Fractals in Trade Duration: Capturing Long-Range Dependence and Heavy Tailedness in Modeling Trade Duration

Wei Sun
Institute for Statistics and Mathematical Economics
University of Karlsruhe, Germany

Svetlozar Rachev
Institute for Statistics and Mathematical Economics
University of Karlsruhe, Germany
and
Department of Statistics and Applied Probability
University of California, Santa Barbara, USA

Frank J. Fabozzi*
Yale School of Management, USA

Petko S. Kalev
Department of Accounting and Finance
Monash University, Australia

*Corresponding author: Frank J. Fabozzi, 858 Tower View Circle, New Hope, PA 18938, U.S.A. Email: FABOZZI321@aol.com. S. Rachev's research was supported by grants from Division of Mathematical, Life and Physical Science, College of Letters and Science, University of California, Santa Barbara, and the Deutschen Forschungsgemeinschaft. W. Sun's research was supported by grants from the Deutschen Forschungsgemeinschaft. P.S. Kalev's research was supported with a NCG grant from the Faculty of Business and Economics, Monash University. Data are supplied by Securities Industry Research Center of Asia-Pacific (SIRCA) on behalf of Reuters.

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Abstract

Several studies that have investigated a few stocks have found that the spacing between consecutive financial transactions (referred to as trade duration) tend to exhibit long-range dependence, heavy tailedness, and clustering. In this study, we empirically investigate whether a larger sample of stocks exhibit those characteristics. Comparing goodness of fit in modeling trade duration data for stable distribution and fractional stable noise based on a procedure applying bootstrap methods developed by the authors, the empirical results in this paper suggest that the fractional stable noise and stable distribution dominate alternative distributional assumptions (i.e., lognormal distribution, fractional Gaussian noise, exponential distribution, and Weibull distribution) in our study.

Key Words: fractional stable noise, point processes, self-similarity, stable distribution, trade duration

JEL Classification: C41, G14

1. Introduction

There is considerable interest in the information content and implications of the spacing between consecutive financial transactions (referred to as trade duration) for trading strategies and intra-day risk management. Market microstructure theory, supported by empirical evidence, suggests that the spacing between trades be treated as a variable to be explained or predicted since time carries information and closely correlates with price volatility (see, Bauwens and Veredas (2004), Diamond and Varrecchia (1987), Engle (2000), Engle and Russell (1998), Hasbrouck (1996), and O'Hara (1995)). Manganelli (2005) finds that returns and volatility directly interact with trade duration and trade order size. Trade durations tend to exhibit long-range dependence, heavy tailedness, and clustering (see, Bauwens and Giot (2000), Dufour and Engle (2000), Engle and Russell (1998), and Jasiak (1998)).

These findings raise two questions that we address in this paper:

1. Can single stochastic processes which capture long-range dependence and heavy tailedness be used in modeling trade duration data ?
2. Can a relatively "powerful" distributional assumption in a relatively "simple" functional structure be used for efficient modeling of trade duration data?

It is necessary to treat long-range dependence, heavy tailedness, and clustering simultaneously in order to obtain more accurate predictions. Rachev and Mittnik (2000) note that for modeling financial data, not only does model structure play an important role, but distributional assumptions influence the modeling accuracy. The stable Paretian distribution[1] can be used to capture characteristics of trade duration since it is rich enough to encompass those stylized facts in such data, such as non-Guassian, heavy tails, long-range dependence, and clustering. Other researchers have shown the advantages of stable distributions in financial modeling (see, Fama (1963), Mittnik and Rachev (1993a, 1993b), Rachev (2003), and Rachev et al. (2005)). Meanwhile several studies have reported that long-range dependence, self-similar processes, and stable distribution are very closely related (see, Samorodnitsky and Taqqu (1994), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Doukhan *et al.* (2003), Racheva and Samorodnitsky (2003), and Rachev *et al.* (2005)).

The Hurst index is also used to model long-range dependence, see Hurst (1951,1955). It quantifies the degree of long-range dependence and measures the self-similarity scaling. Fortunately, one type of self-similar process can possess the Hurst index and stable distribution together (i.e., can capture both long-range dependence and heavy tailedness). This kind of stochastic process is a fractional stable noise generated from fractional stable motion (see, Samorodnitsky and Taqqu (1994)). Therefore, single stochastic processes can capture long-range dependence and heavy tailedness, answering the first question posed above. Based on estimating intensity of point processes, an autoregressive conditional duration (ACD) model is proposed by Engle and Russell (1998) for modeling trade duration with intertemporal correlation. The ACD model is a joint approach combining transition analysis and Engle's

[1] To distinguish between Gaussian and non-Gaussian stable distribution, the latter is usually named stable Paretian distribution or Lévy stable distribution. Referring to it as a stable Paretian distribution highlights the fact that the tails of the non-Gaussian stable density have Pareto power-type decay and Lévy stable is the recognition of pioneering works done by Paul Lévy to the characterization of non-Gaussian stable laws (see Rachev and Mittnik (2000)).

autoregressive conditional heteroscedasticity (ARCH) model. The motivation behind the ACD and the ARCH models is that in financial market events tend to occur in clusters. If fractional stable noise can be subordinated into the functional structure of the ACD model, then the second question can be answered.

In order to answer the two questions posed above, this paper introduces fractional stable noise as the single stochastic processes to model trade duration. In the empirical analysis of this paper, fractional stable noise is subordinated to the ACD model to model trade duration. As to self-similar processes, other single stochastic processes, such as fractional Gaussian noise which captures long-range dependence, are also is introduced as an alternative. Since the stable distribution itself can capture heavy tailedness and long-range dependence, we propose it as an alternative distribution that can better explain trade duration. In the empirical analysis, stable distribution is also subordinated to the ACD model. Some other distributions that are often used in modeling trade duration, such as lognormal distribution, exponential distribution and Weibull distribution, have been selected as alternative distributional assumptions in order to compare goodness of fit with the stable distribution and fractional stable noise. Utilizing two test statistics usually used to evaluate model performance under heavy-tailed assumptions, we examine trade duration for a sample of stocks to compare which distributional assumption fits better. By applying a newly developed test procedure that we formulate, based on a bootstrap method, we obtain empirical results that suggest the fractional stable noise and stable distribution dominate these alternative assumptions with high statistical significance. Comparing goodness of fit in the modeling of trade duration data for stable distribution and fractional stable noise, the empirical results indicate that the ACD model with stable distribution fits better than other combinations, while fractional stable noise itself fits better for the time series of trade duration.

The paper is organized as follows. In Section 2, we introduce two self-similar processes: fractional Gaussian noise and fractional stable noise. The method for estimating the parameters in the underlying process is introduced in Section 3. In Section 4, the methods of simulating fractional Gaussian noise and fractional stable noise are introduced. The empirical study based on trade duration data for 18 of the component stocks of the Dow Jones is reported in Section 5. In that section, we compare the goodness of fit of model under fractional stable noise and stable distribution together with distributions (the lognormal distribution, fractional Gaussian noise, exponential distribution, and Weibull distribution). We summarize our conclusions in Section 6.

2. Specification of the self-similar processes

Self-similarity is defined by Samorodnitsky and Taqqu (1994) as follows. Let T be either $R, R_+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$. The real-valued process $\{X(t), t \in T\}$ is self-similar with Hurst index $H > 0$ (H-*ss*) if for any $a > 0$ and $d \geq 1$, $t_1, t_2, ..., t_d \in T$, satisfying:

$$\Big(X(at_1), X(at_2), ..., X(at_d)\Big) \stackrel{d}{=} \Big(a^H X(t_1), a^H X(t_2), ...a^H X(t_d)\Big). \tag{1}$$

2.1 Fractional Gaussian noise

Mandelbrot and Wallis (1968) first introduced the fractional Brownian motion (FBM) and Samorodnitsky and Taqqu (1994) clarify the definition of FBM as a Gaussian process having self-similarity index

H and stationary increments. Mandelbrot and van Ness (1968) defined the stochastic representation

$$B_H(t) := \frac{1}{\Gamma(H+\frac{1}{2})}\Big(\int_{-\infty}^{0}[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}]dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}}dB(s)\Big), \tag{2}$$

where $\Gamma(\cdot)$ represents the Gamma function:

$$\Gamma(a) := \int_0^\infty x^{a-1}e^{-x}dx,$$

and $0 < H < 1$ is the Hurst parameter. The integrator B is the ordinary Brownian motion. The main difference between fractional Brownian motion and ordinary Brownian motion is that the increments in Brownian motion are independent while in fractional Brownian motion they are dependent. As to the fractional Brownian motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional Gaussian noise (FGN), which is, for $j = 0, \pm1, \pm2, ...$, $Y_j = B_H(j-1) - B_H(j)$.

2.2 Fractional stable noise

Fractional Brownian motion can capture the effect of long-range dependence, but with less power to capture the heavy tailedness. The existence of abrupt discontinuities in financial data, combined with the empirical observation of sample excess kurtosis and unstable variance, confirms the stable Paretian hypothesis in Mandelbrot (1963, 1983). It is natural to introduce stable Paretian distribution in self-similar processes in order to capture both long-range dependence and heavy tailedness. Samorodinitsky and Taqqu (1994) introduce the α-stable *H-sssi* processes $\{X(t), t \in R\}$ with $0 < \alpha < 2$. If $0 < \alpha < 1$, the values of Hurst parameter are $H \in (0, 1/\alpha]$ and if $1 < \alpha < 2$, the values of Hurst parameter are $H \in (0, 1]$. There are many different extensions of fractional Brownian motion to the stable distribution. The most commonly used is the linear fractional stable motion (also called linear fractional Lévy motion), $\{L_{\alpha,H}(a, b; t), t \in (-\infty, \infty)\}$, which is defined by Samorodinitsky and Taqqu (1994) as follows:

$$L_{\alpha,H}(a, b; t) := \int_{-\infty}^{\infty} f_{\alpha,H}(a, b; t, x)M(dx), \tag{3}$$

where

$$f_{\alpha,H}(a, b; t, x) := a\Big((t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\Big) + b\Big((t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}}\Big), \tag{4}$$

and where a, b are real constants, $|a| + |b| > 1$, $0 < \alpha < 2$, $0 < H < 1$, $H \neq 1/\alpha$ and M is an α-stable random measure on R with Lebesgue control measure and skewness intensity $\beta(x)$, $x \in (-\infty, \infty)$ satisfying: $\beta(\cdot) = 0$ if $\alpha = 1$. Samorodinitsky and Taqqu (1994) define linear fractional stable noise expressed by $Y(t)$, and $Y(t) = X_t - X_{t-1}$,

$$\begin{aligned} Y(t) &= L_{\alpha,H}(a, b; t) - L_{\alpha,H}(a, b; t-1) \\ &= \int_R \Big(a\Big[(t-x)_+^{H-\frac{1}{\alpha}} - (t-1-x)_+^{H-\frac{1}{\alpha}}\Big] \\ &+ b\Big[(t-x)_-^{H-\frac{1}{\alpha}} - (t-1-x)_-^{H-\frac{1}{\alpha}}\Big]\Big)M(dx) \end{aligned} \tag{5}$$

where $L_{\alpha,H}(a, b; t)$ is linear fractional stable motion defined by equation (3), and M is a stable random measure with Lebesgue control measure given $0 < \alpha < 2$.[2] In this paper, if there is no special indication, fractional stable noise (fsn) is generated from linear fractional stable motion.

[2]Some properties of these processes have been discussed in Mandelbrot and Van Ness (1968), Maejima and Rachev (1987), Manfields *et al.* (2001), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Racheva and Samorodnitsky (2003), Samorodnitsky (1994, 1996, 1998), and Samorodinitsky and Taqqu (1994).

3. Estimation in self-similar processes

3.1 Estimating the self-similarity parameter in fractional Gaussian noise

Beran (1994) discusses the Whittle estimator of self-similarity parameter. For fractional Gaussian noise, Y_t, let $f(\lambda; H)$ denote the power spectrum of Y after being normalized to have variance 1 and let $I(\lambda)$ the periodogram of Y_t; that is

$$I(\lambda) = \frac{1}{2\pi N} \left| \sum_{t=1}^{N} Y_t \, e^{i\,t\,\lambda} \right|^2 . \tag{6}$$

The Whittle estimator of H is to find $\hat{H}$ that minimizes

$$g(\hat{H}) = \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda; \hat{H})} d\lambda. \tag{7}$$

3.2 Estimating the self-similarity parameter in fractional stable noise

Stoev *et al* (2002) proposed the least-squares (LS) estimator of the Hurst index based on the finite impulse response transformation (FIRT) and wavelet transform coefficients of the fractional stable motion. A FIRT is a filter $v = (v_0, v1, ..., v_p)$ of real numbers $v_t \in \Re, t = 1, ..., p$, and length is $p + 1$. It is defined for X_t by

$$T_{n,t} = \sum_{i=0}^{p} v_i \, X_{n(i+t)}, \tag{8}$$

where $n \geq 1$ and $t \in N$. The $T_{n,t}$ are the FIRT coefficients of X_t (i.e., the FIRT coefficients of the fractional stable motion). The indices n and t can be explained as "scale" and "location". If $\sum_{i=0}^{p} i^r v_i = 0$, for $r = 0, ..., q-1$, but $\sum_{i=0}^{p} i^q v_i \neq 0$, the filter v_i can be said to have $q \geq 1$ zero moments. If $\{T_{n,t}, n \geq 1, t \in N\}$ is the FIRT coefficients of fractional stable motion with the filter v_i that has at least one zero moment, Stoev *et al* (2002) prove the following properties of $T_{n,t}$: (1), $T_{n,t+h} \stackrel{d}{=} T_{n,t}$, and (2), $T_{n,t} \stackrel{d}{=} n^H T_{1,t}$, where $h, t \in N$, $n \geq 1$. We assume that $T_{n,t}$ are available for the fixed scales n_j $j = 1, ..., m$ and locations $t = 0, ..., M_j - 1$ at the scale n_j, since only a finite number, say M_j, of the FIRT coefficients are available at the scale n_j. By using these properties, we have

$$E \log |T_{n_j,0}| = H \log n_j + E \log |T_{1,0}|. \tag{9}$$

The left-hand side of this equation can be approximated by

$$Y_{\log}(M_j) = \frac{1}{M_j} \sum_{t=0}^{M_j-1} \log |T_{n_j,t}|. \tag{10}$$

Then we get

$$\begin{pmatrix} Y_{\log}(M_1) \\ \vdots \\ Y_{\log}(M_m) \end{pmatrix} = \begin{pmatrix} \log n_1 & 1 \\ \vdots & \vdots \\ \log n_m & 1 \end{pmatrix} \begin{pmatrix} H \\ E \log |T_{1,0}| \end{pmatrix} + \begin{pmatrix} \sqrt{M}\Big(Y_{\log}(M_1) - E \log |T_{n_1,0}|\Big) \\ \vdots \\ \sqrt{M}\Big(Y_{\log}(M_m) - E \log |T_{n_m,0}|\Big) \end{pmatrix}. \tag{11}$$

We can express above equation as follows

$$Y = X\theta + \frac{1}{\sqrt{M}} \varepsilon, \tag{12}$$

where ε is the vector showing the difference between $\sqrt{M}Y_{\log}(M_m)$ and $\sqrt{M}E(\log|T_{n_m,0}|)$. Equation (10) shows that the self-similarity parameter H can be estimated by a standard linear regression of the vector Y against the matrix X. Stoev *et al.* (2002) show the details for implementing such a procedure.

3.3 Estimating the parameters of the stable Paretian distribution

The stable distribution requires four parameters for complete description: an index of stability $\alpha \in (0,2]$ also called the tail index, a skewness parameter $\beta \in [-1,1]$, a scale parameter $\gamma > 0$, and a location parameter $\zeta \in \Re$. There is unfortunately no closed-form expression for the density function and distribution function of a stable distribution. Rachev and Mittnik (2000) give the definition for the stable distribution: A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $-1 \leq \alpha \leq 1$, $\gamma \geq 0$ and ζ real such that its characteristic function has the following form:

$$E\exp(i\theta X) = \begin{cases} \exp\{-\gamma^{\alpha}|\theta|^{\alpha}(1 - i\beta(\sin\theta)\tan\frac{\pi\alpha}{2}) + i\zeta\theta\}, & if \quad \alpha \neq 1 \\ \exp\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\sin\theta)\ln|\theta|) + i\zeta\theta\}, & if \quad \alpha = 1 \end{cases} \tag{13}$$

and,

$$\sin\theta = \begin{cases} 1, & if \quad \theta > 0 \\ 0, & if \quad \theta = 0 \\ -1, & if \quad \theta < 0 \end{cases} \tag{14}$$

Stable density is not only support for all of $(-\infty, +\infty)$, but also for a half line. For $0 < \alpha < 1$ and $\beta = 1$ or $\beta = -1$, the stable density is only for a half line.

In order to estimate the parameters of the stable distribution, the maximum likelihood estimation (MLE) method given in Rachev and Mittnik (2000) has been employed. Given N observations, $X = (X_1, X_2, \cdots, X_N)'$ for the positive half line the log-likelihood function is of the form

$$\ln(\alpha, \lambda; X) = N\ln\lambda + N\ln\alpha + (\alpha - 1)\sum_{i=1}^{N}\ln X_i - \lambda\sum_{i=1}^{N}X_i^{\alpha}, \tag{15}$$

which can be maximized using, for example, a Newton-Raphson algorithm. It follows from the first-order condition,

$$\lambda = N\left(\sum_{i=1}^{N}X_i^{\alpha}\right)^{-1} \tag{16}$$

that the optimization problem can be reduced to finding the value for α which maximizes the concentrated likelihood

$$\ln^*(\alpha; X) = \ln\alpha + \alpha\nu - \ln\left(\sum_{i=1}^{N}X_i^{\alpha}\right), \tag{17}$$

where $\nu = N^{-1}\Sigma_{i=1}^{N}\ln X_i$. The information matrix evaluated at the maximum likelihood estimates, denoted by $I(\hat{\alpha}, \hat{\lambda})$, is given by

$$I(\hat{\alpha}, \hat{\lambda}) = \begin{pmatrix} N\hat{\alpha}^{-2} & \sum_{i=1}^{N}X_i^{\hat{\alpha}}\ln X_i \\ \sum_{i=1}^{N}X_i^{\hat{\alpha}}\ln X_i & N\hat{\lambda}^{-2} \end{pmatrix}.$$

It can be shown that, under fairly mild condition, the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ are consistent and have asymptotically a multivariate normal distribution with mean $(\alpha, \lambda)'$(see Rachev and Mittnik (2000)).

Other methods for estimating the parameters of a stable distribution (i.e., the method of moments based on the characteristic function, the regression-type method, and the fast Fourier transform method) are discussed in Stoyanov and Racheva-Iotova (2004a, 2004b, 2004c).

4. Simulation of self-similar processes

4.1 Simulation of fractional Gaussian noise

Paxson (1997) gives a method to generate the fractional Gaussian noise by using the Discrete Fourier Transform of the spectral density. Bardet *et al.* (2003) give a concrete simulation procedure based on this method with respect to alleviating some of the problems faced in practice. The procedure is:

1. Choose an even integer M. Define the vector of the Fourier frequencies $\Omega = (\theta_1, ..., \theta_{M/2})$, where $\theta_t = 2\pi t/M$ and compute the vector $F = f_H(\theta_1), ..., f_H(\theta_{M/2})$, where

$$f_H(\theta) = \frac{1}{\pi}\sin(\pi H)\Gamma(2H+1)(1-\cos\theta)\sum_{t\in\aleph}|2\pi t + \theta|^{-2H-1}$$

   $f_H(\theta)$ is the spectral density of FGN.

2. Generate $M/2$ i.i.d exponential $Exp(1)$ random variables $E_1, ..., E_{M/2}$ and $M/2$ i.i.d uniform $U[0,1]$ random variables $U_1, ..., U_{M/2}$.

3. Compute $Z_t = \exp(2i\pi U_t)\sqrt{F_t E_t}$, for $t = 1, ..., M/2$.

4. Form the M-vector: $\tilde{Z} = (0, Z_1, ...Z_{(M/2)-1}, Z_{M/2}, \overline{Z}_{(M/2)-1}, ..., \overline{Z}_1)$.

5. Compute the inverse FFT of the complex Z to obtain the simulated sample path.

4.2 Simulation of fractional stable noise

Replacing the integral in equation (5) with a Riemann sum, Stoev and Taqqu (2004) generate the approximation of fractional stable noise. They introduce parameters $n, N \in \aleph$, and let the fractional stable noise $Y(t)$ expressed as

$$Y_{n,N}(t) := \sum_{j=1}^{nN}\Big((\frac{j}{n})_+^{H-1/\alpha} - (\frac{j}{n} - 1)_+^{H-1/\alpha}\Big)L_{\alpha,n}(nt - j), \tag{18}$$

where $L_{\alpha,n}(t) := M_\alpha((j+1)/n) - M_\alpha(j/n)$, $j \in \Re$. The parameter n is mesh size and the parameter M is the cut-off of the kernel function. Stoev and Taqqu (2004) describe an efficient approximation involving the Fast Fourier Transformation algorithm for $Y_{n,N}(t)$. Consider the moving average process $Z(m)$, $m \in \aleph$,

$$Z(m) := \sum_{j=1}^{nM} g_{H,n}(j)L_\alpha(m - j), \tag{19}$$

where

$$g_{H,n}(j) := \Big((\frac{j}{n})^{H-1/\alpha} - (\frac{j}{n} - 1)_{+}^{H-1/\alpha} \Big) n^{-1/\alpha}, \tag{20}$$

and where $L_\alpha(j)$ is the series of i.i.d standard stable Paretian random variables. Since $L_{\alpha,n}(j) \overset{d}{=} n^{-1/\alpha} L_\alpha(j), j \in \Re$, equations (18) and (19) imply $Y_{n,N}(t) \overset{d}{=} Z(nt)$, for $t = 1, ..., T$. Then, the computing of $Y_{n,N}(t)$ is moved to focus on the moving average series $Z(m)$, $m = 1, ..., nT$. Let $\tilde{L}_\alpha(j)$ be the $n(N+T)$-periodic with $\tilde{L}_\alpha(j) := L_\alpha(j)$, for $j = 1, ..., n(N+T)$ and let $\tilde{g}_{H,n}(j) := g_{H,n}(j)$, for $j = 1, ..., nN$; $\tilde{g}_{H,n}(j) := 0$, for $j = nN+1, ..., n(N+T)$. Then

$$\{Z(m)\}_{m=1}^{nT} \overset{d}{=} \Big\{ \sum_{j=1}^{n(N+T)} \tilde{g}_{H,n}(j) \tilde{L}_\alpha(n-j) \Big\}_{m=1}^{nT}, \tag{21}$$

because for all $m = 1, ..., nT$, the summation in equation (19) involves only $L_\alpha(j)$ with indices j in the range $-nN \leq j \leq nT - 1$. Using a circular convolution of the two $n(N+T)$-periodic series $\tilde{g}_{H,n}$ and $\tilde{L}_\alpha$ computed by using their Discrete Fourier Transforms (DFT), the variables $Z(n)$, $m = 1, ..., nT$ (i.e., the fractional stable noise), can be generated.

5. Empirical study

In this section, we report the results of empirical tests that investigate the goodness of fit of several candidate distributional assumptions.

5.1 The data

Ultra-high frequency data of 18 Dow Jones index component stocks based on NYSE trading for year 2003 are examined.[3] The companies in the sample are listed in Table 1. Our sample is considerably larger than other studies that have investigated trade duration. Table 2 lists those studies and the stocks included in each one. Note that for the studies that include U.S. stocks, IBM is included in 7 of 11 studies and because the sample size is small, IBM constitutes a major part of those studies. IBM is included in our study also.

The trade durations were calculated for regular trading hours (i.e., overnight trading was not considered). Consistent with Engle and Russell (1998) and Ghysels *et al.* (2004), open trades are deleted in order to avoid effects induced by the opening auction. Therefore trade durations only from 10:00 to 16:00 are considered.

Figures 1 to 6 plot several sampled trade duration series. These figures show data characteristics that are consistent with the data patterns reported in the literature. For the trade durations of each stock in our study, sample period runs were performed from 4 January 2003 to 31 December 2003. We will let N denote the length of the sample, sub-sample series that have been randomly selected by a moving window with length T ($1 \leq T \leq N$). Replacement is allowed in the sampling. Stoev and Taqqu (2004) suggest that $2^{14} - 6,000 = 10,384$ is the optimal length for a fractional stable noise series to be

[3] The data from were provided by The Securities Industry Research Center of Asia-Pacific in Australia. The Dow Jones index consists of 30 stocks. The whole database we developed included data from 1996 to 2003 for stocks that remained in the index over the entire period. Only 18 stocks satisfied that requirement and we use the data of 2003 in this study.

simulated efficiently. Therefore, in the empirical analysis, sub-sample length (i.e., the window length) of $T = 10,384$ was chosen. A total of 684 sub-samples were randomly created.

The trade duration data are distributed asymetrically. All observations of duration are positive numbers. The stable distribution, fractional Gaussian noise, and fractional stable are all defined on both positive and negative supports. In our empirical study, we transfer the asymmetrically distributed series to the symmetrically distributed series before we estimate the corresponding parameters. Note that only the positive numbers of the generated series are considered in our simulation.

5.2 Preliminary tests

Table 1 shows the descriptive statistics of the trade duration data in our study. From the statistics reported in this table, it can be seen that excess kurtosis exists.

Engle (1982) proposes a Lagrange-multiplier test for ARCH phenomenon. A test statistic for ARCH of lag order q is given by

$$X_q \equiv nR_q^2,$$

where R_q^2 is the non-centered goodness-of-fit coefficient of a qth order autoregression of the squared residuals taken from the original regression

$$\hat{u}_t^2 = \omega_0 + \omega_1 \hat{u}_{t-1}^2 + \omega_2 \hat{u}_{t-2}^2 + \cdots + \omega_q \hat{u}_{t-q}^2 + e_t, \tag{22}$$

where $\hat{u}$ is the residual in original regression equation. Under the null hypothesis of the residuals of the original model being normally i.i.d., the ARCH statistic of lag order q follows a χ^2 distribution with q degree of freedom:

$$\lim_{n\to\infty} X_q \sim \chi_q^2.$$

Table 3 shows the test statistics and critical values in which we can reject the null hypothesis that there is no ARCH effect at different lag levels for the duration increments. It is clear that an ARCH effect is exhibited in these data.

The Hurst index $H \in (0, 1)$ usually serves as the measure of the tendency of a process and stands for the self-similarity index in Gaussian stochastic processes. It can be somewhat explained by considering the covariance of two consecutive increments. When $H \in (0, 0.5)$, the increments of a process tend to have opposite signs and thus are more zigzagging due to the negative covariance; when $H \in (0.5, 1)$, the covariance between these two increments is positive and less zigzagging of the process; when $H = 0.5$, the covariance between this two increments is zero. It can be stated as follows: If the Hurst index is less than 0.5, the process displays "anti-persistence" which means that the positive excess return is more likely to be reversed and the performance in the next period is likely to be below the average, or on the contrary, the negative excess return is more likely to be reversed and the performance in the next period is likely to be above the average. If the Hurst index is greater than 0.5, the process displays "persistence" which means that the positive excess return or the negative excess return is more likely to be continued and the performance in the next period is likely to be the same as that in the current period. If the Hurst index is equal to 0.5, the process displays no memory, which means the performance in the next period has equal probability to be below and above the performance in the current period. From Table 1, we find that the Hurst index has no value of 0.5, which indicates that the memory effect occurs in our samples.

The Hurst index for non-Gaussian stable processes has different bounds for "persistence" and "anti-persistence". For tail index $\alpha \in (0, 2)$, when $H \in (0, 1/\alpha)$, the processes exhibit "anti-persistence", and when $H \in (1/\alpha, 1)$, the processes exhibit "persistence". There is no long-range dependence when $\alpha \in (0, 1]$ because the Hurst index is bounded in the interval $(0, 1)$. When $H = 1/\alpha$, depending on the value of α the processes exhibit either no memory or long-range dependence.[4] From Table 1, we find that the Hurst index has no value of $1/\alpha$. Therefore, we find that long-range dependence occurs in our samples.

We use the Ljung-Box-Pierce Q-statistic based on autocorrelation function to test the serial correlation (i.e., the memory effect). The Q-statistic is given as follows:

$$Q :\sim \chi^2_m = N(N+2) \sum_{k=1}^{m} \frac{\rho_k^2}{N-k}, \tag{23}$$

where, N denotes the sample size, m the number of autocorrelation lags included in the statistic, and ρ_k the sample autocorrelation at lag order k which is

$$\rho_k = \frac{\sum_{t=1}^{N-k} y_t y_{t+k}}{\sum_{t=1}^{N} y_t^2}. \tag{24}$$

Ljung and Box (1978) show that the Q-statistic follows the asymptotic chi-square distribution with m degrees of freedom.

Table 4 shows that the null hypothesis that there is no serial correlation can be rejected at different lags. This table shows that the memory effect occurs in each duration series. In order to see when the memory effect vanishes, we compare the Q-statistic with its corresponding critical value. When the quotation of the Q-statistic and the corresponding critical value are less than 1, we cannot reject the null hypothesis that there is no serial correlation. Table 4 also shows such quotations. From this table, all the trade durations exhibit serial correlation. After 500 lags, the memory effect vanishes for 7 stocks and after 1500 lags, the memory effect vanishes for 13 stocks. From the ratios in Table 4, we find that the speed of autocorrelation decay is declining, which confirms the effect of long-range dependence.

5.3 The methodology of finding the best model

In our empirical study, we simulate a series for each distributional assumption with and without subordinating them into the ACD(1,1) structure. Then we compare the goodness of fit of the simulated together with originial trade duration series.

The class of ACD model can be defined as:

$$d_i \;=\; \sigma_i \, u_i, \tag{25}$$

and

$$\sigma_t^2 \;\;=\;\; \kappa + \sum_{i=1}^{p} \gamma_i \, d_{t-i} \;+\; \sum_{j=1}^{q} \theta_j \, \sigma_{t-j}^2, \tag{26}$$

[4] A detailed discussion see Samorodnitsky and Taqqu (1994) and Racheva-Iotova and Samorodnitsky (2003).

u_t can be calculated from d_t/σ_t. We define

$$\tilde{u_t} = \frac{d_t}{\hat{\sigma_t}}, \tag{27}$$

where $\hat{\sigma_t}$ is the estimation of σ_t. In our empirical analysis, an ACD(1,1) model structure is adopted. The objective is to check the statistical characteristics exhibited by trade duration d_t and the error term $\tilde{u_t}$ in ACD(1,1) structure. We simulate d_t and $\tilde{u_t}$ with the ACD(1,1) structure based on the parameters estimated from the empirical series. Then we test the goodness of fit between the empirical series and the simulated series. Six candidate distributional assumptions — lognormal distribution, stable distribution, exponential distribution, Weibull distribution, fractional Gaussian noise, and fractional stable noise — are analyzed for estimation, simulation, and testing.

The Kolmogorov-Simirnov distance (KS) and Anderson-Darling distance (AD) proposed by Rachev and Mittnik (2000) are used as the criterion for the goodness of fit testing. They are defined as following:

$$KS = \sup_{x\in\Re}|F_s(x) - \tilde{F}(x)|, \tag{28}$$

and

$$AD = \sup_{x\in\Re}\frac{|F_s(x) - \tilde{F}(x)|}{\sqrt{\tilde{F}(x)(1-\tilde{F}(x))}}, \tag{29}$$

where $F_s(x)$ denotes the empirical sample distribution and $\tilde{F}(x)$ is the estimated distribution function. The major disadvantage of KS statistic researchers have argued is that it tends to be more sensitive near the center of the distribution than at the tails. But AD statistic can overcome this. The reliability of testing the empirical distribution will be increased with the help of these two statistics, with the KS distance focusing on the deviations around the median of the distribution and the AD distance on the discrepancies in the tails.

5.3 Results

The AD and KS statistics are calculated for the six candidate distributional assumptions. Table 5 reports the descriptive statistics of the computed AD and KS statistics. From Table 5, fractional stable noise and stable distribution exhibit a smaller mean value for the AD and KS statistics in comparison with the other four distributions. Figure 2 shows the boxplot of AD statistics of $\tilde{u_t}$ for the six alternative distributional assumptions investigated. Figure 3 shows the boxplot of AD statistics for d_t. Figure 4 shows the boxplot of KS statistics of $\tilde{u_t}$ for the six alternative distributional assumptions. Figure 5 shows the boxplot of KS statistics of d_t. These figures show that fractional stable noise and stable distribution have a small value of AD and KS statistics, confirming the results reported in Table 5. These results indicate that with or without an ACD(1,1) model structure, the fractional stable noise and stable distribution perform better than the other four tested distributional assumptions based on the criterion for goodness of fit testing.

From Figures 2 to 5, we can see that the fractional stable noise and the stable distribution fit $\tilde{u_t}$ and d_t better than other distributional assumptions. In order to empirically examine our conjecture, we formulate a statistical test procedure. Because we know that smaller AD and KS statistics mean better goodness of fit, in our test we are going to statistically test how significantly "smaller" AD and KS statistics are. The hypothesis test is:

$$\begin{aligned} H_0 &: \mu_{criterion1} - \mu_{criterion2} \geq 0 \\ H_1 &: \mu_{criterion1} - \mu_{criterion2} < 0 \end{aligned} \tag{30}$$

where $\mu_{criterion}$ is the mean value of AD or KS statistics of the candidate distributional assumptions investigated. The distributions of AD and KS values are unknown. All AD or KS values are expressed as i.i.d. random variables $X_1, X_2, ..., X_n$, each with distribution function $F_X(\cdot|\theta)$. A $100(1-\alpha)\%$ upper confidence bound (UCB) is defined as $U(X_1, X_2, ..., X_n)$ for a function of $h(\theta)$ if for every θ,

$$P_\theta\left(h(\theta) \leq U(X_1, X_2, ..., X_n)\right) \geq 1 - \alpha \tag{31}$$

and $(-\infty, U(X_1, X_2, ..., X_n)]$ is the $100(1-\alpha)\%$ upper confidence interval for $h(\theta)$. Similarly, $L(X_1, X_2, ..., X_n)$ is a $100(1-\alpha)\%$ lower confidence bound (LCB) for the function $h(\theta)$ for every θ

$$P_\theta\left(h(\theta) \geq L(X_1, X_2, ..., X_n)\right) \geq 1 - \alpha \tag{32}$$

and $[L(X_1, X_2, ..., X_n), +\infty)$ is the $100(1-\alpha)\%$ lower confidence interval for $h(\theta)$.

As hypothesis testing and confidence intervals are dual concepts, the hypothesis testing in (50) is in fact evaluated by following test rules; that is, (1) if UCB is less than zero, H_0 can be rejected, (2) if LCB is greater than zero, H_0 cannot be rejected, and (3) if H_0 is greater than LCB but at the same time less than UCB, there is no statistically significant conclusion. Employing the bootstrap method introduced in DiCiccio and Efron (1996), the 99% bootstrap confidence intervals are reported in Table 6. From this table, at a high confidence level, fractional stable noise and stable distribution are more suitable to modeling trade duration data with or without support of an ACD(1,1) structure.

In comparing the fractional stable noise and stable distribution, it is unclear as to whether the fractional stable noise is better than the stable distribution or vice versa. Table 7 compares the supporting cases for the fractional stable noise and stable distribution. The fractional stable noise has a slightly better support than stable distribution. The stable distribution has a greater number of supporting cases in comparison to the fractional stable noise in modeling duration data with an ACD(1,1) structure.

6. Conclusions

The empirical research with very few stocks have demonstrated that trade duration data exhibit three characteristics: long-range dependence, heavy tailedness, and clustering. In this paper, we investigate the presence of these characteristics using a larger number of stocks and investigate whether for modeling trade duration data: (1) a single stochastic processes capturing long-range dependence and heavy tailedness and; (2) a relatively powerful distributional assumption in a relatively simple functional structure can be used.

To examine these issues, we introduce fractional stable noise and fractional Gaussian noise to capture long-range dependence and heavy tailedness in modeling the trade duration. In our empirical analysis, we investigate six distributional assumptions (fractional stable noise, fractional Gaussian noise, stable distribution, lognormal distribution, exponential distribution, and Weibull distribution) for modeling the trade duration for 18 Dow Jones index component stocks. By using parameters estimated from the empirical series, we simulate a series for each distributional assumption with and without subordinating them into an ACD(1,1) structure. Then we compare the goodness of fit for these generated series to the empirical series by adopting two test criteria for testing heavy-tailed distributions, the Kolmogorov-Simirnov and Anderson-Darling statistics. A test procedure is formulated based on a bootstrap method, and it is used in order to obtain empirical results.

The above test procedure yields empirical evidence which shows that the stable distribution and fractional stable noise are better in modeling trade duration than the exponential distribution, log-normal distribution, Weibull distribution, and fractional Gaussian noise. The results indicate that residuals from the ACD(1,1) model are more likely to be described by a stable distribution and trade durations exhibit the features of fractional stable noise. That is, stable distribution subordinated with an ACD(1,1) structure and fractional stable noise, demonstrate superior performance in the modeling of trade duration.

We argue that it is critical that the findings reported in this paper be taken into account in modeling trade duration. Many studies have found that stable distribution is a better description of financial data because it can capture heavy tailedness and has a close relationship with long-range dependence. As a self-similar process, fractional stable noise can capture almost all reported stylized facts in financial return data, such as heavy tailedness, long memory, non-Gaussian characters, and clustering. Therefore, if fractional stable noise and stable distribution can be properly employed in financial modeling, more accurate prediction might be realized by well-defined functional models.

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Table 1: Statistical characteristics of trade duration in 2003 for 18 stocks

Stock	$size$	$mean$	$variance$	$skewness$	$kurtosis$	$Hurst_{\tilde{u}_t}$	$Hurst_{d_t}$	$\alpha_{\tilde{u}_t}$	α_{d_t}
Alcoa Inc.	511817	10.8858	2214.9814	69.0315	5388.2732	0.6139	0.7202	1.3291	1.1022
American Express	819489	6.9582	2030.7830	77.7079	6318.8553	0.6379	0.5774	1.5673	1.3316
Caterpillar	571251	10.1333	3363.9955	58.7267	3679.0844	0.6932	0.6289	1.3475	1.0286
E.I.DuPont de Nemours	715515	7.8020	1629.7963	84.7517	7767.4205	0.6107	0.6379	1.3832	1.1930
Walt Disney	813090	6.8123	1354.3835	94.4269	9558.9300	0.6936	0.6943	1.3827	1.2792
Eastman Kodak Co.	475512	11.9756	3591.6955	57.6869	3600.8356	0.6445	0.7108	1.4295	1.1715
General Electric	1188851	4.8295	1470.4087	92.0469	8745.4964	0.4814	0.4725	1.4833	1.4413
General Motors	684082	8.4509	2821.2248	65.0970	4452.0523	0.4647	0.5679	1.2921	1.1606
IBM	986153	5.8425	1921.3140	81.4365	6837.2217	0.5186	0.4952	1.6492	1.2792
Int.Paper Company	606742	9.1403	1786.1335	79.9445	7072.3144	0.6195	0.7110	1.3677	1.0742
Coca-Cola Co.	695948	8.1239	2059.3277	75.4303	6113.4419	0.5676	0.5763	1.3470	1.1606
McDonalds	615302	9.0091	1801.3532	80.1190	7001.8869	0.5548	0.6555	1.3144	1.2437
3M Co.	739258	7.7422	2299.2715	71.6024	5407.2143	0.5819	0.5948	1.3705	1.2349
Altria Group	846140	6.7920	2078.7019	77.6406	6338.5890	0.5396	0.5842	1.3825	1.1282
Merck & Co.	875457	6.6009	2158.2266	76.6211	6145.4793	0.5861	0.5252	1.4886	1.1715
Procter & Gamble	780633	7.2657	1875.1364	80.2896	6796.9486	0.6853	0.5704	1.4437	1.2792
AT&T Inc.	553896	10.1818	2504.3505	67.7552	4979.9100	0.6280	0.6894	1.4133	1.1542
United Technologies	657286	8.5779	2240.6857	72.5846	5659.7341	0.6051	0.6544	1.3176	1.1282

Table 2: Studies of Trade Duration and Stocks Included in Sample

Study	Exchange	Stocks	Model distribution(s)
Bauwens (2005)	Tokyo Stock Exchange	Nippon Steel, Sony, Tokyo Electric, Toyota	Generalized gamma
Bauwens and Veredas (2004)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon	Weibull
Bauwens and Giot (2000)	New York Stock Exchange	Boeing, Walt Disney, IBM	Exponential, Weibull
Bauwens and Giot (2003)	New York Stock Exchange	Walt Disney, IBM	Weibull
Bauwens *et al.* (2004)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon	Burr, Exponential, Generalized gamma, Weibull
Engle and Russell (1998)	New York Stock Exchange	IBM	Exponential, Weibull
Engle and Lunde (2003)	New York Stock Exchange	Bank-American, Walt Disney, General Motors	Exponential
		Federal National Mortagage, McDonald's,	
		Monsanto, Procter & Gamble, Schlumberger	
Feng *et al.* (2004)	New York Stock Exchange	Boeing, Coca Cola, IBM	Exponential, log-Weibull, log-Gamma
Fernandes and Gramming (2003)	New York Stock Exchange	Boeing, Coca Cola, Walt Disney, Exxon, IBM	Burr
Fernandes and Gramming (2005)	New York Stock Exchange	Exxon	Exponential, Weibull, Burr, Generalized gamma
Ghysels *et al.* (2004)	Paris Stock Exchange	Alctel	Exponential, Gamma
Jasiak (1998)	Paris Stock Exchange	Alctel	Weibull
	New York Stock Exchange	IBM	
Zhang *et al.* (2001)	New York Stock Exchange	IBM	Generalized gamma

Table 3: ARCH-test for different lags of increments series generated from trade duration data. Test statistic and critical value are presented.

Stock	$lag1$	$lag2$	$lag3$	$lag5$	$lag10$	$lag15$	$lag25$	$lag50$
Alcoa Inc.	25020	33327	37483	41644	45423	46829	48008	48894
American Express	24976	33297	37441	41573	45281	46619	47668	48292
Caterpillar	24977	33322	37467	41629	45410	46814	47993	48880
E.I.DuPont de Nemours	24995	33329	37484	41647	45436	46851	48046	48970
Walt Disney	24999	33332	37490	41654	45439	46853	48045	48965
Eastman Kodak Co.	24967	33315	37466	41632	45417	46826	48012	48911
General Electric	24990	33328	37488	41653	45435	46848	48036	48945
General Motors	24987	33316	37463	41616	45377	46765	47909	48717
IBM	24967	33290	37430	41559	45256	46583	47614	48204
Int. Paper Company	24985	33325	37480	41645	45429	46841	48029	48936
Coca-Cola Co.	24981	33322	37480	41647	45430	46843	48036	48956
McDonalds	24978	33319	37479	41643	45426	46835	48019	48916
3M Co.	25138	33485	37667	41858	45667	47087	48284	49206
Altria Group	24990	33326	37484	41647	45429	46840	48026	48929
Merck & Co.	24988	33325	37481	41648	45437	46852	48049	48978
Procter & Gamble	25015	33332	37479	41639	45421	46833	48023	48937
AT&T Inc.	24975	33320	37475	41642	45430	46843	48035	48953
United Technologies	24973	33318	37473	41639	45427	46839	48031	48950
Critical Value	3.8415	5.9915	7.8147	11.0700	18.3070	24.9960	37.6520	67.5050

Table 4: Ljung-Box-Pierce Q-test statistic for different lags at α=0.05. The italic numbers show the ratios of Q-test statistics compared with corresponding critical values.

stock	lag10	lag20	lag50	lag100	lag200	lag500	lag1000	lag1500
Alcoa Inc.	760.3	931.7	1379.5	2041.0	3224.5	5912.9	9897.8	13327.0
	41.5	*29.7*	*20.4*	*16.4*	*13.8*	*10.7*	*9.2*	*8.4*
American Express	330.5	336.2	344.5	352.8	372.8	405.6	596.7	691.3
	18.1	*10.7*	*5.1*	*2.8*	*1.6*	*0.7*	*0.6*	*0.4*
Caterpillar	413.9	446.1	516.6	623.5	814.5	1329.9	2093.9	2899.9
	22.6	*14.2*	*7.6*	*5.0*	*3.4*	*2.4*	*1.9*	*1.8*
E.I.DuPont de Nemours	371.6	401.4	455.6	527.1	644.4	840.9	1184.5	1376.5
	20.3	*12.7*	*6.7*	*4.2*	*2.7*	*1.5*	*1.1*	*0.8*
Walt Disney	294.5	319.2	374.8	452.1	606.5	936.2	1475.7	1620.3
	16.1	*10.1*	*5.5*	*3.6*	*2.5*	*1.6*	*1.3*	*1.0*
Eastman Kodak Co.	157.9	177.6	204.4	257.0	351.2	654.6	1078.8	1530.8
	8.6	*5.6*	*3.0*	*2.0*	*1.5*	*1.1*	*1.1*	*0.9*
General Electric	386.8	391.1	393.1	394.8	397.4	406.6	601.1	627.5
	21.1	*12.4*	*5.8*	*3.1*	*1.6*	*0.7*	*0.5*	*0.3*
General Motors	350.8	356.2	365.7	373.3	396.8	513.5	584.7	832.1
	19.1	*11.3*	*5.4*	*3.0*	*1.6*	*0.9*	*0.5*	*0.5*
IBM	159.9	165.7	169.7	174.0	181.2	195.7	357.1	418.0
	8.7	*5.2*	*2.5*	*1.3*	*0.7*	*0.3*	*0.3*	*0.2*
Int. Paper Company	344.3	400.2	519.7	698.0	986.6	1424.1	1846.4	2299.0
	18.8	*12.7*	*7.6*	*5.6*	*4.2*	*2.5*	*1.7*	*1.4*
Coca-Cola Co.	244.2	253.4	266.0	293.4	318.8	366.1	620.3	758.9
	13.3	*8.0*	*3.9*	*2.3*	*1.3*	*0.6*	*0.5*	*0.4*
McDonalds	233.2	256.1	292.2	346.9	442.6	899.3	1016.4	1343.8
	12.7	*8.1*	*4.3*	*2.7*	*1.8*	*1.6*	*0.9*	*0.8*
3M Co.	515.3	528.2	551.2	579.0	631.5	750.7	1044.4	1340.2
	28.1	*16.8*	*8.1*	*4.6*	*2.6*	*1.3*	*0.9*	*0.8*
Altria Group	279.1	284.2	292.6	299.0	319.1	357.7	556.0	617.1
	15.2	*9.0*	*4.3*	*2.4*	*1.3*	*0.6*	*0.5*	*0.3*
Merck & Co.	262.9	268.0	276.1	280.0	285.9	303.0	461.6	535.6
	14.3	*8.5*	*4.0*	*2.2*	*1.2*	*0.5*	*0.4*	*0.3*
Procter & Gamble	517.6	522.9	533.5	541.1	553.0	580.6	774.7	824.3
	28.2	*16.6*	*7.9*	*4.3*	*2.3*	*1.1*	*0.7*	*0.5*
AT&T Inc.	280.2	319.1	398.1	517.9	691.1	990.2	1370.2	1709.4
	15.3	*10.1*	*5.8*	*4.1*	*2.9*	*1.7*	*1.2*	*1.1*
United Technologies	327.1	356.2	417.2	491.2	621.1	834.9	1130.4	1389.9
	17.8	*11.3*	*6.1*	*3.9*	*2.6*	*1.5*	*1.1*	*0.8*
Critical Value	18.3	31.4	67.5	124.3	233.9	553.1	1074.7	1591.2

Table 5: Summary of KS statistics for alternative distributional assumptions, " * " indicates the test for d_t, otherwise for $\tilde{u}_t$. Mean, median, standard deviation ("std"), maximum value ("max"), minimum value ("min") and range of AD and KS statistics are presented in this table.

	AD_{mean}	AD_{median}	AD_{std}	AD_{max}	AD_{min}	AD_{range}	AD^*_{mean}	AD^*_{median}	AD^*_{std}	AD^*_{max}	AD^*_{min}	AD^*_{range}
FGN	2.7056	1.7220	2.7171	15.2900	0.1732	15.1170	4.4772	3.9688	3.0290	19.3230	0.2423	19.0810
fsn	0.4468	0.4207	0.1809	1.1384	0.1066	1.0318	0.4574	0.4307	0.1836	1.2071	0.1014	1.1057
lognormal	2.6100	1.6285	2.7004	15.4960	0.1652	15.3310	4.3202	3.8918	3.0646	19.6470	0.2407	19.4070
stable	0.4460	0.4233	0.1787	1.1384	0.1122	1.0262	0.4583	0.4372	0.1835	1.0795	0.1060	0.9736
exponential	1.0553	1.0516	0.0198	1.1521	0.9719	0.1802	1.0572	1.0526	0.0174	1.1246	1.0254	0.0993
weibull	1.0487	1.0472	0.0220	1.1103	0.9471	0.1632	1.0572	1.0526	0.0174	1.1246	1.0254	0.0993

c

	KS_{mean}	KS_{median}	KS_{std}	KS_{max}	KS_{min}	KS_{range}	KS^*_{mean}	KS^*_{median}	KS^*_{std}	KS^*_{max}	KS^*_{min}	KS^*_{range}
FGN	0.2615	0.2901	0.0866	0.3955	0.0644	0.3311	0.3157	0.3359	0.0715	0.4054	0.0902	0.3152
fsn	0.0381	0.0361	0.0101	0.0887	0.0197	0.0690	0.0493	0.0484	0.0097	0.0920	0.0288	0.0632
lognormal	0.2590	0.2816	0.0858	0.3984	0.0613	0.3371	0.3129	0.3334	0.0717	0.4030	0.0891	0.3139
stable	0.0378	0.0366	0.0099	0.0861	0.0195	0.0666	0.0495	0.0484	0.0101	0.0923	0.0281	0.0641
exponential	0.5265	0.5249	0.0091	0.5579	0.4856	0.0723	0.5278	0.5257	0.0081	0.5586	0.5126	0.0459
weibull	0.5236	0.5230	0.0105	0.5522	0.4729	0.0793	0.5278	0.5257	0.0081	0.5586	0.5126	0.0459

Table 6: Bootstrap 99% confidence intervals for mean of differences in AD and KS statistics, " * " indicates statistics for d_t, otherwise for $\tilde{u}_t$. "FGN" stands for fractional Gaussian noise, "fsn" stands for fractional stable noise, "exp" stands for exponential distribution, "wbl" stands for Weibull distribution.

T:	AD	AD^*	KS	KS^*
$E(T_{stable} - T_{FGN})$	(-2.5223, -1.9851)	(-4.3235 , -3.7124)	(-0.2318 , -0.2157)	(-0.2734 , -0.2594)
$E(T_{stable} - T_{lognormal})$	(-2.4281, -1.8893)	(-4.1642 , -3.5481)	(-0.2292 , -0.2132)	(-0.2706 , -0.2565)
$E(T_{stable} - T_{exp})$	(-0.6278 , -0.5913)	(-0.6175 , -0.5803)	(-0.4895 , -0.4878)	(-0.4793 , -0.4774)
$E(T_{stable} - T_{wbl})$	(-0.6211 , -0.5850)	(-0.6175 , -0.5804)	(-0.4868 , -0.4848)	(-0.4793 , -0.4773)
$E(T_{fsn} - T_{FGN})$	(-2.5236 , -1.9774)	(-4.3204 , -3.7077)	(-0.2315 , -0.2154)	(-0.2736 , -0.2597)
$E(T_{fsn} - T_{lognormal})$	(-2.4265 , -1.8888)	(-4.1638 , -3.5485)	(-0.2290 , -0.2130)	(-0.2708 , -0.2568)
$E(T_{fsn} - T_{exp})$	(-0.6274 , -0.5904)	(-0.6182 , -0.5815)	(-0.4893 , -0.4875)	(-0.4795 , -0.4776)
$E(T_{fsn} - T_{wbl})$	(-0.6203 , -0.5838)	(-0.6185 , -0.5815)	(-0.4866 , -0.4845)	(-0.4795 , -0.4776)
$E(T_{fsn} - T_{stable})$	(-0.0057 , 0.0074)	(-0.0077 , 0.0059)	(-0.0002 , 0.0007)	(-0.0007 , 0.0002)

Table 7: Supporting cases comparison of goodness of fit for fractional stable noise and stable distribution based on AD and KS statistics. Symbol " * " indicates the test for d_t, otherwise the test is for $\tilde{u}_t$. Symbol " $\succ$" means being preferred and "$\sim$" means indifference. Numbers shows the supporting cases to the statement in the first column and the number in parentheses give the proportion of supporting cases in the whole sample.

	AD	AD^*	KS	KS^*
$fsn \succ stable$	327	345	327	362
	(47.81%)	(50.44%)	(47.81%)	(52.93%)
$stable \succ fsn$	344	328	351	318
	(50.29%)	(47.95%)	(51.32 %)	(46.49%)
$fsn \sim stable$	13	11	6	4
	(1.90%)	(1.61 %)	(0.87%)	(0.58%)

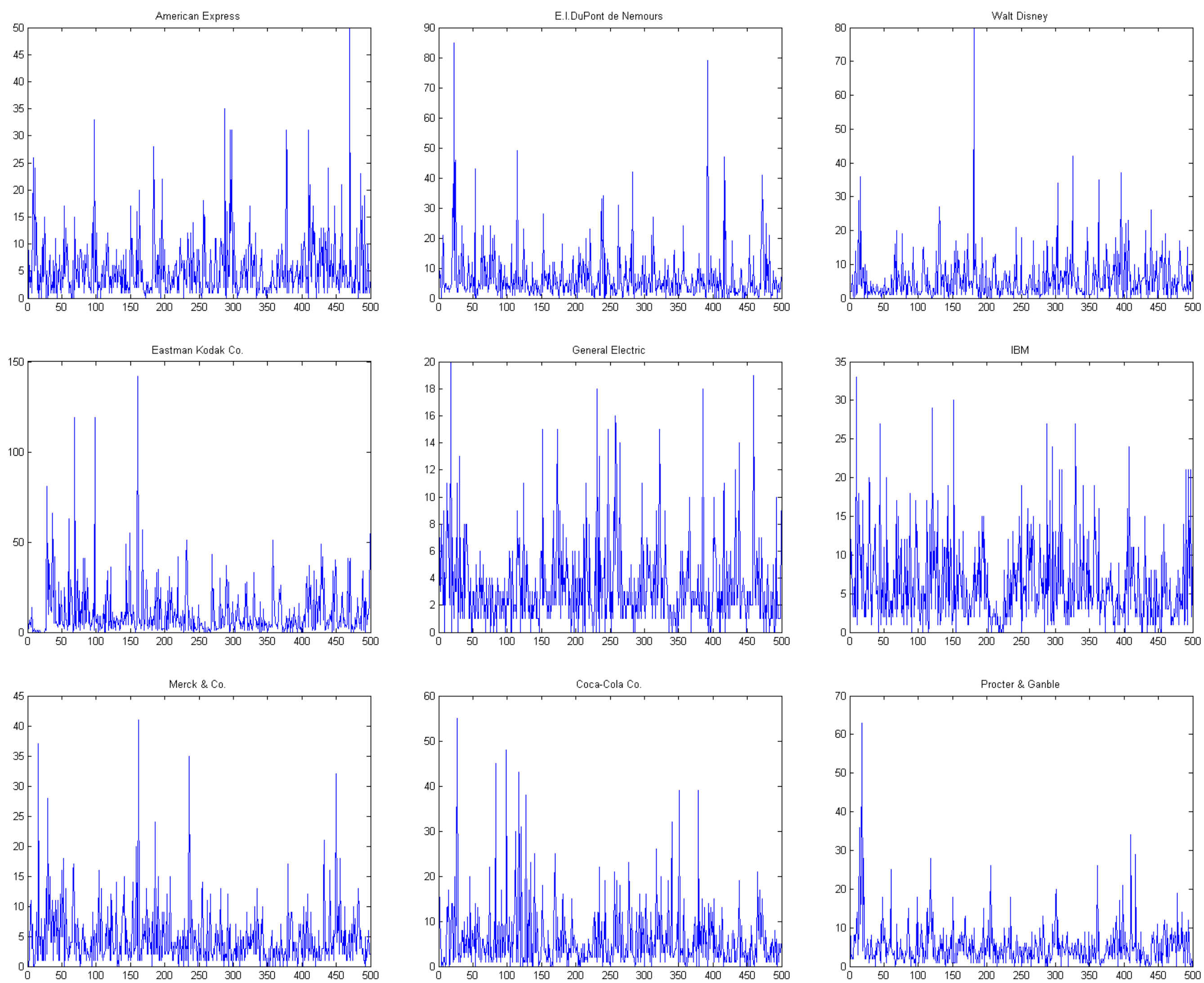

Figure 1: Plot of trade duration for several stocks.

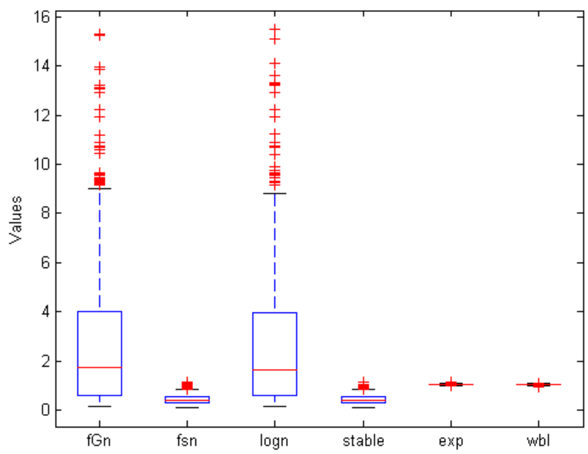

Figure 2: Boxplot of AD statistics for $\tilde{u}_t$ in alternative distributional assumptions.

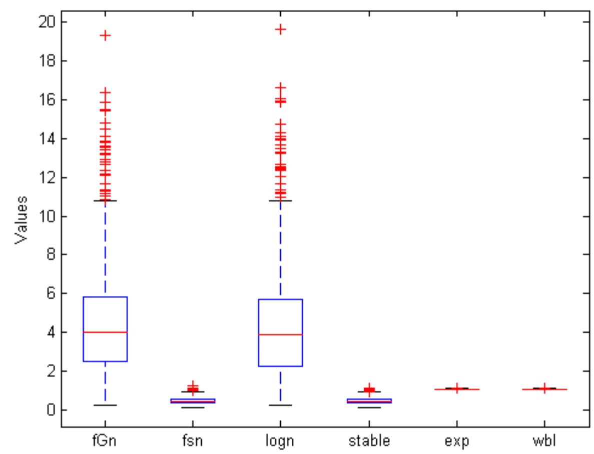

Figure 3: Boxplot of AD* statistics for d_t in alternative distributional assumptions.

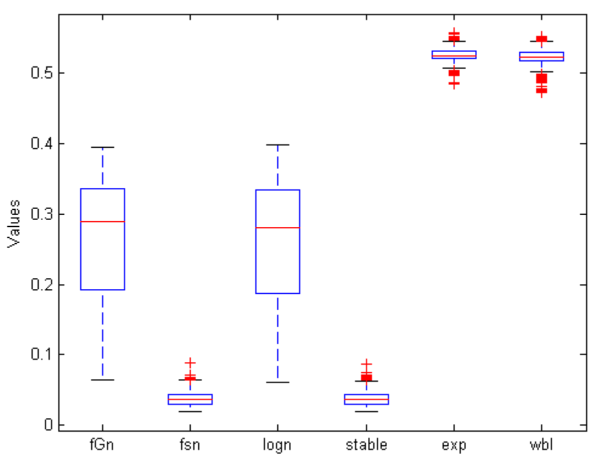

Figure 4: Boxplot of KS statistics for $\tilde{u}_t$ in alternative distributional assumptions.

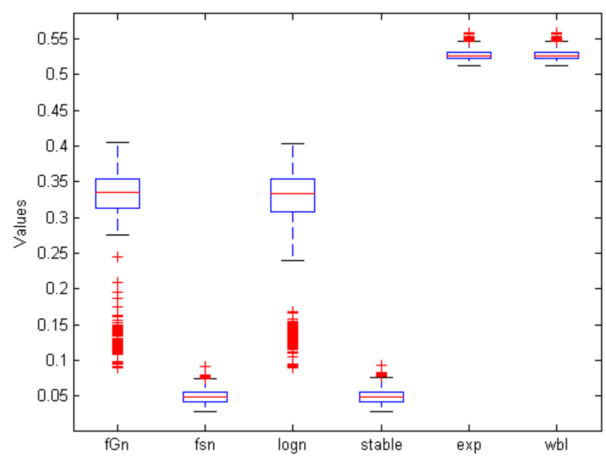

Figure 5: Boxplot of KS* statistics for d_t in alternative distributional assumptions.