Credit Risk: Intensity Based Model

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Intensity Based Model

Firm value model

- The model explains the defaultable term structure of interest rate.
- It is not applicable for large portfolio of corporate bonds.
- The defaults are endogenous:

$$\bar{B}(t, T) = \bar{B}(t, V_t, r_t, T),$$

where $V_t$ is the value of the firm and $r_t$ is default free interest rate.

Intensity based model

- The model is designed for large portfolios of corporate bonds.
- It does not explain defaultable term structure of interest rate.
- It fits term structure of interest rate into market data.
- The defaults are exogenous.

$$\bar{B}(t, T) = \bar{B}(t, N_t, r_t, T),$$

where $N_t$ is the number of defaults in $[0, T]$ in the portfolio. $N_t$ will be modeled by the Non-homogeneous Poisson Process named Cox process.
Poisson Process

Definition (1)

\((N_t)_{t \geq 0}\) is a (simple, homogeneous) Poisson process with an intensity \(\lambda > 0\), iff

i  \(N(0) = 0\)

ii  It has independent and stationary increments.

\(\cdot\)  \((N_{t_i} - N_{t_{i-1}})_{i \geq 1}\) are independent.

\(\cdot\)  \(N_{t_i+s} - N_{t_{i-1}+s} \overset{d}{=} N_{t_i} - N_{t_{i-1}}\) for all \(i\).

\[0 \leq t_0 < t_1 < \cdots < t_n.\]

iii  \(\mathbb{P}(N_{t+s} - N_t = k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s}\) : the probability of \(k\)-defaults in \([t, t+s]\)
Poisson Process

- \( (N_t) \) is a process with a left limit and right continuity.
- It has the following properties.
  - \( \mathbb{P}(N_{t+\Delta t} - N_t = 0) = \frac{(\lambda\Delta t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} = 1 - \lambda \Delta t + o(\Delta t) \)
  - \( \mathbb{P}(N_{t+\Delta t} - N_t = 1) = \frac{\lambda\Delta t}{1!} e^{-\lambda t} = \lambda \Delta t + o(\Delta t) \)
  - \( \mathbb{P}(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t) \)
- \( E[N_t] = \lambda t \): the mean of number of defaults in \([0, t]\). \( \lambda > 0 \) is the default intensity.
- The intensity \( \lambda \) is time independent.
Non-homogeneous Poisson process

**Definition**

\((N_t)_{t\geq 0}\) is a non-homogeneous Poisson process with an intensity 
\(\lambda_t = \lambda(t) > 0, \ t \geq 0\), iff

- i, ii of Definition (1) hold.
- \(\mathbb{P}(N_{t+s} - N_t = k) = \frac{(\int_s^{s+t} \lambda(u)du)^k}{k!} e^{-\int_s^{s+t} \lambda(u)du} : \) the probability of \(k\)-defaults in \([t, t + s]\)

**Asymptotic property**

- \(\mathbb{P}(N_{t+\Delta t} - N_t = 0) = 1 - \lambda_t \Delta t + o(\Delta t)\)
- \(\mathbb{P}(N_{t+\Delta t} - N_t = 1) = \lambda_t \Delta t + o(\Delta t)\)
- \(\mathbb{P}(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t)\)

The default intensity \(\lambda_t\) is, in fact, a random process depending of the macro-economic environment.
Cox Processes

**Definition**

Cox-Process \((N_t)_{t \geq 0}\) is a Poisson process with stochastic intensity \((\lambda_t)_{t \geq 0}\).

- If the intensity \(\lambda_t\) is a random process which gives only one trajectory (random path), say \(\tilde{\lambda}_t\), then \((N_t)_{t \geq 0}\) is a non-homogeneous Poisson process with intensity \(\tilde{\lambda}_t\).
Cox Processes

- In intensity based model, \((\lambda_t)_{t \geq 0}\) is an Itô process with mean reverting property,

\[
d\lambda_t = \mu_\lambda(t)dt + \sigma_\lambda(t)dW_t^\lambda.
\]
on the \(\tilde{\mathbb{P}}\)-risk-neutral world.

- The default-free interest rate (e.g. ECB-rate) is also an Itô process

\[
dr_t = \mu_r(t)dt + \sigma_r(t)dW_t^r
\]
on the \(\tilde{\mathbb{P}}\)-risk-neutral world.

- Here

\[
dW_t^\lambda dW_t^r = \rho dt, \quad -1 < \rho < 1.
\]
Zero Recovery Security Pricing

- Value of a defaultable bond with zero recovery rate.
Zero Recovery Security Pricing

- $\tilde{B}(t, T) = \tilde{B}(t, N_t, r_t, T)$.
  - $N_t$: non-homogeneous Poisson process with intensity $\lambda_t$
- By Itô-formula and Arbitrage Pricing Theory (APT), we obtain

$$\frac{\partial \tilde{B}}{\partial t} + \frac{\partial \tilde{B}}{\partial r} \mu_r(t) + \frac{1}{2} \frac{\partial^2 \tilde{B}}{\partial r^2} \sigma_r^2(t) - \tilde{B}(t, T)(\lambda_t + r_t) = 0$$  \hspace{1cm} (1)

- (1) is a generalization of the PDE for $B(t, T)$, default-free bond, when $\lambda_t = 0$,

$$\frac{\partial B}{\partial t} + \mu_r(t) \frac{\partial B}{\partial r} + \frac{1}{2} \sigma_r^2(t) \frac{\partial^2 B}{\partial r^2} - r_t B(t, T) = 0.$$

The solution

$$B(t, T) = E^\tilde{P}_t \left[ e^{-\int_t^T r_u du} \right].$$

- Hence, the solution of (1) is

$$\tilde{B}(t, T) = B(t, T) e^{-\int_t^T \lambda_u du}.$$
Zero Recovery Security Pricing

- In terms of the yield, we have
  \[ B(t, T) = e^{-Y_{t,T}(T-t)} \quad \bar{B}(t, T) = e^{-\bar{Y}_{t,T}(T-t)} \]

  where \( Y_{t,T} \) is the yield of default free bond and \( \bar{Y}_{t,T} \) is the yield of defaultable bond.

- Spread
  \[ S(t, T) = \bar{Y}_{t,T} - Y_{t,T} = \frac{1}{T-t} \left( \ln B(t, T) - \ln \bar{B}(t, T) \right) \]
  \[ = \frac{1}{T-t} \int_{t}^{T} \lambda_u du. \]

  Note that \( \bar{Y}_{t,T} - Y_{t,T} \geq 0 \), since \( \bar{B}(t, T) \leq B(t, T) \).

- In case \( \lambda_t \equiv \lambda \), \( S(t, T) = \lambda \): the default intensity.
Pricing with Fractional Recovery
Value of a defaultable bond (or portfolio) with fractional recovery rate.
Pricing with Fractional Recovery

- The pricing equation

\[
\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial r} \mu_r(t) + \frac{1}{2} \frac{\partial^2 \bar{B}}{\partial r^2} \sigma_r^2(t) - \bar{B}(t, T)(q\lambda_t + r_t) = 0
\]

- Solution:

\[
\bar{B}(t, T) = B(t, T) e^{-\int_t^T q\lambda_u du}.
\]

- Spread

\[
S(t, T) = \frac{1}{T-t} \int_t^T q\lambda_u du.
\]

- Equation

\[
\bar{B}(t, T) = B(t, T) e^{-\int_t^T q\lambda_u du} = E_t^{\tilde{P}} \left[ e^{-\int_t^T r_u + q\lambda_u du} \right].
\]

implies \( \bar{r}_t = r_t + q\lambda_t \): Defaultable short rate.
Pricing with Stochastic Intensity

Consider the risk-free interest rate and the intensity of the Cox process:

\[ dr_t = \mu_r(t)dt + \sigma_r(t)dW^r_t \]
\[ d\lambda_t = \mu_\lambda(t)dt + \sigma_\lambda(t)(\rho dW^r_t + \sqrt{1 - \rho^2}d\bar{W}_t). \]

The pricing equation

\[
0 = \frac{\partial \tilde{B}}{\partial t} + \mu_r(t)\frac{\partial \tilde{B}}{\partial r} + \mu_\lambda(t)\frac{\partial \tilde{B}}{\partial \lambda} + \frac{1}{2}\sigma^2_r(t)\frac{\partial^2 \tilde{B}}{\partial r^2} \\
+ \rho \sigma_r(t)\sigma_\lambda(t)\frac{\partial^2 \tilde{B}}{\partial r \partial \lambda} + \frac{1}{2}\sigma^2_\lambda(t)\frac{\partial^2 \tilde{B}}{\partial \lambda^2} - (r + q\lambda_t)\tilde{B}.
\]

The final condition is \( \tilde{B}(T, r, \lambda) = 1 \). The boundary conditions are \( \tilde{B} \to 0 \) as \( r, \lambda \to \infty \), and \( \tilde{B} < \infty \) as \( r, \lambda \to 0 \).
Pricing with Stochastic Intensity

Solution:

\[ \bar{B}(t, T) = \mathbb{E}_t^{\mathbb{P}} \left[ e^{-\int_t^T \bar{r}_u du} \right] \]

where \( \bar{r}_u = r_t + q\lambda_t \).

Credit derivative pricing:

\[ F(t, T) = \mathbb{E}_t^{\mathbb{P}} \left[ e^{-\int_t^T \bar{r}_u du} X \right] \]

where \( X \) is the value of a default affected payoff.
Example

Example: CIR model

\[ dr_t = (a_r - b_r r_t)dt + \sigma_r \sqrt{r_t} dW_t \]
\[ d\lambda_t = (a_\lambda - b_\lambda \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t}(\rho dW_t + \sqrt{1 - \rho^2} d\bar{W}_t) \]

- The constant \( a_r, b_r \) and \( \sigma_r \) are calibrated on the default free term structure of interest rate.
- The constant \( a_\lambda, b_\lambda, \sigma_\lambda, \rho, \) and \( q \) should be calibrated from the defaultable term structure of interest rate (= Market data).
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