Orderings and Probability Functionals Consistent with Preferences

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Abstract

This paper unifies the classical theory of stochastic dominance and investor preferences with the recent literature on risk measures applied to the choice problem faced by investors. First we summarize the main stochastic dominance rules used in the finance literature. Then we discuss the connection with the theory of integral stochastic orders and we introduce orderings consistent with investors' preferences. Thus, we classify them, distinguishing several categories of orderings associated with different classes of investors. Finally we show how we can use risk measures and orderings consistent with some preferences to determine the investors' optimal choices.

Key words: Stochastic dominance, Probability functionals, Mellin transform, coherent and convex measures.

JEL Classification: G11, C44, C61

MSC Classification (2000): 60E15, 91B16, 91B28

1. Introduction

In this paper, we classify risk/uncertainty orderings and measures consistent with investors' preferences. In particular, we present a general and unifying framework of the theory of orderings by examining the connection with recent studies on risk measures.

Utility theory classifies the optimal choices for different categories of market agents (for example, risk-averse, non-satiable, non-satiable risk averse) under ideal market conditions. Roughly speaking, in utility theory the ordering of uncertain choices begins with the selection of a finite number of axioms that characterize the preferences for a given class of market agents. The second step of the theory involves representing the preferences of market agents using "utility functionals" that summarize the decision makers' behavior.

Consequently, using the correspondence between the orderings of utility functionals and the orderings of random variables, we can identify the optimal choices for a given class of market agents. The fundamentals of utility theory under uncertainty conditions have been developed by von Neumann and Morgenstern (1953). Many improvements and advancements of the theory have been proposed, see, among others, Tversky, Kahneman (1992), Machina (1982), Yaari (1987), Gilboa and Schmeidler (1989), Schervisch, et al (1990) and Maccheroni et al (2005). In particular, Karni (1985) and Schervisch, et al (1990) have emphasized that investors' choices are strictly dependent on the possible states of the returns. Thus, investors have generally state-dependent utility functions. Moreover, as shown by Castagnoli and LiCalzi (1999) the state-dependent utilities and the target-based approaches are equivalent. In addition expected utility can be reinterpreted in terms of the probability that returns are above a given benchmark (see Bordley and LiCalzi (2000)). Thus, the more appealing benchmark approach is a generalization of the classic von Neumann–Morgenstern approach.

In this paper we study orderings among probability functionals which are induced by an order of preferences. Since investors maximize an expected state-dependent utility function, we analyze stochastic orderings that take into account possible investors' benchmarks. We begin by analyzing the links among continua stochastic dominance orders, survival, and inverse stochastic dominance rules (see Fishburn (1976, 1980) and Muliere and Scarsini (1989)). We tie together the consistency-isotonicity of risk and reward measures with classical orderings and we show how risk/uncertainty measures are used to obtain non-dominated choices. We continue by discussing an extension of classical orderings using probability functionals that satisfy an opportune identity property (see Rachev (1991)) and the basic rules of the theory of integral stochastic orders (see, among others, Müller (1997), Müller and Stoyan (2002)). In the discussion we provide several examples that help clarify the main developments of the proposed analysis. In particular, we describe new orderings that are consistent with a particular ordering of preferences. The ordering of preferences could be characterized either with some axioms that identify the decision makers' preferences or with another order that identifies the preferences of a particular class of investors. Then, we classify the new orderings distinguishing between uncertainty and risk orderings; between orderings and survival/dual/conditional/derivate orderings; between bounded and unbounded orderings, between static and dynamic orderings, among behavioral finance orderings and among different levels of orderings. Furthermore, we provide a methodology to build new orderings starting with some risk and uncertainty measures that satisfy some basic properties. Finally, we show how to obtain non-dominated choices with respect to any ordering.

In the next section, section 2, we examine continua and inverse stochastic dominance rules. In section 3, we describe how to use probability functionals to define new orderings and portfolio risk measures. We provide concluding remarks regarding the potential applications of the paper's findings in section 4.

2. A first classification of the principal stochastic dominance rules and their main implications

Stochastic orderings represent a fundamental starting point to solve the investor's problem of choosing a portfolio under uncertainty. For this reason, it has been widely used in financial economics. In this section, we summarize the main rules of stochastic orderings in a complete probability space $(\Omega, \Im, P)$. We then apply these rules in the general analysis in the sections to follow.

Bounded continua stochastic dominance rules: Fishburn (1976) suggests using the fractional integral applied to the cumulative distribution function $F_X^{(1)} = F_X$ of a random variable *X*, in order to define continuous orderings among different classes of random variables. Let *X*, *Y* be two random variables such that the functions $F_X^{(\alpha)}(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} dF_X(u)$, $F_Y^{(\alpha)}(t)$ are defined on $supp\{X,Y\} \equiv [a,b]$ (where $a,b \in \overline{\mathbb{R}}$ and $a = \inf\{x \mid F_X(x)+F_Y(x)>0\}$, $b = \sup\{x \mid F_X(x)+F_Y(x)<2\}$) that is, $F_X^{(\alpha)}(t) < \infty$, $F_Y^{(\alpha)}(t) < \infty \quad \forall t \in [a,b)$. Then, we say *X* dominates *Y* with respect to the $\alpha\ (\geq 1)$ *bounded* *stochastic dominance order* (namely $X \underset{\alpha}{\overset{b}{\geq}} Y$) if and only if $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for every t belonging to $supp\{X,Y\} \equiv [a,b]$. We also say that X strictly dominates Y with respect to the α bounded order (namely $X \underset{\alpha}{\overset{b}{>}} Y$) iff $X \underset{\alpha}{\overset{b}{\geq}} Y$ and $F_X \neq F_Y$.

Bounded inverse stochastic dominance rules: As suggested by Muliere and Scarsini (1989), we can also describe stochastic dominance rules based on the left inverse of F_X given by $F_X^{(-1)}(p) = \inf\{x \mid F_X(x) \geq p\}$ for every $p \in (0,1]$ and $F_X^{(-1)}(0) = \lim_{p \to 0} F_X^{(-1)}(p)$. Thus, assuming for every $\alpha > 1$ and $p \in [0,1]$, $F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_0^p (p-u)^{\alpha-1} dF_X^{(-1)}(u)$, we say X dominates Y with respect to $\alpha\,(\geq 1)$ *inverse stochastic order* ($X \underset{-\alpha}{\geq} Y$) iff $F_X^{(-\alpha)}(p) \geq F_Y^{(-\alpha)}(p)$ $\forall p \in [0,1]$ and that X strictly dominates Y with respect to α inverse order ($X \underset{-\alpha}{>} Y$) iff $X \underset{-\alpha}{\geq} Y$ and $F_X \neq F_Y$. [1]

Survival stochastic dominance rules: In reliability theory, survival analysis, and actuarial mathematics, orderings are often expressed by the survival functions $\bar{F}_X^{(1)}(x) = P(X > x)$. Thus, assuming for every $\alpha > 1$, $\bar{F}_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} dF_X(u)$ for $t < b$, and $\bar{F}_X^{(\alpha)}(t) = 0$, $\forall t \geq b$, we say X dominates Y with respect to $\alpha\,(\geq 1)$ bounded survival stochastic order ($X \underset{\alpha,\,sur}{\overset{b}{\geq}} Y$) iff $\bar{F}_X^{(\alpha)}(t) \leq \bar{F}_Y^{(\alpha)}(t)$ for every t belonging to $supp\{X,Y\} \equiv [a,b]$. Similarly, assuming for every $p \in [0,1]$ $\bar{F}_X^{(-1)}(p) = -F_X^{(-1)}(p)$, and $\bar{F}_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)} \int_p^1 (u-p)^{\alpha-1} dF_X^{-1}(u)$ for every $\alpha > 1$, stochastic dominance rules are defined for inverse survival orderings. In particular, when, either $\alpha > 1$, or X is a continuous random variable, we have that $\bar{F}_X^{(-\alpha)}(p) = F_{-X}^{(-\alpha)}(1-p)$ for every $p \in [0,1]$ and $\bar{F}_X^{(\alpha)}(u) = F_{-X}^{(\alpha)}(-u)$ for every $u \in [a,b]$.

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[1] In particular, it is well known that $\underset{1}{\overset{b}{\geq}}$ and $\underset{2}{\overset{b}{\geq}}$ orders are equivalent to the respective $\underset{-1}{\geq}$ and $\underset{-2}{\geq}$ orders. However, it is still not clear if, for any α, there is a correspondence between $\underset{\alpha}{\overset{b}{\geq}}$ and the respective inverse order $\underset{-\alpha}{\geq}$. This question should be the subject of future research.

Bounded/unbounded stochastic dominance rules: All the previous dominance rules are usually called *bounded stochastic dominance rules* since they imply an order of fractional integrals of (inverse, survival) distribution functions defined only on the support of the random variables or on a given interval of the real line. Contrary to the bounded orderings, we can define orderings, which we call *unbounded*, that do not consider the support of the compared random variables. Fishburn (1980) states that X (strictly) dominates Y with respect to the α stochastic dominance order; namely, $X \underset{\alpha}{\geq} Y$ ($X \underset{\alpha}{>} Y$) iff $F_X^{(\alpha)}(t) \leq F_Y^{(\alpha)}(t)$ for every real t (and $F_X \neq F_Y$) where $F_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-u)^{\alpha-1} dF_X(u)$. He also demonstrates that the orders $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$ are equivalent for every $\alpha \in [1,2]$. Instead, when $\alpha > 2$, the orders $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$ do not generally coincide. We observe that the definition of unbounded orderings can be easily extended to survival, inverse, and survival and inverse orderings. As a matter of fact, we can extend $F_X^{(-1)}$ to the entire real line $\mathbb{R}$ assuming $F_X^{(-1)}(u) = F_X^{(-1)}(0)$ $\forall u \leq 0$ and $F_X^{(-1)}(t) = F_X^{(-1)}(1)$ $\forall t \geq 1$. Thus, we say X dominates Y with respect to the *unbounded α inverse stochastic order* (unbounded $X \underset{-\alpha}{\geq} Y$) iff $F_X^{(-\alpha)}(u) \geq F_Y^{(-\alpha)}(u)$ for every $u \in \mathbb{R}$, where, with abuse of notation, we still use the same notation to point out the new function $F_X^{(-\alpha)}(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} (u-t)^{\alpha-1} dF_X^{(-1)}(t)$. Similarly, we say X dominates Y with respect to the *unbounded α survival stochastic order* ($X \underset{\alpha,sur}{\geq} Y$) iff $\overline{F}_X^{(\alpha)}(t) \leq \overline{F}_Y^{(\alpha)}(t)$ for every real t where $\overline{F}_X^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{+\infty} (u-t)^{\alpha-1} dF_X(u)$. Moreover, all the above α unbounded orderings imply the analogous α bounded ones.

Dominance rules of Behavioral Finance: In contrast to classical results on expected utility theory, many empirical and theoretical studies support dominance rules of behavioral finance (see Friedman and Savage (1948), Markowitz (1952), Tversky and Kahneman (1992), Levy and Levy (2002), Baucells and Heukamp (2006), Edwards (1996) and the references therein). In particular, given $c, d \in supp\{X, Y\}$, $c \geq d$, we say that *X dominates Y in the sense of prospect α* ($\alpha \geq 1$) *dominance order* ($X \; P \underset{\alpha}{SD} \; Y$) if and only if $\forall y \in (-\infty, c]$, $g_X^{(\alpha)}(y) := \int_{d}^{d+c-y} F_X^{(\alpha)}(u)du \leq g_Y^{(\alpha)}(y)$ and $\tilde{g}_X^{(\alpha)}(y) := \int_{y}^{c} F_X^{(\alpha)}(u)du \leq \tilde{g}_Y^{(\alpha)}(y)$ if and only if $\forall(x,y)\in[0,1]\times(-\infty,c]$, $g_X^{(\alpha)}(x,y):=xg_X^{(\alpha)}(y)+(1-x)\tilde{g}_X^{(\alpha)}(y)\le g_Y^{(\alpha)}(x,y)$. Similarly, we can define the Markowitz' dominance rules and we say that *X dominates Y in the sense of Markowitz'* α ($\alpha\ge 1$) *dominance order* ($X \underset{\alpha}{M\ SD} Y$) if and only if $\forall y\in(-\infty,c]$ $m_X^{(\alpha)}(y):=\int_{-\infty}^{y}F_X^{(\alpha)}(u)du\le m_Y^{(\alpha)}(y)$ and $\tilde{m}_X^{(\alpha)}(y):=\int_{d+c-y}^{+\infty}F_X^{(\alpha)}(u)du\le\tilde{m}_Y^{(\alpha)}(y)$ if and only if $m_X^{(\alpha)}(x,y):=xm_X^{(\alpha)}(y)+(1-x)\tilde{m}_X^{(\alpha)}(y)\le m_Y^{(\alpha)}(x,y)$, $\forall(x,y)\in[0,1]\times(-\infty,c]$. We can easily extend the definition of the previous behavioral finance orderings to inverse and/or bounded orderings. For example, given $c,d\in(0,1)$, $c\ge d$, we define analogous functions $g_X^{(-\alpha)}$, $\tilde{g}_X^{(-\alpha)}$, $m_X^{(-\alpha)}$, $\tilde{m}_X^{(-\alpha)}$ on the interval [0,c] and then $X \underset{-\alpha}{M\ SD} Y$ if and only if $\forall(x,y)\in[0,1]\times[0,c]$, $m_X^{(-\alpha)}(x,y)\le m_Y^{(-\alpha)}(x,y)$; $X \underset{-\alpha}{P\ SD} Y$ if and only if $\forall(x,y)\in[0,1]\times[0,c]$, $g_X^{(-\alpha)}(x,y)\le g_Y^{(-\alpha)}(x,y)$. Further dominance behavioral finance rules can be found in Baucells and Heukamp (2006).

Risk/uncertainty dominance rules: When we address the risk aversion aspect of a choice, we can consider the so called Rothschild-Stiglitz (R-S) order (Rothschild and Stigliz (1970)). It is well known that *X* dominates *Y* in the sense of Rothschild and Stiglitz ($X\ R-S\ge Y$) if and only if a risk-averse investor prefers *X* to *Y*. More generally we state that *X* dominates *Y* in the sense of α - R-S order if and only if $X \underset{\alpha}{\ge} Y$ and $-X \underset{\alpha}{\ge} -Y$, that is if and only if $F_X^{(\alpha)}(x,t):=xF_X^{(\alpha)}(t)+(1-x)\bar{F}_X^{(\alpha)}(t)\le F_Y^{(\alpha)}(x,t)$ for every $x\in[0,1], t\in\mathbb{R}$. Similarly, we state *X* dominates *Y* in the sense of α -inverse (bounded) R-S order (assuming $supp\{X,Y\}\equiv[a,b]$) if and only if, $X \underset{-\alpha}{\ge} Y$ ($X \overset{b}{\underset{\alpha}{\ge}} Y$) and $-X \underset{-\alpha}{\ge} -Y$ ($-X \overset{-a}{\underset{\alpha}{\ge}} -Y$), that is if and only if $F_X^{(-\alpha)}(x,t):=xF_X^{(-\alpha)}(t)+(1-x)\bar{F}_X^{(-\alpha)}(t)\ge F_Y^{(-\alpha)}(x,t)$ $\forall x\in[0,1], \forall t\in[0,1]$ ($F_X^{(\alpha)}(x,t)\le F_Y^{(\alpha)}(x,t)$, $\forall x\in[0,1]$, $\forall t\in[a,b]$). In particular, when α =2, we obtain the classic R-S order. In the ordering literature α -R-S orders are also known as α -concave orders when α is greater than or equal to 2. Clearly, we can also define R-S type rules for orderings of behavioral finance, that is, we say that *X* dominates *Y* in the sense of $\pm\alpha$ - R-S-P(-M) order if and only if $X \underset{\pm\alpha}{P\ SD} Y$ ($X \underset{\pm\alpha}{M\ SD} Y$) and $-X \underset{\pm\alpha}{P\ SD} -Y$ ($-X \underset{\pm\alpha}{M\ SD} -Y$).

Following the operational definition of *risk* and *uncertainty* as perceived by investors (see, among others, Rachev et al (2007)), the previous discussion suggests distinguishing between the orderings with respect to (1) the uncertainty of different positions and (2) the investor's exposure to risk. Generally the strongest risk ordering applied in the finance literature is the strict inequality between random variables (that is, we say that *X* is preferred to *Y* iff *X*>*Y*) which is referred to as the *monotony order*. Therefore stochastic orderings between random variables that are implied by the monotony order are usually called *risk orderings.* Typical risk orderings are $\underset{\pm\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$, $\underset{\pm\alpha}{P\ SD}$, $\underset{\pm\alpha}{M\ SD}$, since if *X*>*Y*, then $X \underset{\pm\alpha}{\geq} (\overset{b}{\underset{\alpha}{\geq}}, \underset{\pm\alpha}{P\ SD}, \underset{\pm\alpha}{M\ SD})$ *Y*. Instead, R-S type orders are used to characterize the different degrees of portfolio uncertainty, and for this reason they are generally called *uncertainty orderings*.

Utility functions: It is interesting to observe that stochastic ordering theory justifies most of the financial applications based on utility theory. In particular, the first and the second orderings are the ones most used. As a matter of fact, we say that $X \underset{\pm 1}{\geq} Y$ if and only if any non-satiable investor prefers X to Y (i.e., $E\left(u(X)\right) \geq E(u(Y))$ for every non-decreasing utility function *u*) and $X \underset{\pm 2}{\geq} Y$ if and only if any non-satiable risk-averse investor prefers *X* to Y (i.e., $E\left(u(X)\right) \geq E(u(Y))$ for every non-decreasing concave utility function *u*). However, all the continuous orderings are characterized by a class of utility functions, i.e., $X \underset{\alpha}{\geq} Y$ if and only if $E\left(u(X)\right) \geq E(u(Y))$ for all utility functions *u* belonging to a given class U_α. In addition, considering the properties of fractional integrals (see Miller and Ross (1993)), any α order implies a β order for every $\beta \geq \alpha \geq 1$ and $U_\alpha \supseteq U_\beta$. Analogous considerations apply to behavioral finance type, R-S type, bounded, survival, and/or inverse orderings (see Fishburn (1976, 1980), Muliere and Scarsini (1989), Levy and Levy (2002), and Ortobelli et al. (2006a) for further details). For example, we say that $X \underset{-\alpha}{\geq} Y$ if and only if $\int_0^1 \phi(x) dF_X^{(-1)}(x) \leq \int_0^1 \phi(x) dF_Y^{(-1)}(x)$ for every ϕ belonging to a given class of functions V^α.

Measures consistent/isotonic with stochastic orderings: This first distinction between different dominance rules could have an important impact on investors. Specifically, to select the set of the admissible choices that are coherent for a given category of investors, we can consider the direct measures $\rho(X)$ (associated with random wealth *X*) that are *consistent* with the order relation ($\underset{\pm\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$, α -(bounded) $R-S \geq$, $\underset{\pm\alpha}{P\ SD}$, $\underset{\pm\alpha}{M\ SD}$); that is, $\rho(X) \leq \rho(Y)$ if X dominates ($\underset{\pm\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$, α -(bounded) $R-S \geq$, $\underset{\pm\alpha}{P\ SD}$, $\underset{\pm\alpha}{M\ SD}$) *Y*. Typically, measures consistent with the monotony order (that is, $\rho(X) \leq \rho(Y)$ if X >*Y*) are a proper risk measure since they take into account downside risk. Thus all the measures consistent with a risk ordering are called *risk measures*. Instead the measures consistent with R-S type orders discriminate between the different levels of uncertainty and for this reason they are generally called *uncertainty measures*. Furthermore, we can order choices by looking at rewards instead of risk. We specify a *reward* measure to be any functional *v* defined on the portfolio returns that is *isotonic* with respect to a given stochastic risk order (for example: $\underset{\pm\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}, P\underset{\pm\alpha}{SD}, M\underset{\pm\alpha}{SD}$). That is, if a given class of investors (e.g., non-satiable, non-satiable risk averse) prefers X to Y, then $v(X) \geq v(Y)$. Therefore, if $\rho(X)$ is a risk measure consistent with a risk ordering, then the measure $-\rho(X)$ is a reward measure that is isotonic with the same order. Consequently, if we characterize consistency with respect to risk orderings, we also implicitly characterize isotonicity. This is why we emphasize consistency within a given order.

Efficient frontiers: The consistency (isotonicity) property is extremely important since it permits an investor to determine the class of optimal choices often referred to as the *efficient frontier* or *space of efficient choices*. For example, suppose we are interested in establishing the optimal choices for a class of investors with risk aversion determined by the parameter α and who want to limit their losses below a fixed wealth benchmark *t*. Then for this class of investors we obtain the best choices by minimizing the risk measure $\rho_{t,\alpha}(X) = E\left((t-X)_+^{\alpha-1}\right)$ that is consistent with the $\underset{\alpha}{\geq}$ order. Similarly, if we are interested in limiting the uncertainty of wealth given by the concentration of the random choices, we can minimize the concentration measure $\tilde{\rho}_\alpha(X) = E\left(\left|X_1 - X_2\right|^{\alpha-1}\right)$ where X_1, X_2 are independent copies of X, that is, an uncertainty measure consistent with α - $R-S \geq$ order.

From this preliminary discussion it follows that there exists many different ways to specify the choices that are available to investors. We distinguish between orders and their inverse/survival orders, bounded and unbounded orders, and risk and uncertainty orders. Moreover, there exists a strong connection between orderings and risk/uncertainty measures that will be more thoroughly treated in the next section.

3. The Theory of Integral Stochastic Orders and Probability Functionals Consistent with Preferences

Most of the previous classification can be generalized using the theory of integral stochastic orders (see, among others, Müller (1997), Müller and Stoyan (2002) and the references therein) and considering probability functionals that satisfy an opportune identity property (see Rachev (1991)). Let us clarify this possible extension.

As discussed in the introduction, decision makers who maximize an expected state dependent utility function implicitly maximize their performance with respect to a given benchmark. Then we generally define functionals on a product space $U = \Lambda \times V$ of joint random variables defined on $(\Omega, \Im, P)$ with values in a Polish space A, where Λ is a non-empty set of the admissible choices while V is the space of the possible benchmarks. However sometimes, the space of benchmarks is not mentioned, since they are not directly used in the comparison between admissible choices. In this case we can simply assume $U = \Lambda$.

In order to define properly functionals associated to orderings, one commonly uses a measurable space (B,M_B). Thus, suppose $\rho : U \times M_B \to \mathbb{R}$ is a functional such that $\rho_X := \rho(X, Z, \bullet) : M_B \to \mathbb{R}$ is a sigma finite signed measure for any fixed random variable X belonging to Λ and any fixed benchmark Z belonging to V. We say that the simple functional ρ is *absolutely continuous* with respect to a sigma finite positive measure μ on (B,M_B), if every signed measure ρ_X, $\forall X \in \Lambda$, $\forall Z \in V$, is *absolutely continuous* with respect to μ. We say that ρ is a *simple functional* over the class Λ when, for any fixed benchmark Z, it satisfies the following identity property: $\forall X, Y \in \Lambda$, $\rho_X = \rho_Y$ iff $P_X = P_Y$ where P_X is the probability distribution measure associated with X and P_Y with Y. If a simple functional over the class Λ is absolutely continuous with respect to a sigma finite positive measure μ, then we say that the functional $f_X := f(X, Z, \bullet) : B \to \mathbb{R}$, $f_X := \dfrac{d\rho_X}{d\mu}$ is *a simple functional derivate* with respect to μ (i.e. simple since $\forall X \in \Lambda$, $\forall Z \in V$, $f_X = f_Y$ μ a.e. iff $P_X = P_Y$).

In many cases, we categorize random variables by taking into account the realization of some events. Suppose there exists a sub-class of sets $C \subseteq N_A$, that we call the *class of conditional sets*, where N_A is the Borel sigma algebra of the Polish space A, and a functional $\rho : U \times C \times M_B \to \mathbb{R}$ such that for every $D \in C$ s.t. $P(X \in D) > 0$, $\rho_{[X/X \in D]} := \rho(X, D, Z, \bullet) : M_B \to \mathbb{R}$ is a sigma finite signed measure $\forall X \in \Lambda$, $\forall Z \in V$ [2]. We say that ρ is a *conditional simple functional* over the class Λ with respect to the class

[2] We denote by [Z/G] any random variable that has as its distribution the conditional distribution of Z given G.

of conditional sets C when, for any fixed benchmark Z, $D\in C$ s.t. $P(X\in D)>0$ and $P(Y\in D)>0$, $\rho_{[X/X\in D]}=\rho_{[Y/Y\in D]}$ if and only if $P_{[X/X\in D]}=P_{[Y/Y\in D]}$. When $C_1\subseteq C_2\subseteq N_A$, any conditional simple functional defined $\rho:U\times C_2\times M_B\to\mathbb{R}$ is said the *extension* of the functional ρ valued on $U\times C_1\times M_B$. In the most common cases, the signed measures ρ_X ($\rho_{[X/X\in D]}$) are associated to bounded variation functions. That is, suppose $B=[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^n$ (with $-\infty\le a_i<b_i\le+\infty$ i=1,…,n) and the measures ρ_X ($\rho_{[X/X\in D]}$) are the Lebesgue-Stieltjes measures induced by a function $\tilde{\rho}_X:=\tilde{\rho}(X,Z,\bullet):B\to\mathbb{R}$ ($\tilde{\rho}_{[X/X\in D]}:=\tilde{\rho}(X,D,Z,\bullet):B\to\mathbb{R}$) of bounded variation (i.e., $\tilde{\rho}_{[X/X\in D]},\tilde{\rho}_X\in BV(\mathbb{R}^n)$), and $\forall\mathbf{x}\in[\mathbf{a},\mathbf{b}]$, $\tilde{\rho}_X(\mathbf{x})=\rho_X[\mathbf{a},\mathbf{x}]$ ($\tilde{\rho}_{[X/X\in D]}(\mathbf{x})=\rho_{[X/X\in D]}[\mathbf{a},\mathbf{x}]$). In the following, when $B=[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^n$, with abuse of notation, we still refer to $\rho:U(\times C)\times B\to\mathbb{R}$ when we use the bounded variation functions $\tilde{\rho}_X$ $\tilde{\rho}_{[X/X\in D]}$ ($\forall X\in\Lambda$). Then, we say that ρ is a (conditional) simple functional over the class Λ when, for any fixed benchmark Z, $\forall X,Y\in\Lambda$, ($\rho_{[X/X\in D]}=\rho_{[Y/Y\in D]}$) $\rho_X=\rho_Y$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^n$ iff ($P_{[X/X\in D]}=P_{[Y/Y\in D]}$) $P_X=P_Y$. Here we report some examples of (conditional) simple functionals where $\Lambda\subseteq L^p$ is a class of real random variables that admit the finite *p-th* moment:

- Suppose $U=\Lambda\subseteq L^p$ with $p>1$ (i.e., we do not consider benchmarks). Then some examples of simple functionals are given by:

  a) for any $\alpha\in[1,p]$, $\rho_X(x,y)=g_X^{(\alpha)}(x,y)$ $\forall(x,y)\in[0,1]\times(-\infty,c]=B$;

  b) for any $\alpha\in[1,p+1]$, $\rho_X(p)=F_X^{(-\alpha)}(p)$ $\forall p\in[0,1]=B$;

  c) for any $\alpha\in[2,p+1]$, $\rho_X(x,t)=F_X^{(\alpha)}(x,t)$ $\forall(x,t)\in[0,1]\times\mathbb{R}=B$.

- Consider $U=\Lambda\subseteq L^p$ with $p>0$ and three possible classes of conditional sets $C_1=\{(a,+\infty)/a\in\mathbb{R}\}$, $C_2=\{(-\infty,a]/a\in\mathbb{R}\}$, $C_3=B_{\mathbb{R}}$ i.e., the Borel sigma algebra on the real line. Then, assuming $B=\mathbb{R}$ and $\alpha\in[1,p+1]$, typical examples of conditional simple functionals are respectively: $\rho_{[X/X>c]}(t)=F^{(\alpha)}_{[X/X>c]}(t):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-y)^{\alpha-1}dF_{[X/X>c]}(y)$, $\forall c\in\mathbb{R}$, $P(X>c)>0$; $\rho_{[X/X\le c]}(t)=F^{(\alpha)}_{[X/X\le c]}(t)$, $\forall c\in R$, $P(X\le c)>0$; $\forall D\in B_{\mathbb{R}}$ $P(X\in D)>0$,

$\rho_{[X/X\in D]}(t)=F^{(\alpha)}_{[X/X\in D]}(t)$. This last conditional simple functional is an extension of the previous ones.

- Suppose the benchmark Z admits a finite mean and for a given $\alpha\in(1,2)$, $Z\underset{-\alpha}{\geq}X$ for every X belonging to $\Lambda\subseteq L^1$ (with $\left|F_Z^{(-2)}(1)\right|<\infty$). Then we can consider the simple functional $\rho_X(u)=F_Z^{(-u)}(1)-F_X^{(-u)}(1)\ \ \forall u\in[\alpha,2]=B$ (see Ortobelli et al. (2006a)).
- Suppose the benchmark $Z\in L^p$ (with $p>1$) and for a given $\alpha\in[1,p)$ $Z\underset{\alpha}{\geq}X$ for every X belonging to $\Lambda\subseteq L^p$. Then we can consider the simple functional $\rho_X(t)=\int_{-\infty}^t\left(F_X^{(\alpha)}(u)-F_Z^{(\alpha)}(u)\right)^q du\ \ \forall t\in\mathbb{R}=B$ for some given $q\geq 1$.
- Suppose the benchmark Z admits a finite mean and dominates in the sense of Rothshild-Stiglitz every X belonging to $\Lambda\subseteq L^1$. Then we can consider the simple functional $\rho_X(x,t)=F_X^{(2)}(x,t)-F_Z^{(2)}(x,t)\ \ \forall(x,t)\in[0,1]\times\mathbb{R}=B$.

Assume that ρ is a simple functional over the class Λ and let Ξ be a non-empty class of measurable real-valued functions defined on B which are integrable with respect to the signed measures ρ_X for any X belonging to Λ. Then, if $\forall X,Y\in\Lambda$, we say *X dominates Y with respect to the class of utility functions* Ξ *and the simple functional* ρ (namely $X\succ_{\rho,\Xi,\Lambda}Y$ or also $\rho_X\succ_{\Xi,\Lambda}\rho_Y$) if and only if for every $f\in\Xi$, $\int_B fd\rho_X\geq\int_B fd\rho_Y$. We call $\succ_{\rho,\Xi,\Lambda}$ *simple integral ordering* among the random variables belonging to Λ and we call Ξ *generator* of the ordering $\succ_{\rho,\Xi,\Lambda}$.

Similarly we can define conditional orderings. Assume a *conditional simple functional* ρ over the class Λ with respect to the class of conditional sets C and let Ξ be a non-empty class of measurable real-valued functions defined on B which are integrable with respect to the signed measures belonging to $\left\{\rho_{[X/X\in D]}/X\in\Lambda,\ D\in C, P(X\in D)>0\right\}$. Then we say X *dominates Y with respect to the class of utility functions* Ξ *and the conditional simple functional* ρ (namely $X\succ_{\rho,\Xi/C,\Lambda}Y$) if and only if for every $D\in C$ s.t. $P(X\in D)>0$ and $P(Y\in D)>0$, $[X/X\in D]\succ_{\rho,\Xi,\Lambda_{/D}}[Y/Y\in D]$, that is, if and only if $\forall D\in C$ (s.t. $P(X\in D)>0$ and $P(Y\in D)>0$) for every $f\in\Xi$, $\int_B fd\rho_{[X/X\in D]}\geq\int_B fd\rho_{[Y/Y\in D]}$. We call $\succ_{\rho,\Xi/C,\Lambda}$ *conditional integral ordering,* (and Ξ, generator of the conditional ordering

$\succ_{\rho,\Xi/C,\Lambda}$). For any conditional integral ordering could exist different possible class of conditional sets $C_1, C_2 \subseteq N_A$ such that the ordering among random variables belonging to Λ does not change, i.e., $X \succ_{\rho,\Xi/C_1,\Lambda} Y$ if and only if $X \succ_{\rho,\Xi/C_2,\Lambda} Y$. We call *maximal class of conditional sets* the largest class of conditional sets $\tilde{C} \supseteq C$ such that the conditional ordering does not change (i.e., $X \succ_{\rho,\Xi/C,\Lambda} Y$ if and only if $X \succ_{\rho,\Xi/\tilde{C},\Lambda} Y$). Moreover, when $C_1 \subseteq C_2 \subseteq N_A$ and we consider the extension to the class C_2 of the *conditional simple functional* ρ valued on class C_1, then $\succ_{\rho,\Xi/C_2,\Lambda}$ implies $\succ_{\rho,\Xi/C_1,\Lambda}$ and the orderings $\succ_{\rho,\Xi_1/C,\Lambda}$, $\succ_{\rho,\Xi_2/C,\Lambda}$ are identical when $C_1 \subseteq C_2 \subseteq \tilde{C}_1$. Similarly, for any (conditional) integral ordering $\succ_{\rho,\Xi,\Lambda}$ there could exist many possible generators. Let $\tilde{\Xi} \supseteq \Xi$ be the largest class of measurable real-valued functions integrable with respect to all signed measures belonging to $\left\{\rho_X / X \in \Lambda\right\}$ ($\left\{\rho_{[X/X\in D]} / X \in \Lambda,\, D \in C, P(X \in D) > 0\right\}$) such that $X \succ_{\rho,\Xi,\Lambda} Y$ ($X \succ_{\rho,\Xi/C,\Lambda} Y$) implies $\int_B f d\rho_X \geq \int_B f d\rho_Y$ ($\forall D \in C$, $P(X \in D) > 0$ and $P(Y \in D) > 0$ $\int_B f d\rho_{[X/X\in D]} \geq \int_B f d\rho_{[Y/Y\in D]}$) for every $f \in \tilde{\Xi}$. We call $\tilde{\Xi}$ the *maximal generator* of the (conditional) integral ordering $\succ_{\rho,\Xi,\Lambda}$ ($\succ_{\rho,\Xi/C,\Lambda}$). Clearly any generator *G* of the (conditional) integral ordering $\succ_{\rho,\Xi,\Lambda}$ ($\succ_{\rho,\Xi/C,\Lambda}$) is a subset of the maximal generator $\tilde{\Xi}$. If $\Xi_2 \subseteq \Xi_1$ then $\succ_{\rho,\Xi_1,\Lambda}$ ($\succ_{\rho,\Xi_1/C,\Lambda}$) implies $\succ_{\rho,\Xi_2,\Lambda}$ ($\succ_{\rho,\Xi_2/C,\Lambda}$) and $\succ_{\rho,\Xi_1,\Lambda}$, $\succ_{\rho,\Xi_2,\Lambda}$ ($\succ_{\rho,\Xi_1/C,\Lambda}, \succ_{\rho,\Xi_2/C,\Lambda}$) are identical when $\Xi_2 \subseteq \Xi_1 \subseteq \tilde{\Xi}_2$. In particular if $B = [c,d] \subseteq \overline{\mathbb{R}}$ and $\succ_{\rho,\Xi,\Lambda}$ is an integral ordering on the class of random variables Λ, we distinguish many other sub-orderings on the class Λ. For example, let us consider the class of functions

$$\Xi_\alpha = \left\{ \phi_\alpha(x) = \int_{x^+}^{d} (s-x)^{\alpha-1} d\tau(s) \Big/ \tau \text{ is a } \sigma-\text{finite signed measure on } B : \phi_\alpha \in \Xi \right\},$$

for any $\alpha > 1$ such that the class of functions Ξ_α is non-empty. Then we get the orderings $X \succ_{\rho,\Xi_\alpha,\Lambda} Y$ if and only if $\int_{[c,d]} f(x) d\rho_X(x) \geq \int_{[c,d]} f(x) d\rho_Y(x)$ for every $f \in \Xi_\alpha$. Since $\Xi_\alpha \subseteq \Xi$, the integral ordering $\succ_{\rho,\Xi,\Lambda}$ implies the integral sub-ordering $\succ_{\rho,\Xi_\alpha,\Lambda}$.

Similarly to integral orderings for random variables we can define integral orderings for stochastic processes. Let us consider a measurable space (B, M_B) and a product space $\tilde{U} = \tilde{\Lambda} \times \tilde{V}$ that is a joint space of right continuous with left limits (RCLL) adapted processes defined on a filtered space $\left(\Omega, \Im, \left(\Im_t\right)_{0 \leq t \leq T}, P\right)$ with values in a Polish space A. Then we define a functional $\rho: \tilde{U} \times M_B \to \mathbb{R}$ such that $\rho_X := \rho(X, Z, \bullet): M_B \to \mathbb{R}$ is a sigma finite signed measure for any fixed stochastic process *X* belonging to $\tilde{\Lambda}$ and any fixed benchmark *Z* belonging to $\tilde{V}$. We say that ρ is a *simple functional for the stochastic processes* of the class $\tilde{\Lambda}$ when, for any fixed benchmark *Z,* it satisfies the following identity property: $\forall X = \{X_t\}_{0 \le t \le T}, Y = \{Y_t\}_{0 \le t \le T} \in \tilde{\Lambda}$, $\rho_X = \rho_Y$ if and only if $P_{X_t} = P_{Y_t} \quad \forall t \in [0, T]$ where P_{X_t} and P_{Y_t} are the probability distribution measures associated at X_t and Y_t respectively. Analogously we say that ρ is a *strong simple functional for the stochastic processes* when the identity property is: $\forall X, Y \in \tilde{\Lambda}$, $\rho_X = \rho_Y$ if and only if the processes $X = \{X_t\}_{0 \le t \le T}$ and $Y = \{Y_t\}_{0 \le t \le T}$ have the same finite dimensional distributions. Then, given a non-empty class of measurable real-valued functions Ξ defined on *B* which are integrable with respect to the signed measures ρ_X for any process *X* belonging to $\tilde{\Lambda}$, we say, $\forall X, Y \in \tilde{\Lambda}$, *X* dominates *Y* with respect to the class of utility functions Ξ and the simple functional ρ (namely $X \succ_{\rho, \Xi, \tilde{\Lambda}} Y$ or also $\rho_X \succ_{\Xi, \tilde{\Lambda}} \rho_Y$) if and only if for every $f \in \Xi$, $\int_B f d\rho_X \ge \int_B f d\rho_Y$.

3.1 FORS Orderings

All the orderings classified in the previous section are some simple integral orderings, that we called FORS orderings[3], where $B = [\mathbf{a}, \mathbf{b}] \subseteq \overline{\mathbb{R}}^n$, (with $-\infty \le a_i < b_i \le +\infty$, i=1,…,*n*) and are characterized by a particular consistency property.

Definition 1 *There exists a FORS ordering induced by an order of preferences $\succ$ on a class of admissible choices Λ if there exists a simple functional $\rho: U \times B \to \mathbb{R}$ (with $B = [\mathbf{a}, \mathbf{b}] \subseteq \overline{\mathbb{R}}^n$, $-\infty \le a_i < b_i \le +\infty$,* i=1,…,*n*) *that is consistent with the order of preferences $\succ$ on the class Λ, i.e., $\forall X, Y \in \Lambda$ $\rho_X \le \rho_Y$ anytime that X is preferred to Y with respect to the order of preferences $\succ$ ($X \succ Y$). In this case X dominates Y in the sense of FORS ordering induced by $\succ$ (namely $X \underset{\succ}{FORS} Y$) if and only if $\rho_X(u) \le \rho_Y(u) \quad \forall u \in B$. We call the simple functional ρ, the FORS functional (measure) associated with the FORS ordering of random variables belonging to Λ.*

In many financial applications it is important to distinguish some characteristics of investor behavior. This is the typical case of behavioral finance orderings where we assume

[3] FORS is an abbreviation from the initial of the surname of the authors of this paper. We will identify the names if the paper is accepted.

there exist m measurable subsets A_i of $B=[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^n$ such that $[\mathbf{a},\mathbf{b}]=\bigcup_{i=1,\dots,m} A_i$ and we consider m probability functionals $\rho_{(i)}:U\times A_i\to\mathbb{R}$ consistent with the order of preference $\succ$. Each probability functional $\rho_{(i)}$ characterizes a particular investor's behavior (see, among others, Levy and Levy (2002)). Suppose, for every $\mathbf{x}\in S=\left\{\mathbf{x}\in\mathbb{R}^m / x_i\geq 0, \sum_{i=1}^m x_i=1\right\}$ and $\forall\mathbf{t}\in[\mathbf{a},\mathbf{b}]$, $\rho_X(\mathbf{x},\mathbf{t}):=\sum_{i=1}^m x_i\rho_{X,(i)}(\mathbf{t})I_{A_i}(\mathbf{t})$ (where $I_{A_i}(\mathbf{t})=\begin{cases}1 \; if \;\; \mathbf{t}\in A_i\\ 0 \; otherwise\end{cases}$) is a simple probability functional on the class of random variables Λ. Then we say that X *dominates* Y in the sense of the *FORS behavioral finance ordering* induced by $\succ$ (*namely* $X \underset{\succ BF}{FORS} Y$) if and only if $\rho_X(\mathbf{x},\mathbf{t})\leq\rho_Y(\mathbf{x},\mathbf{t})$ for every $(\mathbf{x},\mathbf{t})\in S\times[\mathbf{a},\mathbf{b}]$.

Assume that μ is a positive sigma finite measure on $B=[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^n$ and $\forall u\in B$ $\rho_X(u)=\int_{[\mathbf{a},\mathbf{u}]} f_X\, d\mu$ is a FORS functional associated with a FORS ordering, i.e., $f_X=\frac{d\rho_X}{d\mu}$ is the simple functional derivate of ρ w.r.t. μ on B. Then if $f_X\leq f_Y$ μ a.e. on B implies $X \underset{\succ}{FORS} Y$. In particular we can introduce another ordering on the class of random variables Λ that we call μ *derivate order of* ρ. That is we say that X dominates Y in the sense of the μ derivate order of ρ (*namely* $X \; D_\mu \underset{\succ}{FORS} Y$) if and only if $f_X\leq f_Y$ μ a.e. on B. A μ derivate order of the FORS ordering induced by an order of preferences $\succ$ is not a priori induced by the same order of preferences $\succ$, i.e., it is not a priori a FORS ordering.

As for integral orderings we can extend the above definition to conditional orderings and orderings for stochastic processes. Thus, let us assume a *conditional simple functional* over the class Λ with respect to the class of conditional sets C is consistent with the order of preferences $\succ$ on Λ, i.e., $\forall X,Y\in\Lambda$, $\forall D\in C$ s.t. $P(X\in D)>0$ and $P(Y\in D)>0$ $\rho_{[X/X\in D]}\leq\rho_{[Y/Y\in D]}$ anytime $X\succ Y$. Then, we say that X dominates Y in the sense of the conditional FORS ordering induced by $\succ$ on the class of conditional sets C (namely $X \; C-\underset{\succ}{FORS} \; Y$) if and only if $\forall D\in C$ s.t. $P(X\in D)>0$ and $P(Y\in D)>0$ $\rho_{[X/X\in D]}(u)\leq\rho_{[Y/Y\in D]}(u)$ $\forall u\in B$. Analogously for stochastic processes, let us assume a simple functional $\rho:\tilde{U}\times B\to\mathbb{R}$ (i.e. $B=[0,T]\times[\mathbf{a},\mathbf{b}]$ with $[\mathbf{a},\mathbf{b}]\subseteq\overline{\mathbb{R}}^n$ and, $\forall t\in[0,T]$ $\rho_X(t,\bullet)=\rho_Y(t,\bullet)$ almost everywhere if and only if $F_{X_t}=F_{Y_t}$, $\forall t\in[0,T]$) that is consistent with the order of preferences $\succ$ on the class of stochastic processes $\tilde{\Lambda}$, i.e., $\forall X, Y \in \tilde{\Lambda}$ $\rho_X \le \rho_Y$ anytime $X \succ Y$. Thus, we say that *X* dominates *Y* in the sense of dynamic FORS ordering induced by $\succ$ (namely $X \underset{dyn\succ}{FORS} Y$) if and only if $\rho_X(t,u) \le \rho_Y(t,u)$ $\forall (t,u) \in B = [0,T] \times [\mathbf{a}, \mathbf{b}]$.

Clearly given an order of preferences $\succ$ it could happen that many possible FORS orderings can be induced by $\succ$. In this case only the simple FORS functional ρ associated with the FORS ordering permits to identify the ordering. In the following we report some examples of FORS orderings:

- Consider the simple functional $m_X^{(\pm\alpha)}(x,y) = x m_X^{(\pm\alpha)}(y) + (1-x)\tilde{m}_X^{(\pm\alpha)}(y)$ that is consistent with $\pm\alpha$ Markowitz' order. In this case, since $m_X^{(\pm\alpha)}(x,y) \le m_Y^{(\pm\alpha)}(x,y)$, $\forall (x,y) \in [0,1] \times (-\infty, c]$ if and only if $X \underset{\pm\alpha}{M} SD\ Y$, then $\pm\alpha$ Markowitz' order is a particular FORS behavioral finance order like the prospect ordering we have seen in the previous section.
- Typical examples of conditional FORS orderings are the *hazard rate order* and the *likelihood ratio order* which are often used in queueing theory, reliability theory (see Kashnikov and Rachev (1988) and Shaked and Shanthikumar (1994)) and in credit risk modeling. Given two random variables *X* and *Y* with absolutely continuous distribution functions, we say that *X* dominates *Y* with respect to the hazard rate order ($X \underset{hr}{\ge} Y$) if and only if the infinitesimal rate of failure at time *t* (also called *hazard rate*) of *X* is lower or equal than hazard rate of *Y* for any *t*, that is, $r_X(t) := \dfrac{f_X(t)}{1-F_X(t)} \le r_Y(t)$ $\forall t \in \mathbb{R}$, where f_X is the density of *X*. More generally given two real random variables we say $X \underset{hr}{\ge} Y$ if and only if $[X / X \in D] \underset{1}{\ge} [Y / Y \in D]$ for every D belonging to $C_{hr} = \{(a, +\infty) / a \in \mathbb{R}\}$, that is, the hazard rate order is a conditional FORS ordering with simple functional associated $\rho_{[X/X>c]}(t) = F_{[X/X>c]}(t)$ ($\forall c \in \mathbb{R}$, $P(X > c) > 0$, $B = \mathbb{R}$). Further generalizations of hazard rate order can be found in Chapter 4 Kashnikov and Rachev (1988). We say that *X* dominates *Y* with respect to the likelihood ratio order ($X \underset{lr}{\ge} Y$) if and only if $[X / X \in D] \underset{1}{\ge} [Y / Y \in D]$ for every *D* belonging to $C_{lr} = B_{\mathbb{R}}$, that is the likelihood ratio order is a conditional FORS ordering with simple functional associated $\rho_{[X/X \in D]}(t) = F_{[X/X \in D]}(t)$ ($\forall D \in B_{\mathbb{R}}$

$P(X \in D) > 0$, $B = \mathbb{R}$). Since $C_{lr} \supseteq C_{hr}$ and the simple functional on C_{lr} is an extension of the functional on C_{hr}, we get that the likelihood ratio order implies the hazard rate order.

- Suppose the benchmark Z admits a finite mean and for a given $\alpha \in (1,2)$, $Z \underset{-\alpha}{\geq} X$ for every X belonging to $\Lambda \subseteq L^1$ (with $\left| F_Z^{(-2)}(1) \right| < \infty$). Then the simple functional $\rho_X(u) = F_Z^{(-u)}(1) - F_X^{(-u)}(1) \quad \forall u \in [\alpha, 2] = B$ is consistent with α inverse stochastic order. So, we say that *X* dominates *Y* in the sense of this FORS ordering induced by $\underset{-\alpha}{\geq}$, if and only if $\rho_X(t) \leq \rho_Y(t) \quad \forall t \in [\alpha, 2]$.
- Suppose the benchmark $Z \in L^p$ (with $p > 1$) and for a given $\alpha \in [1, p)$ $Z \underset{\alpha}{\geq} X$ for every *X* belonging to $\Lambda \subseteq L^p$. Then the simple functional $\rho_X(t) = \int_{-\infty}^{t} \left(F_X^{(\alpha)}(u) - F_Z^{(\alpha)}(u) \right)^q du$ $\forall t \in \mathbb{R} = B$ (for some given $q \geq 1$) is consistent with the α stochastic order and, we say that *X* dominates *Y* in the sense of this FORS ordering induced by $\underset{\alpha}{\geq}$, if and only if $\rho_X(t) \leq \rho_Y(t) \quad \forall t \in \mathbb{R}$.
- Similar examples can be obtained for dynamic FORS orderings. For example, suppose the benchmark Z satisfies that for every $t \in [0, T]$ and for a given $\alpha > 1$, $Z_t \underset{\alpha}{\geq} X_t$ for every $X = \{X_t\}_{0 \leq t \leq T}$ belonging to $\tilde{\Lambda}$. Then, for some given $q \geq 1$, we can consider the following simple functional $\rho_X(t,s) = \int_{-\infty}^{s} \left(F_{X_t}^{(\alpha)}(u) - F_{Z_t}^{(\alpha)}(u) \right)^q du \quad \forall (t,s) \in [0,T] \times \mathbb{R} = B$ that is consistent with the order $\underset{dyn\ \alpha}{\geq}$ (i.e., $X \succ Y$ ($X, Y \in \tilde{\Lambda}$) if and only if $\forall t \in [0, T]$, $X_t \underset{\alpha}{\geq} Y_t$, namely $X \underset{dyn\ \alpha}{\geq} Y$). So we say that *X* dominates *Y* in the sense of this dynamic FORS ordering induced by $\underset{dyn\ \alpha}{\geq}$, if and only if $\rho_X(t,s) \leq \rho_Y(t,s) \quad \forall (t,s) \in B$.

Most of the discussions and extensions done for FORS orderings for random variables are also valid for conditional FORS orderings and dynamic FORS orderings for stochastic processes. Indeed, conditional FORS orderings refer to FORS orderings for conditional random variables and dynamic FORS orderings refer to FORS orderings for random variables components of stochastic processes being fixed at each time. So, we can pay much more attention to "static, unconditional" FORS orderings among random variables. In particular, when $B = [a,b] \subseteq \overline{\mathbb{R}}$ (with $-\infty \leq a < b \leq +\infty$) we can repeat almost all the previous classification for FORS orderings. Hence, in the following we assume $[a,b] \subseteq \overline{\mathbb{R}}$. Let ρ be a

FORS functional associated with the FORS ordering (induced by an order of preferences $\succ$) of random variables belonging to Λ. Then for every $\alpha \geq 1$, $\forall X, Y \in \Lambda_{(\alpha)} = \left\{ X \in \Lambda : \left| \int_a^b |t|^{\alpha-1} d\rho_X(t) \right| < \infty \right\}$ we say that *X* dominates *Y* in the sense of the α-FORS ordering induced by $\succ$ (namely $X \underset{\succ,\alpha}{FORS} Y$) if and only if $\rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u)$ $\forall u \in [a,b]$ where $\rho_{X,\alpha}(u) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^u (u-t)^{\alpha-1} d\rho_X(t) & if\ \alpha > 1 \\ \rho_X(u) & if\ \alpha = 1 \end{cases}$. That is the FORS ordering implies the existence of a sub-FORS ordering with FORS functional associated $\rho_{(\alpha)} : \Lambda_{(\alpha)} \times V \times [a,b] \to \mathbb{R}$ where $\rho_{X,(\alpha)} := \rho_{X,\alpha}$. We also say that ρ (ρ_X) generates the FORS ordering.

When the order of preferences $\succ$ is isotonic with the monotony order (i.e. $X \succ Y$ if $X>Y$) any FORS ordering induced by $\succ$ is called a *FORS risk ordering*. When $X \underset{\succ,\alpha}{FORS} Y$, $-X \underset{\succ,\alpha}{FORS} -Y$ and the order of preferences $\succ$ is isotonic with the monotony order, we say that *X dominates Y in the sense of* α *FORS uncertainty ordering* induced by the risk order of preferences $\succ$ (namely $X \underset{\succ,unc\ \alpha}{FORS} Y$). These orderings are generally called *FORS uncertainty orderings* and $X \underset{\succ,unc\ \alpha}{FORS} Y$ if and only if

$$\rho_{X,\alpha}(x,t) := x\rho_{X,\alpha}(t) + (1-x)\rho_{-X,\alpha}(t) \leq \rho_{Y,\alpha}(x,t),\ \forall (x,t) \in [0,1] \times [a,b].$$

In particular many of the extensions defined for stochastic dominance orders can be applied to any FORS risk ordering induced by $\succ$. So, when $\forall X \in \Lambda$ the FORS functional ρ_X is monotone, the left inverse of ρ_X is given by $\rho_X^{-1}(x) = \inf \left\{ u \in [a,b] : \rho_X(u) \geq x \right\}$ for any *x* belonging to the range of ρ_X. Thus, we can describe inverse, survival, uncertainty, bounded/unbounded FORS orderings and most of the implications for stochastic dominance orders are still valid (Ortobellli et al (2006a)).

Remark 1 *Given a FORS ordering on a class of random variables* Λ *and its associated FORS measure* ρ_X, *the following implications and definitions hold:*

1) *For every* $\alpha > v \geq 1$ $X \underset{\succ,v}{FORS} Y$ *implies* $X \underset{\succ,\alpha}{FORS} Y$ *and we can write*

$$\rho_{X,\alpha}(t) = \frac{1}{\Gamma(\alpha - v)} \int_a^t (t-u)^{\alpha - v - 1} \rho_{X,v}(u) du \quad \forall \alpha > v \geq 1.$$

That is $\forall \alpha \geq 1$ *the simple functional* $\rho_{(\alpha)}$ *is the Lebesgue derivate order of* $\rho_{(\alpha+1)}$ *, since* $\forall \beta > 1$, $\forall X \in \Lambda_{(\beta)}$, $\rho_{X,\beta}$ *is absolutely continuous w.r.t. the Lebesgue measure.*

2) *We can define a survival ordering using the survival FORS functional* $\overline{\rho}_{X,\alpha}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} d\rho_X(u) & \text{if } \alpha > 1 \\ -\rho_X(t) & \text{if } \alpha = 1 \end{cases}$ *associated with the FORS ordering induced by the order of preferences* $\succ$. *Thus, we say that X dominates Y in the sense of* α*-FORS survival ordering induced by* $\succ$ (*namely* $X \underset{\succ, sur\ \alpha}{FORS} Y$) *iff* $\overline{\rho}_{X,\alpha}(t) \leq \overline{\rho}_{Y,\alpha}(t)$ *for every t belonging to* $[a,b]$. *Using the properties of the fractional integral we can write* $\overline{\rho}_{X,\alpha}(t) = \frac{1}{\Gamma(\alpha-v)} \int_t^b (u-t)^{\alpha-v-1} \overline{\rho}_{X,v}(u) du \quad \forall \alpha > v \geq 1$. *However, in this case we cannot generally say that the results obtained for survival orders are equivalent to those obtained for orders applied to the opposite random variables.*

3) *When the survival ordering is induced by an order of preferences* $\succ$ *isotonic with the monotony order we say that X dominates Y in the sense of* α *FORS uncertainty survival ordering induced by the order of preferences* $\succ$ (*namely* $X \underset{\succ, unc-sur\ \alpha}{FORS} Y$) *if and only if* $X \underset{\succ, sur\ \alpha}{FORS} Y$ *and* $X \underset{\succ, \alpha}{FORS} Y$, *if and only if* $\overline{\rho}_{X,\alpha}(x,t) := x\rho_{X,\alpha}(t) + (1-x)\overline{\rho}_{X,\alpha}(t) \leq \overline{\rho}_{Y,\alpha}(x,t)$ $\forall (x,t) \in [0,1] \times [a,b]$.

4) *Suppose* $|\rho_X(b)| < \infty$, $|\rho_X(a)| < \infty$ *for every X belonging to* Λ. *Then we can extend* ρ_X *on all the real line* $\mathbb{R}$ *by assuming* $\rho_X(u) = \rho_X(a)\ \forall u \leq a$, $\rho_X(u) = \rho_X(b)\ \forall u \geq b$. *We say X unbounded* $\underset{\succ, \alpha}{FORS}$ *dominates Y iff* $\rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u)$ *for every* $u \in \mathbb{R}$ *where, with abuse of notation, we still use the same notation* $\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^u (u-t)^{\alpha-1} d\rho_X(t)\ \forall u \in \mathbb{R}$. *If* $\forall X \in \Lambda$, ρ_X *is monotone, then the unbounded* $\underset{\succ, \alpha}{FORS}$ *order implies bounded* $\underset{\succ, \alpha}{FORS}$ *order.*

5) *For any monotone increasing FORS measure* ρ_X *associated with a FORS ordering, the left inverse* $\rho_X^{-1}(x)$ *is a FORS reward measure (for any point x) and* $-\rho_X^{-1}$ *generates itself a FORS ordering induced by* $\succ$.

FORS orderings are particular integral orderings, since any FORS orderings can be represented in terms of utility functions. As a matter of fact, suppose ρ_X is a FORS measure associated with a FORS ordering $\succ$ on a given class of random variables Λ. Then, given

$X,Y \in \Lambda_{(\alpha)}$, $X \underset{\succ,\alpha}{FORS} Y$ if and only if $\int_a^b \phi(u) d\rho_X(u) \geq \int_a^b \phi(u) d\rho_Y(u)$ for every ϕ belonging to

$$W^{\alpha} = \Big\{ \phi(x) = -\int_{x^+}^{b} (s-x)^{\alpha-1} d\tau(s) - k(b-x)^{\alpha-1} \Big| k \geq 0,\ k=0 \text{ if } b=\infty;$$
$$\tau \text{ is a } \sigma-\text{finite positive measure s.t. } \forall X \in \Lambda_{(\alpha)} \text{ the}$$
$$\text{function } |s-x|^{\alpha-1} \text{ is } d\tau(s) \times d\rho_X(x) \text{ integrable in } [a,b]\times[a,b] \Big\}.$$

Using the properties of the fractional integral, we can generate different levels of orderings every time we have a bounded and monotone FORS functional $\rho_X^{(1)} : [a,b] \to \mathbb{R}$ (with $|b| < +\infty$) associated with a FORS ordering defined on a class of random variables Λ (that we call FORS1 or simply $\succ$). As a matter of fact, $h_X(u) = \Gamma(u-1)\rho_{X,u}^{(1)}(t)$, $\forall u > 1, \forall t \in [a,b]$ is the Mellin transform (valued on the real line) of $\rho_X^{(1)}(t-x) I_{[0,t-a]}(x)$. Thus $\rho_X^{(2)} : [1, p_1] \to \mathbb{R}$ defined by $\rho_X^{(2)}(u) = \rho_{X,u}^{(1)}(b)$ is a simple functional on the class of random variables $\Lambda_{p_1} = \left\{ X \in \Lambda / p_1 > 1 : \left| \rho_{X,p_1}^{(1)}(b) \right| < +\infty \right\}$ (see Ortobelli et al. (2006a) for further details). The functional $\rho_X^{(2)}$ is a FORS2 measure (induced by FORS1 ordering) associated with the following second level of FORS ordering (namely FORS2) defined $\forall X,Y \in \Lambda_{p_1,(\alpha)} = \left\{ Z \in \Lambda_{p_1} : \left| \int_1^{p_1} u^{\alpha-1} d\rho_X^{(2)}(u) \right| < \infty \right\}$, and $\forall \alpha \geq 1$. So, we say that X dominates Y in the sense of a second level α-FORS ordering (namely $X \underset{\succ,\alpha}{FORS2} Y$) if and only if $\rho_{X,\alpha}^{(2)}(u) \leq \rho_{Y,\alpha}^{(2)}(u)$ $\forall u \in [1, p_1]$. Clearly the definition can be extended recursively. As a matter of fact, for *k* = 2,…, *n*, we can consider the FORS*k* measure $\rho_X^{(k)} : [1, p_{k-1}] \to \mathbb{R}$ defined by $\rho_X^{(k)}(u) = \rho_{X,u}^{(k-1)}(p_{k-2})$ (where $p_0 = b$) that is associated with a *k*-th level of ordering, namely FORS*k*, on the class of random variables $\Lambda_{p_{k-1}} = \left\{ X \in \Lambda_{p_{k-2}} / p_{k-1} > 1 : \left| \rho_{X,p_{k-1}}^{(k-1)}(p_{k-2}) \right| < +\infty \right\}$ (where $\Lambda_{p_0} = \Lambda$). Let us consider the following example of second level of α-FORS ordering applied to the dual bounded order:

Example: A second level of FORS ordering can be defined on the class $\Lambda = \left\{ X / \forall \lambda \in [m,n] : \left| F_X^{(-\lambda)}(1) \right| < +\infty \right\}$ assuming the simple functional $_m\rho_X(\lambda) = -\Gamma(\lambda) F_X^{(-\lambda)}(1)$ for every $\lambda \in [m,n]$ ($m > 1$), i.e., the simple functional $_m\rho_X$ is a FORS measure induced by the *m* inverse stochastic dominance order $\underset{-m}{>}$. Thus for every

$m > 1$, the following simple functional ${}_m\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_m^u (u-s)^{\alpha-1} d_m\rho_X(s) \quad \forall u \geq m$ identifies a FORS ordering induced by the order $\underset{-m}{>}$ (that is itself a FORS ordering).

3.2 Basic Properties to Build FORS Orderings Starting with Some Risk and Uncertainty Measures

Most of portfolio theory is based on minimizing a distance from a benchmark or minimizing potential possible losses while keeping constant some portfolio characteristics. Clearly the admissible optimal choices must not be dominated with respect to the order of preferences. Thus, we define as *efficient for a given class of market agents* all the admissible choices that cannot be preferred (dominated) by all the agents of the same class. These preferences are represented by some "probability functionals" that summarize the behavior of market agents. Therefore, the characterization of these probability functionals serves to identify implicitly the investors' preferences. In particular, we require that a probability functional $\mu : U \to \mathbb{R}$ (where we assume the product space $U = \Lambda \times V$ of joint real valued random variables with the space of benchmarks $V \supseteq \Lambda$) satisfies two properties: the i*dentity property* (that is imported from the theory of probability metrics (see Rachev (1991))), and the *consistency property* (that is imported from the theory of orderings (see Shaked and Shanthikumar (1994), Müller and Stoyan (2002)). We introduce these properties in a static framework, but clearly they can be opportunely extended in a dynamic context.

On the consistency property: In terms of probability functionals $\mu : U \to \mathbb{R}$, consistency is defined as:

X dominates Y with respect to a given order of preferences $\succ$ *implies* $\mu(X,Z) \leq \mu(Y,Z)$ *for a fixed arbitrary benchmark* $Z \in V$.

Any probability functional consistent with a given order of preferences $\succ$ is defined as *FORS measure induced by* (*consistent with*) *order* $\succ$. When the probability functional is consistent with an order of preferences $\succ$ (typically an α-FORS ordering) that characterizes (with a parameter α) a given category of market agents, we define the functional α*-FORS measure induced by* (*consistent with*) *preference order* $\succ$. As for ordering theory, there exist two types of FORS *measures*:

• Measures of *risk* (tails, losses) that are induced by an order of preference isotonic with the monotony order, such as $\underset{\pm\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$. We call them *FORS risk measures*. (If we multiply these measures by (-1) we obtain *FORS reward measures*);

• Measures of *uncertainty* that are induced by FORS uncertainty orderings. We call them *FORS uncertainty measures*.

Since in the definition of consistency no rule relative to the benchmark *Z* is prescribed and the space *U* could be assumed equal to Λ, then all the risk/uncertainty measures $\omega:\Lambda\to\mathbb{R}$ can be considered to be a subclass of probability functionals consistent with an order of preferences. Classical examples used in the recent literature (see Atzner et al (1999), Föllmer and Sheid (2002), Rachev et al (2007)) are the convex/coherent risk measures which are consistent with the monotony order. Analogously we can better specify the consistency of convex/coherent FORS risk measures considering its consistency with an α-FORS risk order. So we call *convex* α*-FORS measure* any convex (i.e., $\omega(aX+(1-a)Y)\le a\omega(X)+(1-a)\omega(Y)$, $\forall X,Y\in\Lambda$, $\forall a\in[0,1]$), translation invariant (i.e., $\omega(X+m)=\omega(X)-m$, $\forall X\in\Lambda$ and $m\in\mathbb{R}$) probability functional ω that is consistent with an α-FORS risk order, and we call *coherent* α*-FORS measure* any convex α-FORS measure ω that is positive homogeneous (i.e., $\forall k\ge 0$ $\forall X\in\Lambda$, $\omega(kX)=k\omega(X)$).

Let us consider the following examples:

- Suppose $U=\Lambda$ (i.e., we do not consider the benchmark). Then some examples of FORS measures are given by :

  a) for any fixed $\alpha\ge 1$ and $\forall\beta\in(0,1)$, $\dfrac{-\Gamma(\alpha+1)}{\beta^{\alpha}}F_X^{(-(\alpha+1))}(\beta)$ is a coherent (α+1)-FORS measure consistent with $\underset{-(\alpha+1)}{\ge}$ order;

  b) for any fixed $\alpha>1,\forall p\in\mathbb{R}$ $F_X^{(\alpha)}(p)$; is an α-FORS risk measure consistent with $\underset{\alpha}{\ge}$ order;

  c) for any fixed $\alpha\ge 2$, $\forall(x,t)\in[0,1]\times[0,1]$ $F_X^{(-\alpha)}(x,t)$ is an α-FORS uncertainty measure consistent with inverse α R-S order.

- Suppose $Z\in V$, $U=\Lambda\times V$ and for a given $\alpha\ge 1$, $Z\underset{-\alpha}{\ge}X$ for every *X* belonging to Λ. Then, the measure $\mu(X,Z)=\int_0^t\left(F_Z^{(-\alpha)}(u)-F_X^{(-\alpha)}(u)\right)^q du$ (for some fixed $q\ge 1$, $\forall t\in[0,1]$) is a *FORS risk measure induced by* $\underset{-\alpha}{\ge}$ order.

- Suppose for a given $\alpha \geq 1$, the benchmark $Z \underset{\alpha}{\geq} X$ for every X belonging to $\Lambda \subset L^{\infty}$ (with $b = \max_{X \in \Lambda} \left| ess \sup(\max(X, Z)) \right| < \infty$). Then $\forall u \geq \alpha$ the measure $F_X^{(u)}(b) - F_Z^{(u)}(b)$ is a *FORS risk measure induced by* $\underset{\alpha}{\geq}$ order.

On the identity property: We say that, for any pair of random variables X, Y belonging to $\Lambda \subseteq V$, the probability functional $\mu : \Lambda \times V \to \mathbb{R}$ satisfies an identity property when:

$f(X) = f(Y) \Leftrightarrow \mu(X, Y) = 0$*; where f(X) identifies a set of probabilistic characteristics of the random variable X.*

We distinguish between three main groups of probability functionals depending on certain modifications of the identity property (see Rachev (1991)): primary, simple, and compound. *Primary probability functionals* determine some random variable characteristics. Among many possible primary probability functionals two examples are:

- $\mu(X, Y) = \left| \int_0^p F_X^{(-1)}(u) du - \int_0^p F_Y^{(-1)}(u) du \right|$ that is equal to zero if and only if X and Y present the same conditional value at risk $CVaR_p(X) := \frac{-1}{p} \int_0^p F_X^{(-1)}(u) du$ for a given $p \in (0,1)$;
- $\mu(X, Y) = \left| E(\left| X - E(X) \right|^p) - E(\left| Y - E(Y) \right|^p) \right|$ that is equal to zero if and only if X and Y present the same p-th absolute central moment.

Simple probability functionals identify the distribution (i.e., for any pair of random variables $X, Y \in \Lambda$: $\mu(X, Y) = 0 \Leftrightarrow F_X = F_Y$). Examples of simple probability functionals are the Kolmogorov metric $KS(X, Y) = \sup_{t \in R} \left| F_X(t) - F_Y(t) \right|$ and the Gini-Kantorovich metrics $GK_p(X, Y) = \left[\int_{-\infty}^{+\infty} \left| F_X(t) - F_Y(t) \right|^p dt \right]^{\min(1, 1/p)}$ ($p \geq 0$). *Compound probability functionals* identify the random variable almost surely (i.e., for any pair of random variables $X, Y \in \Lambda : \mu(X, Y) = 0 \Leftrightarrow P(X = Y) = 1$). Examples of compound probability functionals are (for a given $p \geq 0$) the L^p-metrics $\mu_p(X, Y) = E\left(\left| X - Y \right|^p \right)^{\min(1, 1/p)}$ and the Birnbaum-Orlicz metrics $\Theta_p(X, Y) = \left(\int_{-\infty}^{+\infty} \left(P(X \leq t < Y) + P(Y \leq t < X) \right)^p dt \right)^{\min(1, 1/p)}$.

Clearly the identity property can be extended to functional $\rho : U \times M_B \to \mathbb{R}$ (or $\rho : U \times B \to \mathbb{R}$ where (B, M_B) is an opportune measurable space) in order to obtain simple functionals (we gave previously some examples) or primary functionals that identify the distributional tail behavior. Combining the identity property with consistency and using the tools derived from the theory of probability metrics, we can build several FORS measures and orderings. Let us consider the following constructive examples:

- For any fixed $\beta \in (0,1)$, $h_X(v) = \dfrac{-\Gamma(v)}{\beta^v} F_X^{(-v)}(\beta)$ is a primary probability functional

over the class of random variables $\Lambda = \left\{ X / p > 1 : \left| F_X^{(-p)}(\beta) \right| < +\infty \right\}$ and $1 \le v < p$. In addition, $h_X(v)$ is a FORS measure induced by $\underset{-m}{>}$ for every for every $v \in [m, p]$ and a given $m \in [1, p)$, that is coherent when $p > v \ge 2$. Then for every $\alpha \ge 1$, the measure ${}_m h_{X,\alpha}(u) = \dfrac{1}{\Gamma(\alpha)} \int_m^u (u-s)^{\alpha-1} dh_X(s)$ for every $u \in [m, p]$ is a FORS measure induced by $\underset{-m}{>}$ that identifies the distribution of the tail (i.e., the measure ${}_m h_{X,\alpha}(u) = {}_m h_{Y,\alpha}(u)$ for every $u \in [m, p]$ for a given $\alpha \ge 1 \Leftrightarrow F_X^{-1}(q) = F_Y^{-1}(q) \;\; \forall q \le \beta$).

- Suppose the benchmark $Z \in L^p$ (with $p > 1$) and $Z \underset{1}{\ge} X$ for every X belonging to $\Lambda \subseteq L^p$. Then $\rho_X(q) = \int_0^1 \left(F_Z^{-1}(u) - F_X^{-1}(u) \right)^q du \quad \forall q \in [1, p]$ is a simple functional. As a matter of fact, $\rho_X(q) = E_{\tilde{F}} \left(\left(\tilde{Z} - \tilde{X} \right)^q \right)$ where $\tilde{X} = F_X^{-1}(U) \le \tilde{Z} = F_Z^{-1}(U)$ and U is uniformly distributed (0,1), $\tilde{F}(x,z) = \min\left(F_X(x), F_Z(z) \right)$ is the Hoeffding-Frechet bound of the class of all bivariate distribution functions with marginals F_X and F_Z (see Cambanis et al. (1976), Rachev (1991)). Then $\rho_X(q)$, $\forall q \in [1, p]$ is a simple functional, since the moments curve identify univocally the distribution of $\tilde{Z} - \tilde{X}$. Moreover $\forall q \in [1, p]$, $\rho_X(q)$ is a FORS measure induced by the first stochastic dominance order. Thus ρ_X is a FORS measure associated with the new FORS ordering induced $\underset{1}{\ge}$, i.e., $X \; FORS_{\underset{1}{\ge}} \; Y$ if and only if $\rho_X(u) \le \rho_Y(u) \;\; \forall u \in [1, p]$.

[INSERT HERE TABLE 1]

Table 1 summarizes the main FORS measures/orderings classified in the static and dynamic context.

3.3 Efficient Frontier

The previous analysis summarized the main orderings and measures used in the recent literature. In particular, we presented a constructive methodology to build orderings and probability functionals consistent with the preferences of different investors' categories. This feature is fundamental in portfolio theory when we want to select the set of the choices that are not dominated by a given FORS ordering, the so called *efficient frontier*. As a matter of fact, given a FORS measure $\rho : U \times B \to \mathbb{R}$ associated with a given FORS ordering induced by an order of preferences $\succ$, we obtain the set of efficient choices with respect to the FORS ordering by solving the following optimization problem:

$$\min_{X \in \Lambda} \rho_X(t) \quad s.t. \;\; t \in B \tag{1}$$

Clearly, the efficient choices solution to problem (1) are a sub class of the optimal choices with respect to the order of preferences $\succ$.

For example, Rachev et al (2007) have shown that, even in periods where there are major crises in the market, the "aggressive-coherent" behavior of the non-satiable investor who are neither risk averse nor risk loving (those who maximize an aggressive coherent utility functional) allows the investor to increase final wealth much more than by adopting "conservative strategies". Aggressive coherent utility functionals are isotonic with the monotony order. Typical examples are the utility functionals written as $a\rho_{-X} - b\hat{\rho}_X$, where a, b>0 and ρ_X $\hat{\rho}_X$ are two coherent risk measures.. Two procedural steps are necessary to find the efficient choices with respect to an aggressive-coherent ordering:

a) build a FORS ordering using an aggressive coherent utility functional;
b) solving the optimization problem (1).

Thus, for example, consider the Gini-type measure $GT_{(\beta,v)}(X) = \frac{-v}{\beta^v}\int_0^\beta (\beta-u)^{v-1} F_X^{-1}(u)du$ that for every $v \geq 1$ and for every $\beta \in (0,1)$ is a linearizable coherent risk measure (see Ortobelli *et al*. (2006b)) and it is consistent with $\underset{-(v+1)}{\geq}$ order. Using Gini-type measures we can essentially build two different types of FORS orderings. Thus, these three FORS orderings (induced by $\underset{-(\min(v,s)+1)}{\geq}$ order) have FORS measures associated:

1) $\rho_X(t) = GT_{(m,s)}(X) - GT_{(t,v)}(-X) \;\; \forall t \in (0,1) = B$;

2) $\rho_X(t) = GT_{(t,v)}(X) - GT_{(m,s)}(-X) \;\; \forall t \in (0,1) = B$;

3) $\rho_X(t,w) = GT_{(t,v)}(X) - GT_{(w,s)}(-X) \;\; \forall (t,w) \in (0,1)\times(0,1) = B$,

for some fixed $s, v \geq 1$ and $m \in (0,1)$.

With the first ordering we vary the aggressive component, with the second one we vary the coherent component, and the third ordering implies the first two. In the first ordering the efficient choices vary with respect to the investor’s aggressiveness (taking into account how much investors could be more or less risk lover), with the second ordering optimal choices vary with respect to the investor’s aversion to risk. Moreover, using the same techniques developed by Ortobelli et al (2006b), we can easily transform the optimization problem (1) based on the previous simple FORS functionals into linear optimization problems.

4. Concluding Remarks

This paper has a twofold objective: to classify the main orderings used in financial economics literature and to propose a methodology to determine the optimal choices consistent with investors’ preferences. The classification should help to better discriminate among several different orderings. The methodology describes essentially three procedural steps once the investor’s criteria of choice are known: (1) determine a simple functional consistent with the investor’s criteria of choice (and/or preferences); (2) build the ordering and the sub-orderings associated to the simple functional; and, (3) compute the class (and sub-class) of optimal choices minimizing the functional among the admissible choices.

Clearly, many new issues arise from this paper. In particular, although we can better specify the optimization portfolio problem by taking into account the investor's attitude toward risk, we need to understand how to value the effect of different probability functionals. By studying the theoretical characteristics of the different statistics that specify probability functionals, we can discriminate among “ideal” optimization problems. In that case, we can determine what is the best opportunity from the perspective of different market agents. However, we still need to provide a theory that permits testing the efficiency of choices with respect to any ordering.

Moreover, we can distinguish between several different types of human behavior when we introduce the new "time" variable. In particular, one could expect that different measurements of the same payoff at different dates should be related in some way. For this reason, different inter-temporal properties of a dynamic risk (uncertainty) measure must be properly defined. Clearly, most of the work of providing new models and illustrating their use coherently to the theory developed in this paper must still be done and should be the subject of future research.

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Table 1

This table summarizes the main classification of FORS orderings and measures. We distinguish among simple, primary, and compound probability functionals; between static and dynamic orderings; between uncertainty and risk orderings/measures; between orderings and survival/inverse/conditional/derivate orderings; between bounded and unbounded orderings, among behavioral finance orderings and among different levels of orderings.

<table>
<tr><td colspan="3"></td><td>Identity Property</td><td>Consistency property</td></tr>
<tr><td rowspan="3">FORS MEASURES (Risk and Reward)</td><td colspan="2">Random Variables</td><td rowspan="2">Primary
Simple
Compound</td><td rowspan="2"></td></tr>
<tr><td colspan="2">Stochastic Processes</td></tr>
<tr><td colspan="3">α-FORS Risk Measures</td><td>α-FORS Uncertainty Measures</td></tr>
<tr><td colspan="5"></td></tr>
<tr><td colspan="2" rowspan="9">FORS ORDERINGS</td><td colspan="2">Static FORS ordering</td><td>Dynamic FORS ordering</td></tr>
<tr><td colspan="2">FORS ordering</td><td>Inverse FORS ordering</td></tr>
<tr><td colspan="2">FORS ordering</td><td>Survival FORS ordering</td></tr>
<tr><td colspan="2">Bounded FORS ordering</td><td>Unbounded FORS ordering</td></tr>
<tr><td colspan="2">Risk FORS ordering</td><td>Uncertainty FORS ordering</td></tr>
<tr><td colspan="2">FORS ordering</td><td>Conditional FORS ordering</td></tr>
<tr><td colspan="2">FORS ordering</td><td>μ derivate ordering</td></tr>
<tr><td colspan="3">FORS*k* ordering (k-th level of FORS ordering)</td></tr>
<tr><td colspan="3">FORS behavioral finance ordering</td></tr>
</table>