Credit Risk : Intensity Based Model

Prof. Dr. Svetlozar Rachev

Institute for Statistics and Mathematical Economics
University of Karlsruhe and Karlsruhe Institute of Technology (KIT)

Copyright

Prof. Svetlozar (Zari) T. Rachev
Chair of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
Tel. +49-721-608-7535, +49-721-608-2042(s)
Fax: +49-721-608-3811
http://www.statistik.uni-karslruhe.de

Intensity Based Model

- Firm value model
  - The model explains the defaultable term structure of interest rate.
  - it is not applicable for large portfolio of corporate bonds.
  - The defaults are endogenous:

$$\bar{B}(t,T) = \bar{B}(t, V_t, r_t, T),$$

    where V_t is the value of the firm and r_t is default free interest rate.

- Intensity based model
  - the model is designed for large portfolios of corporate bonds.
  - it does not explain defaultable term structure of interest rate.
  - it fits term structure of interest rate into market data.
  - The defaults are exogenous.

$$\bar{B}(t,T) = \bar{B}(t, N_t, r_t, T),$$

    where N_t is the number of defaults in $[0, T]$ in the portfolio. N_t will be modeled by the Non-homogeneous Poisson Process named Cox process.

Poisson Process

Definition (1)

$(N_t)_{t\geq 0}$ is a (simple, homogeneous) Poisson process with an intensity $\lambda > 0$, iff

i $N(0) = 0$

ii It has independent and stationary increments.

- $(N_{t_i} - N_{t_{i-1}})_{i\geq 1}$ are independent.
- $N_{t_i+s} - N_{t_{i-1}+s} \stackrel{d}{=} N_{t_i} - N_{t_{i-1}}$ for all i.

$0 \leq t_0 < t_1 < \cdots < t_n$.

iii $\mathbb{P}(N_{t+s} - N_t = k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s}$: the probability of k-defaults in $[t, t+s]$

Poisson Process

- (N_t) is a process with a left limit and right continuity.
- It has the following properties.
  - $\mathbb{P}(N_{t+\Delta t} - N_t = 0) = \frac{(\lambda\Delta t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} = 1 - \lambda\Delta t + o(\Delta t)$
  - $\mathbb{P}(N_{t+\Delta t} - N_t = 1) = \frac{\lambda\Delta t}{1!} e^{-\lambda t} = \lambda\Delta t + o(\Delta t)$
  - $\mathbb{P}(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t)$
- $E[N_t] = \lambda t$: the mean of number of defaults in $[0, t]$. $\lambda > 0$ is the default intensity.
- The intensity λ is time independent.

Non-homogeneous Poisson process

Definition

$(N_t)_{t\geq 0}$ is a non-homogeneous Poisson process with an intensity $\lambda_t = \lambda(t) > 0$, $t \geq 0$, iff

- i, ii of Definition (1) hold.

iii $\mathbb{P}(N_{t+s} - N_t = k) = \frac{(\int_t^{s+t} \lambda(u)du)^k}{k!} e^{-\int_t^{s+t} \lambda(u)du}$: the probability of k-defaults in $[t, t+s]$

- Asymptotic property
  - $\mathbb{P}(N_{t+\Delta t} - N_t = 0) = 1 - \lambda_t \Delta t + o(\Delta t)$
  - $\mathbb{P}(N_{t+\Delta t} - N_t = 1) = \lambda_t \Delta t + o(\Delta t)$
  - $\mathbb{P}(N_{t+\Delta t} - N_t \geq 2) = o(\Delta t)$
- The default intensity λ_t is, in fact, a random process depending of the macro-economic environment.

Cox Processes

Definition

Cox-Process $(N_t)_{t\geq 0}$ is a Poisson process with stochastic intensity $(\lambda_t)_{t\geq 0}$.

- If the intensity λ_t is a random process which gives only one trajectory (random path), say $\tilde{\lambda}_t$, then $(N_t)_{t\geq 0}$ is a non-homogeneous Poisson process with intensity $\tilde{\lambda}_t$.

Cox Processes

- In intensity based model, $(\lambda_t)_{t\geq 0}$ is an Itô process with mean reverting property,

$$d\lambda_t = \mu_\lambda(t)dt + \sigma_\lambda(t)dW_t^\lambda.$$

  on the $\tilde{\mathbb{P}}$-risk-neutral world.
- The default-free interest rate (e.g. ECB-rate) is also an Itô process

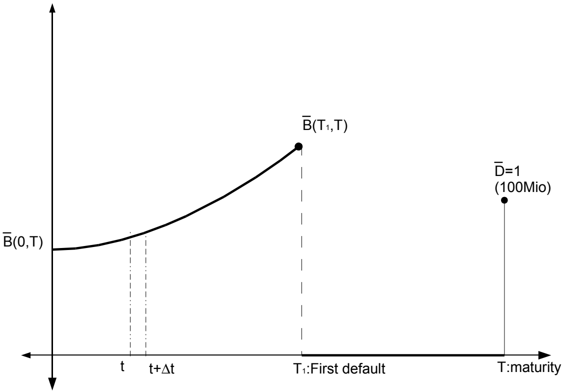
$$dr_t = \mu_r(t)dt + \sigma_r(t)dW_t^r$$

  on the $\tilde{\mathbb{P}}$-risk-neutral world.
- Here

$$dW_t^\lambda dW_t^r = \rho dt, \quad -1 < \rho < 1.$$

Zero Recovery Security Pricing

- Value of a defaultable bond with zero recovery rate.



Zero Recovery Security Pricing

- $\bar{B}(t,T) = \bar{B}(t, N_t, r_t, T)$.
  N_t : non-homogeneous Poisson process with intensity λ_t
- By Itô-formula and Arbitrage Pricing Theory (APT), we obtain

$$\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial r}\mu_r(t) + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2(t) - \bar{B}(t,T)(\lambda_t + r_t) = 0 \qquad (1)$$

- (1) is a generalization of the PDE for $B(t,T)$, default-free bond, when $\lambda_t = 0$,

$$\frac{\partial B}{\partial t} + \mu_r(t)\frac{\partial B}{\partial r} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 B}{\partial r^2} - r_t B(t,T) = 0.$$

  The solution

$$B(t,T) = E_t^{\tilde{\mathbb{P}}}\left[e^{-\int_t^T r_u du}\right].$$

- Hence, the solution of (1) is

$$\bar{B}(t,T) = B(t,T)e^{-\int_t^T \lambda_u du}.$$

Zero Recovery Security Pricing

- In terms of the yield, we have

$$B(t,T) = e^{-Y_{t,T}(T-t)} \quad \bar{B}(t,T) = e^{-\bar{Y}_{t,T}(T-t)}$$

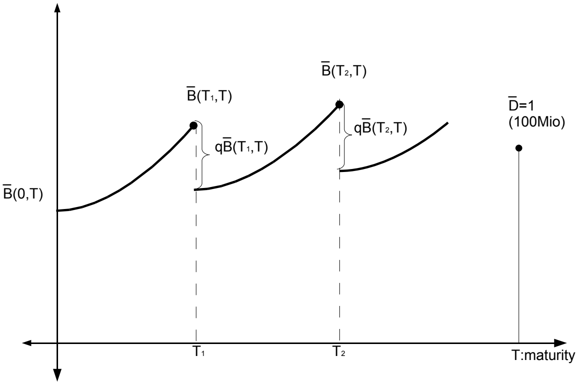
  where $Y_{t,T}$ is the yield of default free bond and $\bar{Y}_{t,T}$ is the yield of defaultable bond.
- Spread

$$\begin{aligned} S(t,T) = \bar{Y}_{t,T} - Y_{t,T} &= \frac{1}{T-t}\left(\ln B(t,T) - \ln \bar{B}(t,T)\right) \\ &= \frac{1}{T-t}\int_t^T \lambda_u du. \end{aligned}$$

  Note that $\bar{Y}_{t,T} - Y_{t,T} \geq 0$, since $\bar{B}(t,T) \leq B(t,T)$.
- In case $\lambda_t \equiv \lambda$, $S(t,T) = \lambda$: the default intensity.

Pricing with Fractional Recovery

Value of a defaultable bond (or portfolio) with fractional recovery rate.



Pricing with Fractional Recovery

- The pricing equation

$$\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial r}\mu_r(t) + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2(t) - \bar{B}(t,T)(q\lambda_t + r_t) = 0$$

- Solution:

$$\bar{B}(t,T) = B(t,T)e^{-\int_t^T q\lambda_u du}.$$

- Spread

$$S(t,T) = \frac{1}{T-t}\int_t^T q\lambda_u du.$$

- Equation

$$\bar{B}(t,T) = B(t,T)e^{-\int_t^T q\lambda_u du} = E_t^{\tilde{\mathbb{P}}}\left[e^{-\int_t^T r_u + q\lambda_u du}\right].$$

implies $\bar{r}_t = r_t + q\lambda_t$: Defaultable short rate.

Pricing with Stochastic Intensity

- Consider the risk-free interest rate and the intensity of the Cox process :

$$dr_t = \mu_r(t)dt + \sigma_r(t)dW_t^r$$
$$d\lambda_t = \mu_\lambda(t)dt + \sigma_\lambda(t)(\rho dW_t^r + \sqrt{1-\rho^2}d\bar{W}_t).$$

- The pricing equation

$$0 = \frac{\partial \bar{B}}{\partial t} + \mu_r(t)\frac{\partial \bar{B}}{\partial r} + \mu_\lambda(t)\frac{\partial \bar{B}}{\partial \lambda} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 \bar{B}}{\partial r^2}$$
$$+ \rho\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2 \bar{B}}{\partial r \partial \lambda} + \frac{1}{2}\sigma_\lambda^2(t)\frac{\partial^2 \bar{B}}{\partial \lambda^2} - (r + q\lambda_t)\bar{B}.$$

- The final condition is $\bar{B}(T, r, \lambda) = 1$. The boundary conditions are $\bar{B} \to 0$ as $r, \lambda \to \infty$, and $\bar{B} < \infty$ as $r, \lambda \to 0$

Pricing with Stochastic Intensity

- Solution:

$$\bar{B}(t,T) = E_t^{\tilde{\mathbb{P}}}\left[e^{-\int_t^T \bar{r}_u du}\right]$$

  where $\bar{r}_u = r_t + q\lambda_t$.

- Credit derivative pricing:

$$F(t,T) = E_t^{\tilde{\mathbb{P}}}\left[e^{-\int_t^T \bar{r}_u du} X\right]$$

  where X is the value of a default affected payoff.

Example

Example : CIR model

$$dr_t = (a_r - b_r r_t)dt + \sigma_r \sqrt{r_t} dW_t$$
$$d\lambda_t = (a_\lambda - b_\lambda \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t}(\rho dW_t + \sqrt{1-\rho^2} d\bar{W}_t)$$

- The constant a_r, b_r and σ_r are calibrated on the default free term structure of interest rate.
- The constant a_λ, b_λ, σ_λ, ρ, and q should be calibrated from the defaultable term structure of interest rate (= Market data).

References

D. Lando (2004).
Credit Risk Modeling
Princeton Series in Finance

P. J. Schönbucher (2000).
The Pricing of Credit Risk and Credit Derivatives
http://www.schonbucher.de/papers/bookfo.pdf

S. Trück and S. T. Rachev (2005).
Credit Portfolio Risk and PD Confidence Sets through the Business Cycle
https://www.statistik.uni-karlsruhe.de/download/tr_credit_portfolio_risk.pdf