\Delta t))_{t=0,\Delta t, 2\Delta t, \cdots, (N-1)\Delta t},$$

using N independent pairs of (N_1, N_2).
We generate S-scenarios $(V^{(j)}(t+\Delta t), r^{(j)}(t+\Delta t))_{t=0,\Delta t, 2\Delta t, \cdots, (N-1)\Delta t}$, $j = 1, 2, \cdots, J$. (e.g. $J = 10000$)

Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$



Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

The value of the corporate bond under scenario s is

$$\begin{aligned}\bar{B}^{(j)}(t,T,\rho) &= 1_{\tau^{(j)}<T} e^{-\int_t^{\tau} r^{(j)}(u)du}(V^{(j)}(\tau)-C) \\ &\quad + 1_{\tau^{(j)}\geq T} e^{-\int_t^{T} r^{(j)}(u)du} min(\bar{D}, V^{(j)}(T))\end{aligned}$$

Given value $\rho \in [-1,1]$, we get the M-C value of the corporate bond

$$\bar{B}(t,T,\rho) = \frac{1}{J}\sum_{j=1}^{J} \bar{B}^{(j)}(t,T,\rho) \tag{12}$$

* ρ has to be calibrated.

Calibration of ρ

- For every $\rho^{(m)} = -1, -1 + \Delta\rho, \cdots, -1 + M\Delta\rho = 1$ (e.g. $M = 200$), we calculate $\bar{B}(0, T, \rho^{(m)})$ using (12).
- For given "credit rating", say BBB, as the credit rating of our firm, we can have data for the market prices $\bar{B}^{market}(0, T_i)$, $i = 1, 2, \cdot, I$.
- We find that ρ^* on the lattice for ρ, such that minimize the following error

$$\sum_{i=1}^{I} \left(\frac{\bar{B}^{market}(0, T_i) - \bar{B}(0, T_i, \rho^{(m)})}{\bar{B}^{market}(0, T_i)} \right)^2$$

Valuation of credit derivatives under Merton's model

Ex1: The option of a corporate bond.
Final payoff: $F(T_1, T_2) = \max(S \cdot B(T_1, T_2) - \bar{B}(T_1, T_2), 0)$, where $0 < t < T_1 < T_2$ (European contingent claim)

$$F(t, T_1, T_2) = E_{\tilde{\mathbb{P}}}\left[e^{-\int_t^{T_1} r(u)du} F(T_1, T_2) | \mathcal{F}_t \right]$$

Having the M-C engine, generate $(V^{(j)}(t), r^{(j)})$, $j = 1, \cdot, J$, then we compute

$$F^{(j)}(t, T_1, T_2) = e^{-\int_t^{T_1} r^{(j)}(u)du} \max(S \cdot B(T_1, T_2) - \bar{B}^{(j)}(T_1, T_2), 0).$$

Finally the M-C value is

$$F(t, T_1, T_2) = \frac{1}{J} \sum_{j=1}^{J} F^{(j)}(t, T_1, T_2).$$

Valuation of credit derivatives under Merton's model

Credit Spread:

- (Default free) Yield Curve: $Y(t,T) = -\frac{1}{T-t}\log B(t,T)$ where $B(t,T) = E[e^{-\int_t^T r(u)du} | \mathcal{F}_t]$ is obtained from the default free TSIR.
- Defaultable Yield Curve: $\bar{Y}(t,T) = -\frac{1}{T-t}\log \bar{B}(t,T)$ where $\bar{B}(t,T)$ is the defaultable bond price given by (8).

Because

$$\bar{B}(t,T) \leq B(t,T),$$

Credit Spread $S(t,T) := \bar{Y}(t,T) - Y(t,T) \geq 0.$

Valuation of credit derivatives under Merton's model

Ex2: Caplet.

- Insurance against potential future Credit Spread.
- It is designed for someone who want to have a protection on nominal principal L (say 10Mio).
- The terminal value at $T_1 < T$:
  $F(T_1, T) = L\delta(T_1, T) \max\left(S(T_1, T) - \bar{S}, 0\right)$ where $\bar{S}$ is fixed, and $\delta(T_1, T)$ is the year fraction between T_1 and T.

Since we have M-C scenario for $\bar{B}^{(j)}(t, T)$ and $B^{(j)}(t, T)$, we have also

$$S^{(j)}(t, T) = \bar{Y}^{(j)}(t, T) - Y^{(j)}(t, T) = -\frac{1}{T - t}(\log \bar{B}^{(j)}(t, T) - \log B^{(j)}(t, T))$$

we have $F^{(j)}(T_1, T) = L\delta(T_1, T) \max\left(S^{(j)}(T_1, T) - \bar{S}, 0\right)$, and hence

$$F(t, T) = \frac{1}{J}\sum_{j=1}^{J} e^{-\int_t^{T_1} r^{(j)}(u)du} F^{(j)}(T_1, T)$$

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