Credit Risk : Firm Value Model

Prof. Dr. Svetlozar Rachev

Institute for Statistics and Mathematical Economics
University of Karlsruhe and Karlsruhe Institute of Technology (KIT)
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Risk

- Market Risk (30% of the total risk)
  - economic factors: current state of the economy
  - government bond rate: default free rate
  - FX-rate (Currency)
  - unemployment rate

- Credit Risk (40% of the total risk)
  - loans
  - defaultable bonds (Corporate bonds)
  - defaultable fixed income securities

- Operational Risk (30% of the total risk)
  - business lines
  - event types
  - management
  - internal / external fraud
Contents

- **Firm value model**
  - The defaults are endogenous
  - Option pricing method
  - APT (Arbitrage Pricing Theory) : Stochastic calculus, risk neutral valuation and no-arbitrage markets

- **Intensity based model**
  - The defaults are exogenous.
  - The model is designed for large portfolios of corporate bonds.

- **Rating based model**
  - Markov chains
  - Rating agencies - nationally recognized statistical rating organizations (NRSROs)

- **Credit derivatives**
  - Description
  - Valuation
Merton’s Firm Value Model

- The defaultable bond and the stock price are derivatives with underlying the value of the firm.
- The default time is endogeneous for the model.
Example

Firm value: \( V_t = D_t + E_t = B_t + S_t \)

Merton:
- What is \( B_t \) the value of the corporate bond at time \( t \)?
- What is \( S_t \) the stock value at time \( t \)?
Simplest Model

Suppose $V_t$ is the value of the firm at $t$. Geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t$$

(1)

where $(W_t)_{t \geq 0}$ is the Brownian motion on the market measure $\mathbb{P}$ (natural world).

Let $r_t$ be the risk free rate at $t$, and assume $r_t \equiv r$.

The bank account:

$$b_t = b_0 e^{rt}, \quad b_0 = 1.$$  \hspace{1cm} (2)

Discount factor:

$$\beta_t = \frac{1}{b_t} = e^{-rt}$$

*(1) and (2) : classical Black-Scholes model for option pricing.*
Let $B_t = \tilde{B}(t, T)$. At maturity $t = T$,

$$B(T, T) = \begin{cases} \bar{D}, & V_T > \bar{D} \\ V_T, & V_T \leq \bar{D} \end{cases}$$

$$= \min(V_T, \bar{D})$$

$$= \bar{D} - \max(\bar{D} - V_T, 0).$$

Hence

$$\tilde{B}(t, T)$$

$$= \text{the value of European contingent claim with } B(T, T) = \min(V_T, \bar{D})$$

$$= E_{\tilde{P}} \left[ e^{-r(T-t)} \tilde{B}(T, T) | \mathcal{F}_t \right]$$

$$= \text{discounted final payoff under risk-neutral measure } \tilde{P} \text{ given } \mathcal{F}_t = \sigma(w_u, u \leq t)$$
Recall that under $\tilde{\mathbb{P}}$ (risk-neutral world),

$$dV_t = rV_t dt + \sigma V_t d\tilde{W}_t$$

(3)

where $(\tilde{W}_t)_{t \geq 0}$ is the Brownian motion on $\tilde{\mathbb{P}}$.

* On the natural world, $\tilde{W}_t = W_t + \theta t$ where $\theta = (\mu - r)/\sigma$ is the market price of risk.

The solution for (3) is

$$V_t = V_0 e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}_t}, \quad t \geq 0. \quad (4)$$

By (4), given information $\mathcal{F}_t$,

$$V_T = V_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma (\tilde{W}_T - \tilde{W}_t)} \quad \text{on } \mathbb{P}.$$
$B_t$ under Black-Scholes model

Therefore,

$$\bar{B}(t, T) = e^{-r(T-t)} E_{\tilde{\mathbb{P}}} \left[ \bar{B}(T, T) | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} E_{\tilde{\mathbb{P}}} \left[ \bar{D} - \max(\bar{D} - V_T, 0) | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} \bar{D} - e^{-r(T-t)} E_{\tilde{\mathbb{P}}} \left[ \max(\bar{D} - V_T, 0) | \mathcal{F}_t \right].$$

By the Black-Scholes put option price formula,

$$\bar{B}(t, T) = e^{-r(T-t)} \bar{D} - e^{-r(T-t)} \bar{D} N(-d_2) + V_t N(-d_1)$$

$$= e^{-r(T-t)} \bar{D}(1 - N(-d_2)) + V_t N(-d_1)$$

$$= e^{-r(T-t)} \bar{D} N(d_2) + V_t N(-d_1)$$  \hspace{1cm} (5)

where

$$d_1 = \frac{\ln(V_t/\bar{D}) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma \sqrt{T - t}$$

and $N(x)$ is the cumulative density function of the standard normal distribution.
Answer 1: Having (5) and $V_t = \bar{B}(t, T) + S_t$, we have

$$S_t = V_t - \bar{B}(t, T)$$

$$= V_t (1 - N(-d_1)) - e^{-r(T-t)} \bar{D}N(d_2)$$

$$= V_t N(d_1) - e^{-r(T-t)} \bar{D}N(d_2)$$

where

$$d_1 = \frac{\ln(V_t/\bar{D}) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma \sqrt{T - t}$$
$S_t$ under Black-Scholes model

Answer 2: If $\bar{B}(t, T)$ is unknown, we view $S_t$ as European contingent claim on $V_t$. By Black-Scholes theory, we have

$$S_t = E_{\tilde{P}} \left[ e^{-r(T-t)} S_T | \mathcal{F}_t \right].$$

At the terminal time (i.e. at the maturity $T$),

$$S_T = \begin{cases} V_T - \bar{D}, & V_T > \bar{D} \\ 0, & V_T \leq \bar{D} \end{cases} = \max(V_T - \bar{D}, 0).$$

Therefore,

$$S_t = E_{\tilde{P}} \left[ e^{-r(T-t)} \max(V_T - \bar{D}, 0) | \mathcal{F}_t \right] = V_t N(d_1) - e^{-r(T-t)} \bar{D} N(d_2)$$

* Note : $\bar{B}(t, T) = V_t - S_t$. 

**Prof. Dr. Svetlozar Rachev (KIT)**

Firm Value Model
Remark

Under Merton’s model, regardless how complex a defaultable instrument is, price of a “structural instrument” at time $t$ is given by

$$\bar{F}_t = \text{price of the structural instrument}$$

$$= \bar{F}(\bar{B}(t, T), t),$$

$$\bar{B}(t, T) = \bar{B}(V_t, t, T) \approx \bar{F}(V_t, t, T)$$

which is an European contingent claim on $V_t$. 
Q: How do the bond holders hedge their risk?
A: The bond holders are “long” in the bond. The only security they can use for hedging is the stock. The stock is the only security available for trade.

- They can buy or sell the stock.
- Let $\Delta_t$ (“Delta” position at $t$) be the number of stock shares bought (or sold) at $t$.
- The bond holders form a riskless portfolio.

$$\Pi_t = 1 \cdot \tilde{B}(t, T) + \Delta_t \cdot S_t$$

(=riskless, like risk free bank account, complete immunization, perfect hedge)

* Bond holders typically (in US) immunize 7% of their holding.
Hedge

The hedge strategy \((a_t, b_t) = (1, \Delta_t)\) should be self-financing

\[
\Pi_t = a_t \bar{B}(t, T) + b_t S_t
\]

\[
= \Pi_0 + \int_0^t a_s d\bar{B}(s, T) + \int_0^t b_s dS_s
\]

\[
\text{total gain from keeping the bond} \quad \text{total gain from trading the stock}
\]

Then

\[
d\Pi_t = a_t d\bar{B}(t, T) + b_t dS_t = d\bar{B}(t, T) + \Delta_t dS_t
\]

Because the bond holders want full immunization, i.e.

\[
\Pi_t = C_t e^{rt} : \text{like a bank account (no randomness)}.
\]

So,

\[
d\Pi_t = [ \cdots ] dt + 0 dW_t
\]

\[
\text{no risk}
\]
Hedge

Under the Merton’s model

\[ \bar{B}(t, T) = \bar{B}(V_t, t, T) = \bar{B}(V_t, t) \]
\[ S_t = S(V_t, t) = V_t - \bar{B}(V_t, t) \]

Thus

\[
d\Pi_t = 1 \cdot d\bar{B}(V_t, t) + \Delta_t dS(V_t, t)
\]
\[
= [ \cdots ] dt + 0dW_t
\]

(Instantaneously risk free portfolio).
Hedge

By the Ito formula and

\[ dV_t = \mu V_t dt + \sigma V_t dW_t, \quad (dV_t)^2 = \sigma^2 V_t^2 dt, \]

we obtain

\[
\begin{align*}
    d\tilde{B}(V_t, t) &= \frac{\partial \tilde{B}}{\partial t} dt + \frac{\partial \tilde{B}}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 \tilde{B}}{\partial V^2} (dV_t)^2 \\
    &= \left( \frac{\partial \tilde{B}}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 \tilde{B}}{\partial V^2} + \mu V_t \frac{\partial \tilde{B}}{\partial V} \right) dt + \left( \sigma V_t \frac{\partial \tilde{B}}{\partial V} \right) dW_t
\end{align*}
\]

and

\[
\begin{align*}
    dS(V_t, t) &= \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial V} dV_t + \frac{1}{2} \frac{\partial^2 S}{\partial V^2} (dV_t)^2 \\
    &= \left( \frac{\partial S}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 S}{\partial V^2} + \mu V_t \frac{\partial S}{\partial V} \right) dt + \left( \sigma V_t \frac{\partial S}{\partial V} \right) dW_t.
\end{align*}
\]
Hedge

Therefore,

\[ d\Pi_t = \left[ \frac{\partial \tilde{B}}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 \tilde{B}}{\partial V^2} + \mu V_t \frac{\partial \tilde{B}}{\partial V} + \Delta_t \left( \frac{\partial S}{\partial t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 S}{\partial V^2} + \mu V_t \frac{\partial S}{\partial V} \right) \right] dt \]

\[ + \sigma V_t \left( \frac{\partial \tilde{B}}{\partial V} + \Delta_t \frac{\partial S}{\partial V} \right) dW_t \]

and hence we obtain

\[ \Delta_t = -\frac{\partial \tilde{B}}{\partial S} \frac{\partial S}{\partial V}. \]

Perfect Hedge!!
Generalization of Merton’s model

In general \( V(t) \) follows Itô process (Continuous diffusion process):

\[
dV(t) = \mu(t) V(t) dt + \sigma(t) V(t) dW(t) \quad \text{on } \mathbb{P} \text{ natural world}
\]

\[
b(t) = b_0 e^{\int_0^t r(u) du}, \quad r(t) : \text{FRB or ECB rate, } \mathcal{F}(t) \text{adapted}, b_0 = 1.
\]

Suppose, there exists unique equivalent martingale measure \( \tilde{\mathbb{P}} \). Then every security (portfolio) price \( P(t) \) after discounting with \( b(t) \) is \( \tilde{\mathbb{P}} \)-martingale. i.e.

\[
\frac{P(t)}{b(t)} = E_{\tilde{\mathbb{P}}} \left[ \frac{P(s)}{b(s)} | \mathcal{F}(t) \right], \quad 0 < t < s. \quad (6)
\]

then it implies, on \( \tilde{\mathbb{P}} \),

\[
dP(t) = r(t) P(t) dt + \underbrace{\text{historical diffusion coefficient}} \quad d\tilde{W}(t)
\]

\[= \text{known from historical data}\]
Generalization of Merton’s model

In our case $P(t) = V(t)$. Hence

$$dV(t) = r(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t)$$

where $\tilde{W}(t)$ is a Brownian motion on $\tilde{P}$. (Note that the process $(\sigma(t))_{t \geq 0}$ is estimated from historical data.)
Generalization of Merton’s model

The interest rate on $\tilde{P}$ will have general form

$$dr(t) = \mu_r(t)dt + \sigma_r(t)d\tilde{W}_r(t)$$

where $\tilde{W}_r(t)$ is a Brownian motion on $\tilde{P}$.

**Example** (CIR (Cox Ingersol Ross) model.) In real application, $\mu_r(t)$ and $\sigma_r(t)$ have simple form.

$$dr(t) = (a_r - b_r r(t)) dt + \sigma_r \sqrt{r(t)} d\tilde{W}(t)$$

(7)

:mean reverting Ornstein-Uhlenbeck process. We estimate $a_r > 0$, $b_r > 0$, and $\sigma_r > 0$ by calibrating the default free term structure interest rate, that is we found the best $a^*$, $b^*$ and $\sigma^*$ so that

$$B(0, T_i) = E_{\tilde{P}} \left[ e^{\int_0^t r_{ar, br, \sigma_r(u)}du} \right], i = 1, 2, \ldots, M$$

are as closed as possible to the market prices $B^{\text{market}}(0, T_i)$ at time $t = 0$ (today).
Generalization of Merton’s model

Recall

\[ dV(t) = r(t) V(t) dt + \sigma(t) V(t) d\tilde{W}(t) \]
\[ dr(t) = \mu_r(t) dt + \sigma_r(t) d\tilde{W}_r(t). \]

Here \( \tilde{W}(t) \) and \( \tilde{W}_r(t) \) are Brownian motions on \( \tilde{P} \) and they are correlated

\[ d\tilde{W}(t) d\tilde{W}_r(t) = \rho dt, \]

where \( \rho \) is correlation coefficient with \( -1 \leq \rho \leq 1 \). More precisely,

\[ \langle \tilde{W}, \tilde{W}_r \rangle_t = \rho t \]
Generalization of Merton’s model

- Typically \( \rho < 0 \): when the interest rate \( r(t) \) goes up, the firm cannot easily borrow money and default are more likely, and hence the firm value \( V(t) \) goes down.
- \( \rho \) must be calibrated from defaultable term structure interest rate.
- In some cases, in practice, \( \rho \) is estimated from historical data with the hope that the model is flexible enough to avoid arbitrages.
General Default Boundary

Value of the default free zero with maturity $T$ evaluated at $t$ ($0 \leq t \leq T$):

$$B(t, T) = E_{\tilde{P}} \left[ e^{-\int_t^T r(u) du} | \mathcal{F}_t} \right]$$

where $r(t)$ is default free Term Structure Interest Rate.
Case 1:

- At time $\tau < T$ (stopping time), $V(\tau)$ hits the boundary, i.e. $V(\tau) = S \cdot B(\tau, T)$.

- Then at $\tau$ the bond holders sell the company cost $C > 0$, and get $\bar{B}(\tau, T)$: the value of corporate (defaultable) bond.

$$\bar{B}(\tau, T) = V(\tau) - C$$

Remark:

$$\bar{B}(\tau, T) = V(\tau) - C = S \cdot B(\tau, T) - C = (S - \bar{C})B(\tau, T)$$

where $\bar{C}$ is relative cost (e.g. 0.05) such that $\bar{C} \cdot B(\tau, T) = C$. 
Case 2:
- Then at $T$, $\bar{B}(T, T)$ is the value of corporate bond at maturity $T$.

$$B(T, T) = \begin{cases} \bar{D} & V_T > \bar{D} \\ V_T & V_T \leq \bar{D} \end{cases} = \min(\bar{D}, V_T)$$
Valuation of the Corporate Bond

Since the market is complete with unique equivalent martingale measure $\tilde{P}$, for any payoff $P_s$ at $s > 0$, we have present value at time $t < s$ as

$$P_t = E_{\tilde{P}} \left[ e^{-\int_t^s r(u)du} P_s | F_t \right]$$

In our case, $P_t = \bar{B}(t, T)$ is the value of the corporate bond at $t$. Hence

$$\bar{B}(t, T) = E_{\tilde{P}} \left[ 1_{\tau < T} e^{-\int_t^\tau r(u)du} (V(\tau) - C) + 1_{\tau \geq T} e^{-\int_t^T r(u)du} min(\bar{D}, V(T)) | F_t \right]$$

(8)
Monte-Carlo Valuation of (8) on $\tilde{P}$

We have to do M-C simulation:

\[
\begin{aligned}
    dV(t) &= r(t)V(t)\,dt + \sigma(t)V(t)\,dW_V(t) \\
    dr(t) &= \mu_r(t)\,dt + \sigma_r(t)\,dW_r(t) \\
    d\tilde{W}(t)d\tilde{W}_r(t) &= \rho\,dt
\end{aligned}
\]

(9)

- We simulate $r(t)$, the default free Term Structure Interest Rate, and obtain the value $B(t, T)$, $0 \leq t \leq T \leq T^*$. ($T^*$: Time horizon. e.g. 30 years)

- Thus we know the boundary $S \cdot B(t, T)$ for $0 \leq t \leq T$. 
Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

Simulate the joint process $(V(t), r(t))$ on $\tilde{\mathbb{P}}$.

Discrete version of (9):

$$
\begin{align*}
V(t + \Delta t) &= V(t) + r(t)V(t)\Delta t + \sigma(t)V(t)(W_V(t + \Delta t) - W_V(t)) \\
r(t + \Delta t) &= r(t) + \mu_r(t)\Delta t + \sigma_r(t)(W_r(t + \Delta t) - W_r(t)) \\
corr(W_V(t + \Delta t) - W_V(t), W_r(t + \Delta t) - W_r(t)) &= \rho \Delta t
\end{align*}
$$

(10)

where $t = 0, \Delta t, 2\Delta t, \cdots, (N - 1)\Delta t$.

Remark: (10) converges with probability 1 to (9) as $\Delta t \to 0$ if (9) has unique strong solution. The drift and diffusion coefficients must be linear.
Monte-Carlo Valuation of (8) on $\tilde{P}$

For $t = 0$,

\[
\begin{align*}
V(\Delta t) &= V(0) + r(0) V(0) \Delta t + \sigma(0) V(0) \sqrt{\Delta t} \varepsilon_V, \\
r(\Delta t) &= r(0) + \mu_r(0) \Delta t + \sigma_r(0) \sqrt{\Delta t} \varepsilon_r
\end{align*}
\]

Furthermore,

\[
corr(\sqrt{\Delta t} \varepsilon_V, \sqrt{\Delta t} \varepsilon_r) = \rho \Delta t
\]

\[
\Rightarrow corr(\varepsilon_V, \varepsilon_r) = \rho
\]

where $\varepsilon_V \sim N(0, 1)$ and $\varepsilon_r \sim N(0, 1)$. 
Monte-Carlo Valuation of (8) on $\tilde{P}$

To simulate the pair $(\varepsilon_V, \varepsilon_r)$, we simulate two independent standard normal random variables $(N_1, N_2)$ (i.e. $N_1 \sim N(0, 1)$, $N_2 \sim N(0, 1)$, and $\text{corr}(N_1, N_2) = 0$), we set

\begin{align*}
\varepsilon_V &:= N_1 \\
\varepsilon_r &:= \rho N_1 + \sqrt{1 - \rho^2} N_2.
\end{align*}

Then

\begin{align*}
E[\varepsilon_V] = E[\varepsilon_r] &= 0, \\
\text{Var}[\varepsilon_V] = \text{Var}[\varepsilon_r] &= 1, \\
\text{corr}(\varepsilon_V, \varepsilon_r) &= \rho.
\end{align*}
Monte-Carlo Valuation of (8) on $\hat{\mathbb{P}}$

We continue using independent pairs $(N_1, N_2)$ for every step, then we obtain one scenario for

$$(V(t + \Delta t), r(t + \Delta t))_{t=0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta t},$$

using $N$ independent pairs of $(N_1, N_2)$.

We generate $S$-scenarios $(V^{(j)}(t + \Delta t), r^{(j)}(t + \Delta t))_{t=0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta t}$, $j = 1, 2, \cdots, J$. (e.g. $J = 10000$)
Monte-Carlo Valuation of (8) on \( \tilde{\mathbb{P}} \)

\[ V(s)(t) \]

\[ V^{(s)}(t) \]

\[ S \cdot B(0,T) \]

\[ \tau \]

\[ D=1 \]

\[ S \ (S=0.4 \ or \ 0.5) \]

Default boundary
Monte-Carlo Valuation of (8) on $\tilde{\mathbb{P}}$

The value of the corporate bond under scenario $s$ is

$$\tilde{B}^{(j)}(t, T, \rho) = 1_{\tau^{(j)} < T} e^{-\int_t^T r^{(j)}(u) du} \left( V^{(j)}(\tau) - C \right) + 1_{\tau^{(j)} \geq T} e^{-\int_t^T r^{(j)}(u) du} \min(\tilde{D}, V^{(j)}(T))$$

Given value $\rho \in [-1, 1]$, we get the M-C value of the corporate bond

$$\tilde{B}(t, T, \rho) = \frac{1}{J} \sum_{j=1}^J \tilde{B}^{(j)}(t, T, \rho)$$

(12)

* $\rho$ has to be calibrated.
Calibration of $\rho$

- For every $\rho^{(m)} = -1, -1 + \Delta \rho, \ldots, -1 + M\Delta \rho = 1$ (e.g. $M = 200$), we calculate $\bar{B}(0, T, \rho^{(m)})$ using (12).
- For given “credit rating”, say BBB, as the credit rating of our firm, we can have data for the market prices $\bar{B}^{\text{market}}(0, T_i)$, $i = 1, 2, \ldots, I$.
- We find that $\rho^*$ on the lattice for $\rho$, such that minimize the following error

$$
\sum_{i=1}^{I} \left( \frac{\bar{B}^{\text{market}}(0, T_i) - \bar{B}(0, T_i, \rho^{(m)})}{\bar{B}^{\text{market}}(0, T_i)} \right)^2
$$
Valuation of credit derivatives under Merton’s model

Ex1: The option of a corporate bond.
Final payoff: \( F(T_1, T_2) = \max(S \cdot B(T_1, T_2) - \bar{B}(T_1, T_2), 0) \), where \( 0 < t < T_1 < T_2 \) (European contingent claim)

\[
F(t, T_1, T_2) = E_\tilde{P} \left[ e^{-\int_t^{T_1} r(u)du} \max(S \cdot B(T_1, T_2) - \bar{B} \cdot B(T_1, T_2), 0) \bigg| \mathcal{F}_t \right]
\]

Having the M-C engine, generate \((V^{(j)}(t), r^{(j)}), j = 1, \ldots, J\), then we compute

\[
F^{(j)}(t, T_1, T_2) = e^{-\int_t^{T_1} r^{(j)}(u)du} \max(S \cdot B(T_1, T_2) - \bar{B}^{(j)}(T_1, T_2), 0).
\]

Finally the M-C value is

\[
F(t, T_1, T_2) = \frac{1}{J} \sum_{j=1}^{J} F^{(j)}(t, T_1, T_2).
\]
Valuation of credit derivatives under Merton’s model

Credit Spread:

- (Default free) Yield Curve: \( Y(t, T) = -\frac{1}{T-t} \log B(t, T) \) where
  \[ B(t, T) = E\left[e^{-\int_t^T r(u)du} | \mathcal{F}_t}\right] \] is obtained from the default free TSIR.
- Defaultable Yield Curve: \( \bar{Y}(t, T) = -\frac{1}{T-t} \log \bar{B}(t, T) \) where \( \bar{B}(t, T) \) is the defaultable bond price given by (8).

Because

\[ \bar{B}(t, T) \leq B(t, T), \]

Credit Spread \( S(t, T) := \bar{Y}(t, T) - Y(t, T) \geq 0. \)
Valuation of credit derivatives under Merton’s model
Ex2: Caplet.

- Insurance against potential future Credit Spread.
- It is designed for someone who want to have a protection on
nominal principal $L$ (say 10Mio).
- The terminal value at $T_1 < T$:
  $$F(T_1, T) = L\delta(T_1, T) \max \left( S(T_1, T) - \bar{S}, 0 \right)$$
  where $\bar{S}$ is fixed, and
  $\delta(T_1, T)$ is the year fraction between $T_1$ and $T$.

Since we have M-C scenario for $\bar{B}^{(j)}(t, T)$ and $B^{(j)}(t, T)$, we have also

$$S^{(j)}(t, T) = \bar{Y}^{(j)}(t, T) - Y^{(j)}(t, T) = -\frac{1}{T - t} \left( \log \bar{B}^{(j)}(t, T) - \log B^{(j)}(t, T) \right)$$

we have $F^{(j)}(T_1, T) = L\delta(T_1, T) \max \left( S^{(j)}(T_1, T) - \bar{S}, 0 \right)$, and hence

$$F(t, T) = \frac{1}{J} \sum_{j=1}^{J} e^{-\int_{t}^{T_1} r^{(j)}(u) du} F^{(j)}(T_1, T)$$
References

D. Lando (2004).
Credit Risk Modeling
Princeton Series in Finance

The Pricing of Credit Risk and Credit Derivatives
http://www.schonbucher.de/papers/bookfo.pdf

Credit Portfolio Risk and PD Confidence Sets through the Business Cycle
https://www.statistik.uni-karlsruhe.de/download/tr_credit_portfolio_risk.pdf