The Modified Tempered Stable Distribution, GARCH Models and Option Pricing

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Abstract

We first introduce a new variant of the tempered stable distribution, named the modified tempered stable(MTS) distribution and we use it to develop the GARCH option pricing model with the MTS innovations. This model allows the description of some stylized phenomena in the empirical observation of financial markets such as volatility clustering, skewness and heavy tails of the return distribution. Secondly, we estimate the model parameters with S&P 500 index data and we use the parameters to calculate the option prices with Monte Carlo simulation. Finally, we compare the model prices to the market prices.

Keywords: Option pricing, GARCH process, Tempered stable distribution, Volatility clustering

1 Introduction

Since Black and Scholes(1973) introduced the pricing and hedging theory for the option market, their model has been the most popular model for option pricing. However, the model which assumes homoskedasticity and the lognormality cannot explain stylized phenomena such as skewness, heavy tails and volatility clustering of the stock returns, which are observed in stock prices.

To explain the stylized phenomena, Mandelbrot (1963a,b) was the first to use a non-normal Lévy process as an asset price process. Hurst, Platen and Rachev (1999) used a model based on stable processes to price options. However, stable distributions have infinite moments of the second or higher order because of the heavy distributional tails. To have more adaptability, a new class of Lévy processes called the tempered stable (TS) process has been introduced under different names including: "truncated Lévy flight" (Koponen (1995)), "KoBoL" process (Boyarchenko and Levendorskiĭ (2000)) , and "CGMY" process (Carr et al. (2002)). In order to obtain a closed form solution of the European option price, Carr et al. (2002) used the generalized Fourier transform for the distribution of the stock price under the assumption of the Markov property. However, the Markov property is often rejected by the empirical evidence as in the case of volatility clustering, for example.

The GARCH option pricing models have been developed to price options under the assumption of volatility clustering. GARCH models of Duan (1995), Heston and Nandi (2000) are remarkable works on the non-Markovian structure of asset returns but the authors disregarded the conditional leptokurtosis and skewness of asset returns. Duan et al. (2004) enhanced the classical GARCH model by adding jumps to the innovation processes. Furthermore, Menn and Rachev (2005a,b) introduced an enhanced GARCH model with innovations which follow the smoothly truncated stable (STS) distribution, which has a finite variance and at the same time allows the conditional leptokurtosis and skewness.

In this paper, we introduce a variant of the tempered stable distributions, called a modified tempered stable (MTS) distribution and we apply it to the GARCH option pricing model.

The MTS distribution is obtained by taking an α-stable law with $\alpha \in (0,1)$ and multiplying the Lévy measure by a modified Bessel function of the second kind on each half of the real axis. It is infinitely divisible, has a closed form characteristic function and finite moments of all orders, and behaves asymptotically like the 2α-stable distribution near zero and like the α-TS distribution on the tails.

The GARCH option pricing model presented in this paper follows the method introduced by Menn and Rachev (2005a,b). However, instead of STS innovations, we assume the innovations of the classical GARCH model follow the MTS distribution with zero mean and unit variance. As in the case of the STS distribution, the MTS distribution is able to describe the leptokurtosis and skewness. In contrast to the STS distribution, its Laplace transform is analytic, so it is more tractable. Moreover, it is infinitely divisible and its characteristic function provides a concrete method to find the equivalent martingale measure by applying a general result of equivalence of measures for Lévy processes presented by Sato (1999).

The remainder of this paper is organized as follows: Section 2 introduces the MTS distribution and the GARCH option pricing model. The characteristic function, cumulant and asymptotic behavior of the MTS distribution are presented in the first subsection, followed by measure changes in the case of the MTS distributions. The GARCH model with the MTS innovations is introduced in the third subsection. Parameter estimation and comparison of model prices to market prices of options are presented in Section 3. Section 4 is a summary of our conclusions.

2 The Model

2.1 Tempered Stable Distributions

Before introducing the MTS distribution and the MTS-GARCH model, let us review the tempered stable distribution. It is well known that the α-stable distributions have infinite p-th moments for all $p > \alpha$. This is due to the fact that its Lévy density decays polynomially. Tempering of the tails with the exponential rate is one choice to ensure finite moments. The Tempered Stable (TS) distribution is obtained by taking a symmetric α-stable distribution and multiplying the Lévy measure with exponential functions on each half of the real axis. Indeed, it is defined in the following.

Definition 2.1. An infinitely divisible distribution is called a *tempered stable (TS) distribution* with parameter $(C_1, C_2, \lambda_+, \lambda_-, \alpha)$, or *$\alpha$-tempered stable ($\alpha$-TS)*, if its Lévy triplet (σ^2, ν, γ) is given by $\sigma = 0$, $\gamma \in \mathbb{R}$ and

$$\nu(dx) = \left(\frac{C_1 e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{C_2 e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0} \right) dx, \tag{2.1}$$

where $C_1, C_2, \lambda_+, \lambda_- > 0$ and $\alpha < 2$.

This process was first constructed by Koponen (1995) under the name truncated Lévy flights. In particular, if $C_1 = C_2 = C > 0$, then this distribution is called the CGMY distribution which has been used in Carr et al. (2002) for financial modeling.

In the above definition, λ_+ and λ_- give the tail decay rates, α describes the jumps near zero, and C_1 and C_2 determine the arrival rate of jumps for a given size.

The characteristic function $\phi_{TS}(u)$ of a tempered stable distribution is given by

$$\phi_{TS}(u) = \exp\left(iu\mu + C_1\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) + C_2\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha) \right), \tag{2.2}$$

for some $\mu \in \mathbb{R}$. Moreover, ϕ_{TS} can be extended to the region $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \lambda_+ \wedge \lambda_-\}$. Using the characteristic function, we obtain cumulants

$$c_m(X) = \frac{d^m}{du^m} \log \phi_{TS}(u) \Big|_{u=0}$$

of all orders. The proof can be found in Carr et al. (2002), Cont and Tankov (2004) and Kim (2005).

Proposition 2.2. *Let X be a tempered stable distributed random variable whose characteristic function is given by* (2.2)*. The cumulant $c_n(X)$ of X is given by*

$$c_n(X) = \Gamma(n-\alpha)C_1\lambda_+^{\alpha-n} + (-1)^n\Gamma(n-\alpha)C_2\lambda_-^{\alpha-n}, \quad \textit{for} \quad n \in \mathbb{N}, \quad n \geq 2,$$

and $c_1(X) = \mu + \Gamma(1-\alpha)C_1\lambda_+^{\alpha-1} - \Gamma(1-\alpha)C_2\lambda_-^{\alpha-1}$.

2.2 The Modified Tempered Stable Distributions

In this section, we introduce a variant of the tempered stable distribution named modified tempered stable (MTS) distribution. The MTS distribution is defined as follows:

Definition 2.3. An infinitely divisible distribution is said to be an *α-modified tempered stable (α-MTS) or modified tempered stable (MTS) distribution* if its Lévy triplet is given by

$$\begin{aligned}
\sigma^2 &= 0 \\
\nu(dx) &= C\left(\frac{{\lambda_+}^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda_+ x)}{x^{\alpha+\frac{1}{2}}}1_{x>0} + \frac{{\lambda_-}^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda_-|x|)}{|x|^{\alpha+\frac{1}{2}}}1_{x<0}\right)dx \\
\gamma &= \mu + C\left(\frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}\left(\lambda_+^{2\alpha-1}-\lambda_-^{2\alpha-1}\right) - \lambda_+^{\alpha-\frac{1}{2}}K_{\alpha-\frac{1}{2}}(\lambda_+) + \lambda_-^{\alpha-\frac{1}{2}}K_{\alpha-\frac{1}{2}}(\lambda_-)\right),
\end{aligned}$$

where $C>0$, $\lambda_+,\lambda_->0$, $\mu\in\mathbb{R}$, $\alpha\in(-\infty,1)\setminus\{\frac{1}{2}\}$ and $K_p(x)$ is the modified Bessel function of the second kind. We denote an MTS distributed random variable X by $X\sim MTS(\alpha,C,\lambda_+,\lambda_-,\mu)$. The Lévy measure $\nu(dx)$ is called the *MTS Lévy measure with parameter* $(\alpha,C,\lambda_+,\lambda_-)$.

The MTS distribution is obtained by taking a symmetric α-stable distribution by $\alpha\in(0,1)$ and multiplying the Lévy measure with $\sqrt{|x|}\lambda^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda|x|)$ on each half of the real axis. The measure can be extended to the case of $\alpha\le 0$. If $\alpha=\frac{1}{2}$ then γ may not defined so we remove it. The following result shows that $\nu(dx)$ is a Lévy measure.

Proposition 2.4. *Let ν be a Borel measure on $\mathbb{R}$ such that $\nu(0)=0$ and*

$$\nu(dx) = C\left(\frac{{\lambda_+}^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda_+ x)}{x^{\alpha+\frac{1}{2}}}1_{x>0} + \frac{{\lambda_-}^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda_-|x|)}{|x|^{\alpha+\frac{1}{2}}}1_{x<0}\right)dx, \tag{2.3}$$

where $C>0$, $\lambda_+,\lambda_->0$, and $\alpha<1$. Then the measure ν is a Lévy measure on $\mathbb{R}$.

Proof. It suffices to show that

$$\int_0^\infty (x^2\wedge 1)\frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}}dx<\infty.$$

We first note that

$$\int_0^\infty x^2\exp\left(-\frac{(\lambda x)^2}{4t}\right)dx = \frac{4t^{\frac{3}{2}}}{\lambda^3}\int_0^\infty y^{\frac{1}{2}}e^{-y}dy = \frac{4}{\lambda^3}t^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right) = \frac{2\sqrt{\pi}}{\lambda^3}t^{\frac{3}{2}}.$$

Hence we have

$$\begin{aligned}\int_0^\infty x^2 \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx &= \frac{1}{2}\left(\frac{\lambda}{2}\right)^{\alpha+\frac{1}{2}} \int_0^\infty \int_0^\infty x^2 \exp\left(-\frac{(\lambda x)^2}{4t}\right) dx\ e^{-t} t^{-(\alpha+\frac{3}{2})} dt \\ &= \frac{1}{2}\left(\frac{\lambda}{2}\right)^{\alpha+\frac{1}{2}} \frac{2\sqrt{\pi}}{\lambda^3} \int_0^\infty e^{-t} t^{-\alpha} dt \\ &= \frac{\lambda^{\alpha-\frac{5}{2}}\sqrt{\pi}}{2^{\alpha+\frac{1}{2}}} \Gamma(1-\alpha).\end{aligned}$$

Therefore, we have

$$\int_0^\infty (x^2 \wedge 1) \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx \le \int_0^\infty x^2 \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx < \infty.$$

□

The following result follows from (A.2) and (A.3) in the Appendix.

Proposition 2.5. *Let*

$$f(x) = C\left(\frac{{\lambda_+}^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_+ x)}{x^{\alpha+\frac{1}{2}}} 1_{x>0} + \frac{{\lambda_-}^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_- |x|)}{|x|^{\alpha+\frac{1}{2}}} 1_{x<0}\right),$$

where $C > 0$, $\lambda_+, \lambda_- > 0$ *and* $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$. *Then*

$$f(x) \sim 2^{\alpha-\frac{1}{2}} C\Gamma\left(\alpha + \frac{1}{2}\right) \frac{1}{x^{2\alpha+1}}, \quad \text{as } x \to 0, \tag{2.4}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}} C\lambda_+^\alpha \frac{e^{-\lambda_+ x}}{x^{\alpha+1}}, \quad \text{as } x \to \infty, \tag{2.5}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}} C\lambda_-^\alpha \frac{e^{-\lambda_- |x|}}{|x|^{\alpha+1}}, \quad \text{as } x \to -\infty. \tag{2.6}$$

Remark 2.6. *If* $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$, *the* 2α*-stable,* 2α*-TS and* α*-MTS distribution have the same asymptotic behavior at the zero neighborhood, but the tails of the* α*-MTS distribution are thinner than those of the* 2α*-stable and fatter than those of the* 2α*-TS distribution.*

The characteristic function of the MTS distribution is given in the following result.

Theorem 2.7. *Let* $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. *Then the characteristic function of* X *is given by*

$$\phi_X(u; \alpha, C, \lambda_+, \lambda_-, \mu) = \exp(iu\mu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-)),$$

where for $u \in \mathbb{R}$,

$$
\begin{aligned}
&G_R(u;\alpha,C,\lambda_+,\lambda_-)\\
&= \begin{cases} \sqrt{\pi}2^{-\alpha-3/2}C\Gamma(-\alpha)\left((\lambda_+^2+u^2)^\alpha-\lambda_+^{2\alpha}+(\lambda_-^2+u^2)^\alpha-\lambda_-^{2\alpha}\right) & \text{if } \alpha\neq 0\\ \sqrt{\pi}2^{-3/2}C\left(\log\left(\frac{\lambda_+^2}{\lambda_+^2+u^2}\right)+\log\left(\frac{\lambda_-^2}{\lambda_-^2+u^2}\right)\right) & \text{if } \alpha=0 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&G_I(u;\alpha,C,\lambda_+,\lambda_-)\\
&=\frac{iuC\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}\left(\lambda_+^{2\alpha-1}F\left(1,\frac{1}{2}-\alpha;\frac{3}{2};-\frac{u^2}{\lambda_+^2}\right)-\lambda_-^{2\alpha-1}F\left(1,\frac{1}{2}-\alpha;\frac{3}{2};-\frac{u^2}{\lambda_-^2}\right)\right),
\end{aligned}
$$

where F is the hypergeometric function. Moreover, ϕ_X can be extended to the region $\{z\in\mathbb{C} : |\mathrm{Im}(z)|<\lambda_+\wedge\lambda_-\}$.

Corollary 2.8. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-; \mu)$. Then the Laplace transform of X is given by*

$$
E[\exp(uX)] = \exp\left(u\mu + G_R(-iu;\alpha,C,\lambda_+,\lambda_-)+G_I(-iu;\alpha,C,\lambda_+,\lambda_-)\right) \tag{2.7}
$$

for $u\in\mathbb{C}$ with $|\mathrm{Re}(u)|<\lambda_+\wedge\lambda_-$.

Using the characteristic function, we obtain the cumulants of all orders.

Proposition 2.9. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$ with $\alpha\in(-\infty,1)\setminus\{\frac{1}{2}\}$. The cumulants $c_m(X)$ of X are given as follows :*

$$
c_m(X) = \begin{cases} \mu + 2^{-\alpha-\frac{1}{2}}C\Gamma\left(\frac{1}{2}-\alpha\right)\left(\lambda_+^{2\alpha-1}-\lambda_-^{2\alpha-1}\right) & \text{if } m=1\\ 2^{m-\alpha-\frac{3}{2}}\left(\frac{m-1}{2}\right)!C\Gamma\left(\frac{m}{2}-\alpha\right)\left(\lambda_+^{2\alpha-m}-\lambda_-^{2\alpha-m}\right) & \text{if } m=3,5,7\cdots\\ 2^{-\alpha-\frac{3}{2}}\sqrt{\pi}\frac{m!}{\left(\frac{m}{2}\right)!}C\Gamma\left(\frac{m}{2}-\alpha\right)\left(\lambda_+^{2\alpha-m}+\lambda_-^{2\alpha-m}\right) & \text{if } m=2,4,6\cdots \end{cases} \tag{2.8}
$$

Remark 2.10. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$.*

1. *By Proposition 2.9, we obtain the mean, variance, skewness and excess kurtosis of X which*

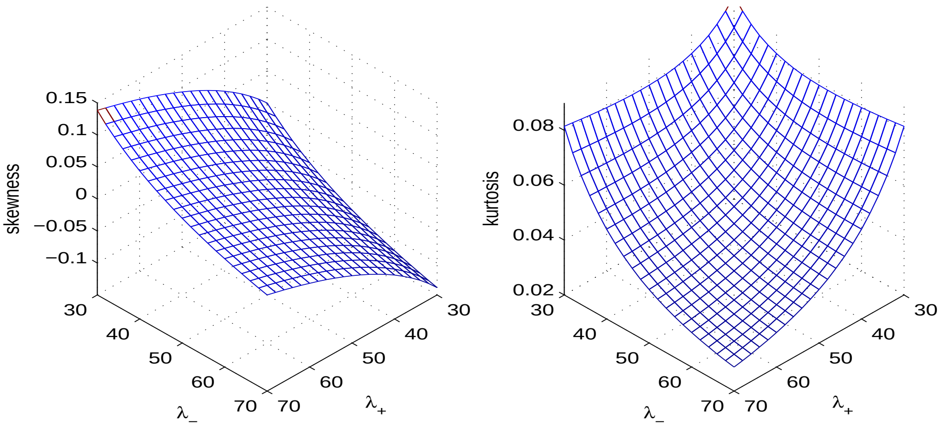

Figure 1: Skewness and Excess Kurtosis of MTS distributions : dependence on λ_+ and λ_-. Parameters : $\alpha = 0.7$, $C = 0.02$, $\mu = 0$, $t = 1$.

are given as follows :

$$E[X] = c_1(X) = \mu + 2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2} - \alpha\right)\left(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1}\right)$$

$$\mathrm{Var}(X) = c_2(X) = \frac{\sqrt{\pi}C}{2^{\alpha+\frac{1}{2}}}\Gamma\left(1-\alpha\right)\left(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}\right)$$

$$s(X) = \frac{c_3(X)}{c_2(X)^{\frac{3}{2}}} = \frac{2^{\frac{1}{2}\alpha+\frac{9}{4}}\Gamma\left(\frac{3}{2}-\alpha\right)\left(\lambda_+^{2\alpha-3} - \lambda_-^{2\alpha-3}\right)}{\pi^{\frac{3}{4}} C^{\frac{1}{2}}\left(\Gamma\left(1-\alpha\right)\left(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}\right)\right)^{\frac{3}{2}}}$$

$$k(X) = \frac{c_4(X)}{c_2(X)^2} = \frac{3 \cdot 2^{\alpha+\frac{3}{2}}\Gamma\left(2-\alpha\right)\left(\lambda_+^{2\alpha-4} + \lambda_-^{2\alpha-4}\right)\big)}{\sqrt{\pi}C\left(\Gamma\left(1-\alpha\right)\left(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}\right)\right)^2}.$$

2. *Figure 1 illustrates the dependence of skewness $s(X)$ and excess kurtosis $k(X)$ on λ_+ and λ_- when α and C are fixed.*

3. *λ_+ and λ_- control the rate of decay on the positive and negative part, respectively. If $\lambda_+ > \lambda_-$ ($\lambda_+ < \lambda_-$), then the distribution is skewed to the left (right). Moreover, if $\lambda_+ = \lambda_-$, then it is symmetric. Figure 2 illustrates this fact.*

4. *C controls the kurtosis of the distribution. If C increases, then the peakness of the distribution decreases. Figure 3 shows the effect of C.*

5. *Figure 4 shows that as α decreases, the distribution has fatter tails and increased peakness.*

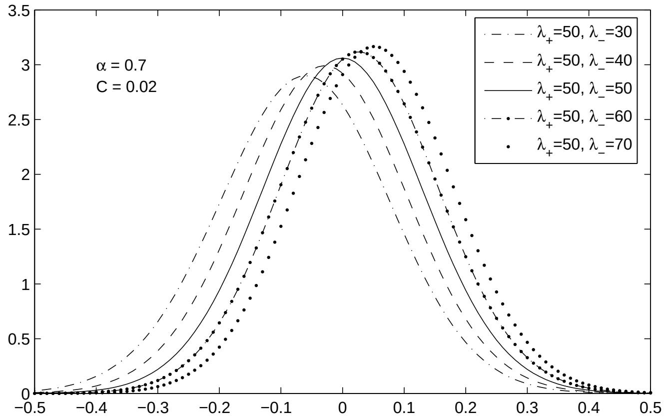

Figure 2: Probability density of the MTS distributions: dependence on λ_+ and λ_-. Parameters : $\lambda_+ = 50$, $\lambda_- \in \{30, 40, 50, 60, 70\}$, $\alpha = 0.7$, $C = 0.02$, $\mu = 0$.

Indeed, we can show that the Lévy process corresponding to the MTS distribution has finite activity if $\alpha < 0$ and infinite activity if $\alpha \geq 0$. Moreover it has finite variation if $\alpha < \frac{1}{2}$ and infinite variation if $\alpha > \frac{1}{2}$ (Kim (2005)).

If we put

$$C = 2^{\alpha+\frac{1}{2}} \left(\sqrt{\pi}\Gamma\left(1-\alpha\right)\left(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}\right)\right)^{-1}$$

and

$$\mu = -2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2} - \alpha\right)\left(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1}\right),$$

then $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$ has zero mean and unit variance. In this case, we say that the random variable X has the *standard MTS distribution*, and denote $X \sim stdMTS(\alpha, \lambda_+, \lambda_-)$.

2.3 Measure Change On Modified Tempered Stable Distributions

To apply the MTS distributions to no-arbitrage option pricing, we would need to determine the equivalent martingale measure (EMM). In this section, we review a general result of equivalence of measures presented by Sato (1999) and apply it to the MTS distribution.

Theorem 2.11. *Let $(X, \mathbf{P})$ and $(X, \mathbf{Q})$ be two infinitely divisible random variables on $\mathbb{R}$ with Lévy triplet (σ^2, ν, γ) and $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ respectively. Then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if and only if the*

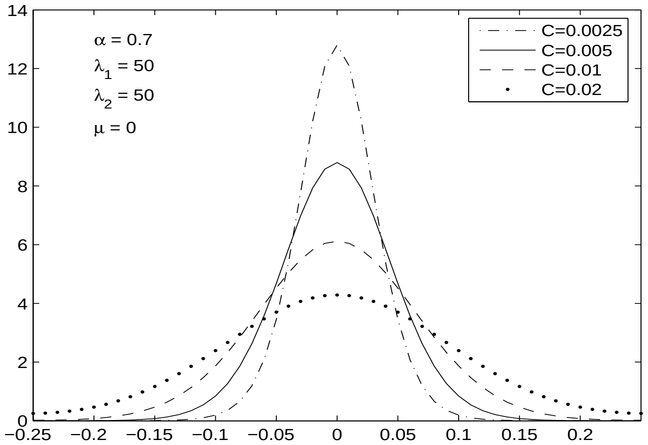

Figure 3: Probability density of the MTS distributions: dependence on C.
Parameters : $C \in \{0.0025, 0.005, 0.01, 0.02\}$, $\alpha = 0.7$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

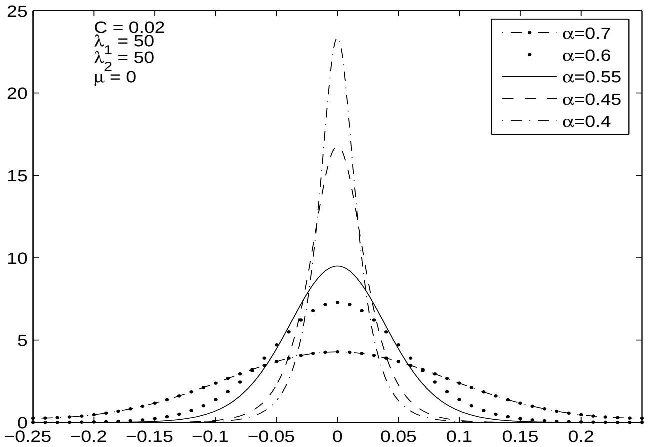

Figure 4: Probability density of the MTS distributions: dependence on α.
Parameters : $\alpha \in \{0.4, 0.45, 0.55, 0.6, 0.7\}$, $C = 0.02$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

Lévy triplet satisfies

$$\sigma^2 = \tilde{\sigma}^2, \tag{2.9}$$

$$\int_{-\infty}^{\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty, \tag{2.10}$$

where $\psi(x) = \log\left(\frac{\tilde{\nu}(dx)}{\nu(dx)}\right)$. *If* $\sigma^2 = 0$ *then*

$$\tilde{\gamma} - \gamma = \int_{|x|\le 1} x(\tilde{\nu} - \nu)(dx). \tag{2.11}$$

When $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent, the Radon-Nikodym derivative is*

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^U \tag{2.12}$$

where $(U, \mathbf{P})$ *is an infinitely divisible random variable with Lévy triplet* $(\sigma_U^2, \nu_U, \gamma_U)$ *given by*

$$\begin{cases} \sigma_U^2 = \sigma^2 \eta^2 \\ \nu_U = \nu \circ \psi^{-1} \\ \gamma_U = -\dfrac{\sigma^2\eta^2}{2} - \displaystyle\int_{-\infty}^{\infty} (e^y - 1 - y 1_{|y|\le 1}) \nu_U(dy) \end{cases}. \tag{2.13}$$

Here η *is such that*

$$\tilde{\gamma} - \gamma - \int_{|x|\le 1} x(\tilde{\nu} - \nu)(dx) = \sigma^2 \eta$$

if $\sigma > 0$ *and zero if* $\sigma = 0$.

Proof. This is a particular case of Theorem 33.1 and Theorem 33.2 in Sato (1999). □

Proposition 2.12. *Let* $(X, \mathbf{P})$ *and* $(X, \mathbf{Q})$ *be two MTS distributed random variables on* $\mathbb{R}$ *with parameters* $(\alpha, C, \lambda_+, \lambda_-, \mu)$ *and* $(\tilde{\alpha}, \tilde{C}, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{\mu})$*, respectively. Then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent if and only if* $C = \tilde{C}$, $\alpha = \tilde{\alpha}$ *and* $\tilde{\mu} = \mu$. *When* $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent, the Radon-Nikodym derivative is*

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^U \tag{2.14}$$

where $(U, \mathbf{P})$ *is an infinitely divisible random variable with Lévy triplet* $(\sigma_U^2, \nu_U, \gamma_U)$ *given by*

$$\begin{cases} \sigma_U^2 = 0 \\ \nu_U = \nu \circ \psi^{-1} \\ \gamma_U = -\displaystyle\int_{-\infty}^{\infty} (e^y - 1 - y 1_{|y|\le 1})(\nu \circ \psi^{-1})(dy) \end{cases}, \tag{2.15}$$

where $\psi(x) = \log\left(\frac{\tilde{\lambda}_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_+ x)}{\lambda_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_+ x)}\right) 1_{x>0} - \log\left(\frac{\tilde{\lambda}_-^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_- x)}{\lambda_-^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_- x)}\right) 1_{x<0}$.

2.4 The MTS-GARCH Option Pricing Model

The MTS-GARCH stock price model is defined over a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ which is constructed as follows: Consider a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid real random variables on a sequence of probability spaces $(\Omega_t, \mathbf{P}_t)_{t\in\mathbb{N}}$, such that $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ on the space $(\Omega_t, \mathbf{P}_t)$. Now we define

$$
\begin{aligned}
\Omega &:= \prod_{t\in\mathbb{N}} \Omega_t, \\
\mathfrak{F}_t &:= \otimes_{k=1}^t \sigma(\varepsilon_k) \otimes \mathfrak{F}_0 \otimes \mathfrak{F}_0 \cdots, \\
\mathfrak{F} &:= \sigma\left(\cup_{t\in\mathbb{N}} \mathfrak{F}_t\right), \\
\mathbb{P} &:= \otimes_{t\in\mathbb{N}} \mathbf{P}_t
\end{aligned}
$$

where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ-algebra generated by ε_k on Ω_k.

We propose the following model for the stock price dynamics:

$$
\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t\sigma_t - g(\sigma_t; \alpha, \lambda_+, \lambda_-) + \sigma_t\varepsilon_t, \quad t \in \mathbb{N}, \tag{2.16}
$$

where S_t denotes the price of the underlying asset at time t, r_t and d_t denote the risk free rate and dividend rate for the period $[t-1, t]$, and λ_t is a $\mathfrak{F}_{t-1}$ measurable random variable. S_0 is the present observed price. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is the characteristic exponent of the Laplace transform for the distribution $stdMTS(\alpha, \lambda_+, \lambda_-)$, i.e. $g(x; \alpha, \lambda_+, \lambda_-) = \log(E_{\mathbf{P}_t}[\exp(x\varepsilon_t)])$. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is defined if $x \in (-\lambda_-, \lambda_+)$ and its value can be obtained from (2.7) if $|x| < \lambda_+ \wedge \lambda_-$, and by numerical calculation if $x \in \{x \in (-\lambda_-, \lambda_+) \mid |x| \geq \lambda_+ \wedge \lambda_-\}$. The one period ahead conditional variance σ_t^2 follows a GARCH(1,1) process with a restriction $0 < \sigma_t < \lambda_+$, i.e.

$$
\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2) \wedge (\lambda_+^2(1-\epsilon)), \quad t \in \mathbb{N}, \quad \varepsilon_0 = 0, \tag{2.17}
$$

where the coefficients α_0, α_1 and β_1 are non-negative, $\alpha_1 + \beta_1 < 1$, $\alpha_0 > 0$ and ϵ is a positive real number with $\epsilon \ll 1$. Clearly σ_t is $\mathfrak{F}_{t-1}$-measurable and hence the process $(\sigma_t)_{t\in\mathbb{N}}$ is predictable. Moreover, the conditional expectation $E[\hat{S}_t/\hat{S}_{t-1}|\mathfrak{F}_{t-1}]$ equals $\exp(r_t + \lambda_t\sigma_t)$ where $\hat{S}_t = S_t \exp\left(\sum_{k=1}^t d_k\right)$ is the stock price considering re-investment of the dividends, thus λ_t can be interpreted as the market price of risk.

Remark 2.13. *If ε_t equals the standard normal distributed random variable for all $t \in \mathbb{N}$, g is to be the Laplace transform of ε_t and we ignore the restriction $\sigma_t < \lambda_+$, then the model becomes 'the normal GARCH model' introduced by Duan(1995).*

Proposition 2.14. *Let $t \in \mathbb{N}$ be fixed and $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ under $\mathbf{P}_t$. Suppose positive real numbers $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfy the equation*

$$\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2} = \tilde{\lambda}_+^{2\alpha-2} + \tilde{\lambda}_-^{2\alpha-2}. \tag{2.18}$$

Let

$$k = 2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2} - \alpha\right)\left(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1} - \tilde{\lambda}_+^{2\alpha-1} + \tilde{\lambda}_-^{2\alpha-1}\right), \tag{2.19}$$

where

$$C = 2^{\alpha+\frac{1}{2}}\left(\sqrt{\pi}\Gamma(1-\alpha)(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2})\right)^{-1}.$$

Then, there is a probability measure $\mathbf{Q}_t$ equivalent to $\mathbf{P}_t$, such that $(\varepsilon_t + k) \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$.

Assumption (A) (i) There exist $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfying equations (2.18) and $\tilde{\lambda}_+ \geq \lambda_+$. (ii) The market price of risk λ_t is given by $\lambda_t = k - (g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) - g(\sigma_t; \alpha, \lambda_+, \lambda_-))/\sigma_t$, for each $0 \leq t \leq T$, where k is defined as (2.19).

Under Assumption (A), let $\mathbf{Q}_t$ be the measure described in Proposition 2.14.

Definition 2.15. *Let $T \in \mathbb{N}$ be the time horizon. Define a new measure $\mathbb{Q}$ on $\mathfrak{F}_T$ equivalent to measure $\mathbb{P}$, with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ where the density process $(Z_t)_{0\leq t\leq T}$ is defined according to*

$$\begin{aligned}
&Z_0 \equiv 1, \\
&Z_t := \frac{d(\mathbf{P}_1 \otimes \cdots \otimes \mathbf{P}_{t-1} \otimes \mathbf{Q}_t \otimes \mathbf{P}_{t+1} \otimes \cdots \otimes \mathbf{P}_T)}{d\mathbb{P}} Z_{t-1}, \quad t = 1, 2, \cdots, T.
\end{aligned}$$

Lemma 2.16. *The measure $\mathbb{Q}$ satisfies the following requirements:*

(a) The discount asset price process $(e^{-r_t}\hat{S}_t)_{1\leq t\leq T}$ is a $\mathbb{Q}$-martingale w.r.t. the filtration $(\mathfrak{F}_t)_{1\leq t\leq T}$.

(b) We have

$$\mathrm{Var}_{\mathbb{Q}}\left(\log\left(\frac{S_t}{S_{t-1}}\right)\Big|\mathfrak{F}_{t-1}\right) \overset{\text{a.s.}}{=} \mathrm{Var}_{\mathbb{P}}\left(\log\left(\frac{S_t}{S_{t-1}}\right)\Big|\mathfrak{F}_{t-1}\right), \quad 1 \leq t \leq T$$

(c) The stock price dynamics under $\mathbb{Q}$ can be written as

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_t \xi_t, \quad 1 \le t \le T$$

where $(\xi_t)_{1\le t\le T}$ is a sequence of real random variables on Ω_t satisfying $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ under $\mathbf{Q}_t$ for $1 \le t \le T$. The variance process under $\mathbb{Q}$ has the form

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - k)^2 + \beta_1 \sigma_{t-1}^2) \wedge (\lambda_+^2(1-\epsilon)), \ t \in \mathbb{N}, \ \xi_0 = 0.$$

Proof. Let $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$, $t \in \mathbb{N}$.

(a) Using Definition 2.15, the Laplace transform of the MTS distribution and the measurability of σ_t with respect to $\mathfrak{F}_{t-1}$, we obtain

$$\begin{aligned}
E_{\mathbb{Q}}[\hat{S}_t | \mathfrak{F}_{t-1}] &= E_{\mathbb{Q}}[\hat{S}_{t-1} \exp(r_t + \lambda_t \sigma_t - g(\sigma_t; \alpha, \lambda_+, \lambda_-) + \sigma_t \varepsilon_t) | \mathfrak{F}_{t-1}] \\
&= E_{\mathbb{Q}}[\hat{S}_{t-1} \exp(r_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_t (k + \varepsilon_t)) | \mathfrak{F}_{t-1}] \\
&= \hat{S}_{t-1} \exp(r_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)) E_{\mathbb{Q}}[\exp(\sigma_t(\varepsilon_t + k)) | \mathfrak{F}_{t-1}] \\
&= \hat{S}_{t-1} \exp(r_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)) E_{\mathbb{Q}}[E_{\mathbf{Q}_t}[\exp(\sigma_t(\varepsilon_t + k)) | \sigma_t] | \mathfrak{F}_{t-1}] \\
&= \hat{S}_{t-1} \exp(r_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)) E_{\mathbb{Q}}[\exp(g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)) | \mathfrak{F}_{t-1}] \\
&= \hat{S}_{t-1} \exp(r_t)
\end{aligned}$$

(b) Since $\mathrm{Var}_{\mathbb{Q}}(\varepsilon_t + k | \mathfrak{F}_{t-1}) \overset{\text{a.s.}}{=} 1 \overset{\text{a.s.}}{=} \mathrm{Var}_{\mathbb{P}}(\varepsilon_t + k | \mathfrak{F}_{t-1})$, we can prove the equality.

(c) Let $\xi_t = \varepsilon_t + k$. Then $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ under $\mathbf{Q}_t$ for $1 \le t \le T$, and the following equality holds :

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_t(\varepsilon_t + k) = r_t - d_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_t \xi_t.$$

In the variance process, ε_{t-1} has to be replaced by $\xi_{t-1} - k$ in order to get the desired result. $\square$

The stock price dynamics under $\mathbb{Q}$ which is stated in Lemma 2.16 (c) is called *the MTS-GARCH risk neutral price process.* The arbitrage free price of a call option with strike price K and maturity T is given by

$$C_t = \exp\left(-\sum_{k=t+1}^{T} r_k\right) E_{\mathbb{Q}}[(S_T - K)^+ | \mathfrak{F}_t] \tag{2.20}$$

where the stock price S_T at time T is given by

$$S_T = S_t \exp\left(\sum_{k=t+1}^{T} \left((r_k - d_k) - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_k \xi_k\right)\right).$$

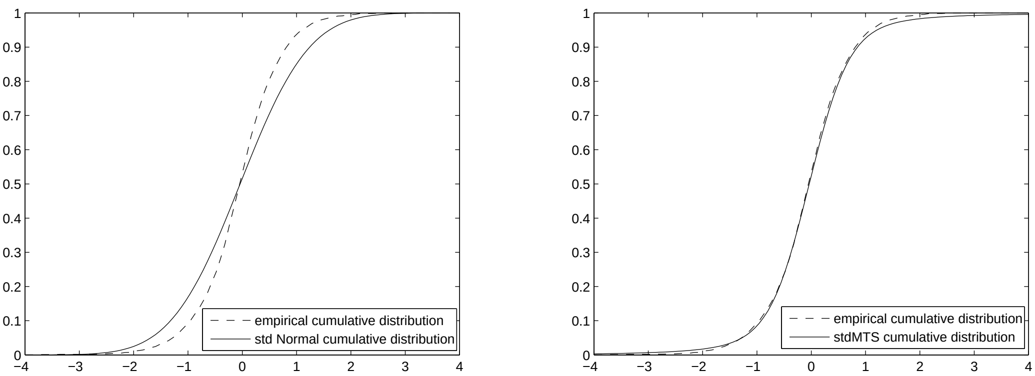

Figure 5: Empirical cumulative distribution, and standard normal and standard MTS cumulative distributions for residual of Data1.

3 Empirical Analysis

3.1 Parameter Estimation

This section is devoted to the maximum likelihood estimation (MLE) of the normal-GARCH and MTS-GARCH models. To simplify the estimation we impose a constant market price of risk λ. Our estimation procedure is as follows: (i) we estimate the parameters α_0, α_1, β_1 and the market price of risk λ from the normal-GARCH model, and (ii) we fix α_0, α_1, β_1 and λ and then estimate α, λ_+ and λ_- from the MTS-GARCH model. Here we assume that $\sigma_0^2 = \alpha_0/(1-\alpha_1-\beta_1)$.

In our empirical study, we use three sets of S&P 500 closing prices. One set (named Data1) consists of 3643 closing prices from Jun 1, 1988 to March 25, 2003, another set (named Data2) consists of 3891 closing prices from Jun 1, 1988 to March 25, 2004, and the third set (named Data3) consists of 4133 closing prices from Jun 1, 1988 to March 25, 2005. Dividend data on a daily basis is extracted from the corresponding S&P 500 total return which is available from the Chicago Board Options Exchange (CBOE). For the estimation of the risk free rate, we use the historical data of LIBOR one month rates. Having estimated the parameters, we will calculate call option prices with Monte Carlo simulation, and compare the model prices to the market prices.

We list the estimated GARCH parameters and the parameters for the standard MTS distribution in Table 1 and in Table 2 respectively. Comparisons of the empirical cumulative distribution to the standard MTS and standard normal cumulative distributions based on the residuals from

Table 1: Estimated GARCH parameters

	β_1	α_1	α_0	λ
Data1	0.8898	0.1058	8.3988E−06	0.0485
Data2	0.8936	0.1020	8.0491E−06	0.0514
Data3	0.8999	0.0959	7.2829E−06	0.0500

Table 2: Estimated parameters of the standard MTS distribution

	α	λ_+	λ_-
Data1	0.8010	0.1424	0.1269
Data2	0.8048	0.1390	0.1263
Data3	0.8064	0.1414	0.1229

Data1 are provided in Figure 5.

For the assessment of the goodness-of-fit, we use two tools: the χ^2-test and the Kolmogrov-Smirnov test. Moreover, we calculate the Anderson-Darling statistic to evaluate better the tail fit.

We define the null hypotheses as follows :

H_0^{normal}: The residuals follow the standard normal distribution.

H_0^{MTS}: The residuals follow the standard MTS distribution.

For our χ^2-test, n is the sample size, and $\mathcal{P} = \{A_1, A_2 \cdots, A_m\}$ is the partition with an equal width of the support of the distributions, where we have excluded the cells for which the expected value $nP[\varepsilon_t \in A_j]$ is greater than 5. Let N_k $(k = 1, 2, \cdots, m)$ be the number of observations x_i falling into the set A_k. We will compare the empirical frequency distribution N_k/n with the theoretical frequency distribution $\pi_k = \mathbf{P}(X \in A_k)$ under the null hypotheses through the Pearson statistic

$$\hat{\chi}^2 = \sum_{k=1}^{m} \frac{(N_k - n\pi_k)^2}{n\pi_k}.$$

General theory says that the Pearson statistic $\hat{\chi}^2$ follows (asymptotically) a χ^2-distribution with $m - 1 - h$ degrees of freedom under the null hypotheses, where h is a number of parameters to be estimated. We choose $-2.48 + 0.08(j-1)$, $j = 1, \cdots, 63$ as the midpoints of A_j for the standard normal innovations, and $-2 + 0.08(j-1)$, $j = 1, \cdots, 53$ as the midpoints of A_j for the standard MTS innovations. In H_0^{normal}, the degrees of freedom are $63 - 1 = 62$. However, in the standard

Table 3: χ^2 test statistic

	Standard Normal		standard MTS	
	χ^2	p-value	χ^2	p-value
Data1	730.8932	0	58.6301	0.1631
Data2	765.8026	0	60.8117	0.1200
Data3	807.8602	0	61.2749	0.1121

Table 4: Kolmogrov-Smirnov (KS) and Anderson-Darling (AD) statistics

	standard normal		standard MTS	
	KS	AD	KS	AD
Data1	0.1019	87.8897	0.0178	0.1219
Data2	0.1004	85.7667	0.0177	0.1153
Data3	0.1001	102.4948	0.0169	0.1098

MTS case, we have three parameters to be estimated, so we take $53 - 4 = 49$ degrees of freedom. Table 3 provides the χ^2-test statistics and their corresponding p-values ($\mathbf{P}(\chi^2_{m-1-h} > \hat{\chi}^2)$). Given that we have $\chi^2_{49,0.95} = 66.3386$, $\chi^2_{49,0.99} = 74.9195$, $\chi^2_{62,0.95} = 81.3810$ and $\chi^2_{62,0.99} = 90.8015$, H_0^{MTS} is not rejected and H_0^{normal} is rejected.

Table 4 provides the Kolmogrov-Smirnov and Anderson-Darling statistics for each data set. The Kolmogrov-Smirnov statistic is defined as $KS = \sup_{x_i} |F(x_i) - \hat{F}(x_i)|$ where F is the cumulative distribution function and $\hat{F}$ is the empirical cumulative distribution function for a given observation $\{x_i\}$. According to Table 5, which provides the critical values of KS, H_0^{normal} is rejected and H_0^{MTS} is not rejected at .15 significance level for each of the data sets. In Table 4, we also provide the Anderson-Darling statistic measuring the distance between F and $\hat{F}$, which is given by

$$AD = \sup_{x_i} \frac{|F(x_i) - \hat{F}(x_i)|}{\sqrt{F(x_i)(1 - F(x_i))}} .$$

We can see that the AD value of the standard MTS case is significantly smaller than that of the standard normal case.

3.2 Comparison of Model and Market Price on the S&P 500 Option Market

In this section, we examine the pricing performance of the MTS-GARCH option pricing model. We will use the market quotes of the European calls on the S&P 500 index, which are traded on

Table 5: Critical values of KS

		$P(KS > D) = c$			
	n	$c = 0.15$	$c = 0.10$	$c = 0.05$	$c = 0.01$
Data1	3643	0.0189	0.0202	0.0225	0.0270
Data2	3891	0.0183	0.0196	0.0218	0.0261
Data3	4133	0.0177	0.0190	0.0212	0.0254

the CBOE. We take a set of closing prices on March 25, 2003 expiring on April 19, 2003. Since we do not have an efficient analytical form of the option price (2.20), the option prices are determined by Monte Carlo simulation.

We calculate the option prices under three models. The first one is the classical Black-Scholes model. The second and the third are the normal-GARCH and MTS-GARCH models. As mentioned above, the risk neutral GARCH price process is given by

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t - \log E_{\mathbb{Q}}[\exp(\sigma_t\xi_t)] + \sigma_t\xi_t, \quad 1 \le t \le T,$$

where $\xi_t, t \in \mathbb{N}$ are iid rv's with the standard normal and standard MTS distributions in the normal-GARCH model and MTS-GARCH model, respectively. The variance process is given by

$$\sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - \lambda_t)^2 + \beta_1\sigma_{t-1}^2,\ t \in \mathbb{N}, \quad \xi_0 = 0$$

and

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k)^2 + \beta_1\sigma_{t-1}^2) \wedge (\lambda_+^2(1-\epsilon)),\ t \in \mathbb{N},\ \xi_0 = 0$$

in the normal-GARCH model and the MTS-GARCH model, respectively.

Since the last date of Data1 is March 25, 2003, the estimated parameters from Data1 is used to determine the option prices. We take historical volatility $\sigma = 0.1633$ of Data1 for the volatility of the Black-Scholes model, and the parameters of Data1 in Table 1 are used for the normal-GARCH model with $\lambda_t = \lambda$. We need the risk neutral parameters $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfying Assumption (A) for the MTS-GARCH model. Let us use the parameters calibrated in the previous section are the estimators of the risk neutral parameters. That is, we generate the sample path with the parameters α_0, α_1, and β_1 of Data1 in Table 1 and $k = 0.0485$ which is the value for λ of Data1 in Table 1. The risk neutral innovation ξ_t is given by $\xi_t \sim stdMTS(0.8010, 0.1424, 0.1269)$, $t \in \mathbb{N}$, whose parameters are of Data1 in Table 2. While, the hidden market parameters λ_+ and λ_- and the process of the market price of risk $(\lambda_t)_{t\ge 0}$ will satisfy Assumption (A).

Table 6: Errors between the market and model prices of options

	RMSE	AAE	APE(%)
Black-Scholes	6.0658	4.8154	17.5629
Normal-GARCH	4.2995	3.5636	12.9972
MTS-GARCH	2.7042	2.4344	8.8790

There are several reasonable ways to choose the initial variance σ_0^2, which strongly influences the model behavior. One possibility is to take the last value in the estimated series or alternatively to choose $\sigma_0^2 = \alpha_0/(1-\alpha_1-\beta_1)$ which is a fraction of the stationary variance. Here, we choose the first method to reflect the current market condition.

To measure the performances of the three models, we will use the root mean square error (RMSE), average absolute error (AAE) and average absolute error as percentage of the mean price (APE). These are defined as follows (Schoutens (2003)):

$$
\begin{aligned}
\text{RMSE} &= \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}},\\
\text{AAE} &= \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}},\\
\text{APE} &= \frac{1}{\text{mean option price}} \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}}.
\end{aligned}
$$

The values of the errors for each model are presented in Table 6, which suggest that the MTS-GARCH model provides the best results.

4 Conclusion

This paper introduces an alternative class of tempered stable distributions which we call Modified Tempered Stable (MTS) distribution model. It has fatter tails than those of TS distribution and thinner tails than those of α-stable distribution. It is sufficiently flexible in describing the skewness and kurtosis of asset returns and has all moments finite. Next we introduced an enhanced GARCH-model, namely the MTS-GARCH model, by applying MTS innovations to the classical GARCH model. As a result, the MTS-GARCH is a time series model equipped with volatility clustering, the leverage effect and the conditional skewness and leptokurtosis of the returns. The risk neutral measure is obtained by applying a change of measure to the MTS distribution.

We obtain encouraging results from the empirical study. Modeling the innovation with MTS laws, we improved goodness-of-fit statistic for the GARCH model on the S&P 500 index data. Furthermore, the χ^2 and Kolmogrov-Smirnov tests rejected the normal hypothesis and did not reject the MTS hypothesis, and the Anderson-Darling statistic of the MTS-GARCH model is significantly smaller than that of the normal-GARCH model. In the S&P 500 option market, the performance of the MTS-GARCH model is better than any other models we have examined. The skewness and fat tails of the MTS innovations seem to generate significant improvements in the empirical results. Consequently, the MTS-GARCH model can be a more realistic model than the normal-GARCH model.

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A Appendices

A.1 Special Functions

The *modified Bessel function of the second kind* is defined as

$$K_p(x) = \frac{\pi}{2\sin p\pi}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k-p}}{k!\Gamma(k-p+1)} - \sum_{k=0}^{\infty}\frac{(x/2)^{2k+p}}{k!\Gamma(k+p+1)}\right). \tag{A.1}$$

Its asymptotic behavior can be described as follows

$$K_p(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}, \quad p \geq 0, \quad x \to \infty \tag{A.2}$$

and

$$K_p(x) \sim \frac{\Gamma(p)}{2}\left(\frac{2}{x}\right)^p, \quad p > 0, \quad x \to 0^+. \tag{A.3}$$

The integral representation of $K_p(x)$ is given by

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-p-1}\, dt$$

and recurrence formula is given by

$$\frac{d}{dx}(x^p K_p(x)) = -x^p K_{p-1}(x). \tag{A.4}$$

The following lemma is useful.

Lemma A.1. *If $\mu - p > -1$ and $a > 0$ then*

$$\int_0^\infty x^\mu K_p(ax)dx = \frac{2^{\mu-1}}{a^{\mu+1}}\Gamma\left(\frac{1+\mu+p}{2}\right)\Gamma\left(\frac{1+\mu-p}{2}\right).$$

Proof. Andrews (1998), p299. □

Now define the hypergeometric function. Before defining it, let us introduce an useful notation

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,3,\cdots, \quad a \in \mathbb{R} \tag{A.5}$$

called the *Pochhammer symbol.* This symbol can also be defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 0,1,2,3,\cdots. \tag{A.6}$$

The function

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \quad |x| < 1 \tag{A.7}$$

is called the *hypergeometric function.*

Lemma A.2 (Andrews (1998), p367)**.** *For $k = 1, 2, 3\cdots$,*

$$\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k(b)_k}{(c)_k}F(a+k,b+k;c+k;x). \tag{A.8}$$

A.2 Proofs of Theorem 2.7 and Proposition 2.9

For proof of Theorem 2.7, we need the following results.

Lemma A.3. *Let $\lambda > 0$. Then*

$$\lambda^{\alpha+\frac{1}{2}}\int_0^1 \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha-\frac{1}{2}}}dx = \begin{cases} \frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}}\Gamma\left(\frac{1}{2}-\alpha\right) - \lambda^{\alpha-\frac{1}{2}}K_{\alpha-\frac{1}{2}}(\lambda) & \text{if } \alpha < \frac{1}{2} \\ \infty & \text{if } \alpha \geq \frac{1}{2} \end{cases}.$$

Proof. By the integral representation of $K_{\alpha+\frac{1}{2}}(\lambda x)$, we have

$$\begin{aligned}\lambda^{\alpha+\frac{1}{2}}\int_0^1 \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha-\frac{1}{2}}}dx &= \frac{\lambda^{2\alpha+1}}{2^{\alpha+\frac{3}{2}}}\int_0^1\int_0^\infty x\exp\left(-t-\frac{(\lambda x)^2}{4t}\right)t^{-\alpha-\frac{3}{2}}dt\,dx \\ &= \frac{\lambda^{2\alpha+1}}{2^{\alpha+\frac{3}{2}}}\int_0^\infty\left(\int_0^1 x\exp\left(-\frac{(\lambda x)^2}{4t}\right)dx\right)e^{-t}t^{-\alpha-\frac{3}{2}}\,dt.\end{aligned}$$

Since

$$\int_0^1 x\exp\left(-\frac{(\lambda x)^2}{4t}\right)dx = \frac{2t}{\lambda^2}\left(1-\exp\left(-\frac{\lambda^2}{4t}\right)\right),$$

we have

$$\begin{aligned}\lambda^{\alpha+\frac{1}{2}}\int_0^1\frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha-\frac{1}{2}}}dx &= \frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}}\int_0^\infty\left(1-\exp\left(-\frac{\lambda^2}{4t}\right)\right)e^{-t}t^{-\alpha-\frac{1}{2}}\,dt \\ &= \frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}}\int_0^\infty e^{-t}t^{-\alpha-1/2}\,dt - \lambda^{\alpha-\frac{1}{2}}\frac{1}{2}\left(\frac{\lambda}{2}\right)^{\alpha-\frac{1}{2}}\int_0^\infty\exp\left(-t-\frac{\lambda^2}{4t}\right)t^{-\alpha-\frac{1}{2}}\,dt \\ &= \begin{cases}\frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}}\Gamma\left(\frac{1}{2}-\alpha\right)-\lambda^{\alpha-\frac{1}{2}}K_{\alpha-\frac{1}{2}}(\lambda) & \text{if } \alpha<\frac{1}{2} \\ \infty & \text{if } \alpha \geq \frac{1}{2}\end{cases}.\end{aligned}$$

□

Lemma A.4. *Let $u^2 < \lambda^2$.*

1. If $\alpha < \frac{1}{2}$, then

$$\begin{aligned}&\sum_{n=1}^\infty\frac{(iu)^n}{n!}\lambda^{\alpha+\frac{1}{2}}\int_0^\infty x^{n-\alpha-\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda x)dx \\ &= \begin{cases} 2^{-\alpha-3/2}\sqrt{\pi}\Gamma(-\alpha)((\lambda^2+u^2)^\alpha-\lambda^{2\alpha}) \\ +iu2^{-\alpha-1/2}\lambda^{2\alpha-1}\Gamma(\frac{1}{2}-\alpha)F\left(1,\frac{1}{2}-\alpha;\frac{3}{2};-\frac{u^2}{\lambda^2}\right) & \text{if } \alpha\neq 0 \\ \sqrt{\pi}2^{-3/2}\log\left(\frac{\lambda^2}{\lambda^2+u^2}\right) \\ +iu2^{-1/2}\lambda^{-1}\Gamma(\frac{1}{2})F\left(1,\frac{1}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\right) & \text{if } \alpha = 0\end{cases}.\end{aligned}\tag{A.9}$$

2. If $\alpha \in (\frac{1}{2}, 1)$, then

$$\begin{aligned}\text{(A.10)}\qquad &\sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} x^{n-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x) dx \\ &= \frac{\sqrt{\pi}}{2^{\alpha+\frac{3}{2}}} \Gamma(-\alpha)((\lambda^2+u^2)^{\alpha} - \lambda^{2\alpha}) \\ &\quad + \frac{iu\lambda^{2\alpha-1}\Gamma(\frac{1}{2}-\alpha)}{2^{\alpha+\frac{1}{2}}} \left(F\left(1, \frac{1}{2}-\alpha; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) - 1 \right).\end{aligned}$$

Proof. 1. Suppose $\alpha < \frac{1}{2}$. By Lemma A.1, we have

$$\begin{aligned}\text{(A.11)}\qquad &\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} x^{n-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x) dx \\ &= \frac{\lambda^{2\alpha}}{2^{\alpha+\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{2iu}{\lambda}\right)^n \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(\frac{n-2\alpha}{2}\right).\end{aligned}$$

Now decompose equation (A.11) into real and imaginary parts, then we have

$$\begin{aligned}\text{(A.12)}\qquad &\frac{\lambda^{2\alpha}}{2^{\alpha+\frac{3}{2}}} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2u}{\lambda}\right)^{2n} \Gamma\left(n+\frac{1}{2}\right) \Gamma(n-\alpha) \right. \\ &\quad \left. + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{2u}{\lambda}\right)^{2n+1} \Gamma(1+n) \Gamma\left(n-\alpha+\frac{1}{2}\right) \right).\end{aligned}$$

Since $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}$, by the binomial theorem, the real part becomes

$$\begin{aligned}\text{(A.13)}\qquad &\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2u}{\lambda}\right)^{2n} \Gamma\left(n+\frac{1}{2}\right) \Gamma(n-\alpha) \\ &= \begin{cases} \sqrt{\pi}\Gamma(-\alpha)\lambda^{-2\alpha} \left((\lambda^2+u^2)^{\alpha} - \lambda^{2\alpha}\right) & \text{if } \alpha \neq 0 \\ \sqrt{\pi} \log\left(\frac{\lambda^2}{\lambda^2+u^2}\right) & \text{if } \alpha = 0 \end{cases}.\end{aligned}$$

Since $(2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n!$ and $\Gamma(n-\alpha+\frac{1}{2}) = (\frac{1}{2}-\alpha)_n\Gamma(\frac{1}{2}-\alpha)$ by (A.6), the imaginary part equals

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{2u}{\lambda}\right)^{2n+1} \Gamma(1+n) \Gamma\left(n-\alpha+\frac{1}{2}\right) \\ &= \frac{2u}{\lambda}\Gamma\left(\frac{1}{2}-\alpha\right) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(1)_n(\frac{1}{2}-\alpha)_n}{\left(\frac{3}{2}\right)_n} \left(-\frac{u^2}{\lambda^2}\right)^n \\ \text{(A.14)}\qquad &= \frac{2u}{\lambda}\Gamma\left(\frac{1}{2}-\alpha\right) F\left(1, \frac{1}{2}-\alpha; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right).\end{aligned}$$

Substituting (A.13) and (A.14) into (A.12), we obtain (A.9).

2. Let $\alpha \in (\frac{1}{2}, 1)$. Then we have, by Lemma A.1,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} x^{n-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x) dx \\
&= \frac{\lambda^{2\alpha}}{2^{\alpha+\frac{3}{2}}} \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{2iu}{\lambda}\right)^n \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(\frac{n-2\alpha}{2}\right) \\
&= \frac{\lambda^{2\alpha}}{2^{\alpha+\frac{3}{2}}} \left(\sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{2iu}{\lambda}\right)^n \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(\frac{n-2\alpha}{2}\right) - \frac{2iu}{\lambda} \Gamma\left(\frac{1}{2} - \alpha\right) \right).
\end{aligned}$$

By a similar argument as the one provided above, we can show (A.10). □

Now, we prove the main result.

proof of Theorem 2.7. Let

$$H(\alpha, \lambda, u) = \int_0^{\infty} (e^{iux} - 1 - iux 1_{|x| \leq 1}) \lambda^{\alpha+\frac{1}{2}} \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx,$$

where $\lambda > 0$ and $|iu| < \lambda$. Let $\alpha < \frac{1}{2}$. Then, we have

$$H(\alpha, \lambda, u) = \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} (e^{iux} - 1) \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx - iu\lambda^{\alpha+\frac{1}{2}} \int_0^1 \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha-\frac{1}{2}}} dx.$$

By Lemma A.3 and the series expansion of the exponential function, we have

$$\begin{aligned}
H(\alpha, \lambda, u) = &\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} x^{n-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x) dx \\
&- iu \left(\frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}} \Gamma\left(\frac{1}{2} - \alpha\right) - \lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda) \right).
\end{aligned}$$

By Lemma A.4, we obtain that

$$\begin{aligned}
H(\alpha, \lambda, u) = &\sqrt{\pi} 2^{-\alpha-\frac{3}{2}} \Gamma(-\alpha)((\lambda^2 + u^2)^{\alpha} - \lambda^{2\alpha}) 1_{\alpha \neq 0} \\
&+ \sqrt{\pi} 2^{-3/2} \log\left(\frac{\lambda^2}{\lambda^2 + u^2}\right) 1_{\alpha=0} \\
&+ \frac{iu\lambda^{2\alpha-1}\Gamma(\frac{1}{2} - \alpha)}{2^{\alpha+\frac{1}{2}}} F\left(1, \frac{1}{2} - \alpha; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) \\
&- iu \left(\frac{\lambda^{2\alpha-1}}{2^{\alpha+\frac{1}{2}}} \Gamma\left(\frac{1}{2} - \alpha\right) - \lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda) \right).
\end{aligned}$$

Let $\alpha \in (\frac{1}{2}, 1)$. Then, we have

$$H(\alpha, \lambda, u) = \lambda^{\alpha+\frac{1}{2}} \int_0^{\infty} (e^{iux} - 1 - iux) \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx + iu\lambda^{\alpha+\frac{1}{2}} \int_1^{\infty} \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha-\frac{1}{2}}} dx.$$

Since we can show that $\lambda^{\alpha+\frac{1}{2}} \int_1^\infty x^{-\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x)dx = \lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda)$ for $\alpha \in (\frac{1}{2}, 1)$, we have

$$H(\alpha, \lambda, u) = \lambda^{\alpha+\frac{1}{2}} \int_0^\infty (e^{iux} - 1 - iux) \frac{K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}} dx + iu\lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda).$$

By the series expansion of the exponential function, we obtain

$$H(\alpha, \lambda, u) = \sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha+\frac{1}{2}} \int_0^\infty x^{n-\alpha-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x)dx + iu\lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda).$$

By Lemma A.4, we have

$$\begin{aligned} H(\alpha, \lambda, u) = & \frac{\sqrt{\pi}}{2^{\alpha+\frac{3}{2}}} \Gamma(-\alpha)((\lambda^2 + u^2)^\alpha - \lambda^{2\alpha}) \\ & + \frac{iu\lambda^{2\alpha-1}\Gamma(\frac{1}{2} - \alpha)}{2^{\alpha+\frac{1}{2}}} F\left(1, \frac{1}{2} - \alpha; \frac{3}{2}; -\frac{u^2}{\lambda^2}\right) \\ & - iu\left(\frac{\Gamma(\frac{1}{2} - \alpha)}{2^{\alpha+\frac{1}{2}}} \lambda^{2\alpha-1} - \lambda^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda)\right). \end{aligned}$$

So, for $\alpha \in (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $|iu| < \lambda_+ \wedge \lambda_-$, we have

$$\begin{aligned} & iu\gamma + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx) \\ & = iu\gamma + CH(\lambda_+, \alpha, u) + C\overline{H(\lambda_-, \alpha, u)} \\ & = iu\mu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-). \end{aligned}$$

¿From the Lévy -Kintchine formula, we obtain the desired characteristic function in the theorem. The characteristic function $\phi_X(u)$ can be extended via analytic continuation to the region $\{ z \in \mathbb{C} \ : \ |\text{Im}(z)| < \lambda_+ \wedge \lambda_- \}$. □

proof of Proposition 2.9. We first note that, if h is an infinitely differentiable function, then we have, for $n \in \mathbb{N}$ and $k \in \mathbb{R}$,

$$\frac{d^{2n+1}}{du^{2n+1}}(uh(ku^2))\Big|_{u=0} = 2^n \cdot 1 \cdot 3 \cdots (2n+1)k^n h^{(n)}(0),$$

and

$$\frac{d^{2n}}{du^{2n}}(uh(ku^2))\Big|_{u=0} = 0.$$

¿From this note and (A.8) , we obtain that

$$
\begin{aligned}
&\frac{d^{2n+1}}{du^{2n+1}}\left(uF\left(1,\frac{1}{2}-\alpha;\frac{3}{2};\frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0} \\
&= 2^n\cdot 1\cdot 3\cdots(2n+1)\lambda^{-2n}\frac{(1)_n(\frac{1}{2}-\alpha)_n}{(\frac{3}{2})_n}F\left(1+n,\frac{1}{2}-\alpha+n;\frac{3}{2}+n;0\right) \\
&= \frac{(2n+1)!(\frac{1}{2}-\alpha)_n}{(\frac{3}{2})_n\lambda^{2n}} \\
&= \left(\frac{2}{\lambda}\right)^{2n} n!\frac{\Gamma\left(n+\frac{1}{2}-\alpha\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)}
\end{aligned}
$$

and

$$
\frac{d^{2n}}{du^{2n}}\left(uF\left(1,\frac{1}{2}-\alpha;\frac{3}{2};\frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0}=0.
$$

Hence we have, for $m\in\mathbb{N}$,

$$
\begin{aligned}
&\frac{d^{m}}{du^{m}}\left(uF\left(1,\frac{1}{2}-\alpha;\frac{3}{2};\frac{u^2}{\lambda^2}\right)\right)\Big|_{u=0} \\
&\qquad=\begin{cases} \left(\frac{2}{\lambda}\right)^{m-1}\left(\frac{m-1}{2}\right)!\frac{\Gamma\left(\frac{m}{2}-\alpha\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)} & \text{if } m=1,3,5,\cdots \\ 0 & \text{if } m=2,4,6,\cdots \end{cases}
\end{aligned}
\tag{A.15}
$$

On the other hand, if $\alpha\neq 0$, we have

$$
\frac{d^{2n}}{du^{2n}}((\lambda^2-u^2)^\alpha-\lambda^{2\alpha})\big|_{u=0}=\frac{(2n)!}{n!}(-\alpha)_n\lambda^{2(\alpha-n)}=\frac{(2n)!}{n!}\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)}\lambda^{2(\alpha-n)}
$$

and

$$
\frac{d^{2n+1}}{du^{2n+1}}((\lambda^2-u^2)^\alpha-\lambda^{2\alpha})\big|_{u=0}=0,
$$

so we obtain that

$$
\frac{d^{m}}{du^{m}}((\lambda^2-u^2)^\alpha-\lambda^{2\alpha})\Big|_{u=0}=\begin{cases} 0 & \text{if } m=1,3,5,\cdots \\ \frac{m!}{(\frac{m}{2})!}\frac{\Gamma(\frac{m}{2}-\alpha)}{\Gamma(-\alpha)}\lambda^{2\alpha-m} & \text{if } m=2,4,6,\cdots \end{cases}. \tag{A.16}
$$

For $m\in\mathbb{N}$ and $\alpha\neq 0$, the cumulant $c_m(X)$ is given by

$$
\begin{aligned}
c_m(X) &= \frac{d^m}{du^m}(\log E[e^{uX}])\big|_{u=0} \\
&= \frac{d^m}{du^m}(\mu u)+\frac{\sqrt{\pi}C\Gamma(-\alpha)}{2^{\alpha+\frac{3}{2}}}\sum_{j=1}^{2}\left[\frac{d^m}{du^m}((\lambda_j^2-u^2)^\alpha-\lambda_j^\alpha)\right]_{u=0} \\
&\quad -\frac{C\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}\sum_{j=1}^{2}\left[(-1)^j\lambda_j^{2\alpha-1}\frac{d^m}{du^m}\left(uF\left(1,\frac{1}{2}-\alpha;\frac{3}{2};\frac{u^2}{\lambda_j^2}\right)\right)\right]_{u=0}.
\end{aligned}
\tag{A.17}
$$

Substituting (A.15) and (A.16) into (A.17), we obtain (2.8).

Moreover, since we have

$$\frac{d^m}{du^m}\log\left(\frac{\lambda^2}{\lambda^2+u^2}\right)\Big|_{u=0}=\begin{cases}0 & \text{if } m=1,3,5,\cdots\\ \frac{m!}{\left(\frac{m}{2}\right)!}\Gamma\left(\frac{m}{2}\right)\lambda^{-m} & \text{if } m=2,4,6,\cdots\end{cases},$$

we obtain (2.8) by the similar arguments given above. □

A.3 Proofs of Proposition 2.12

Lemma A.5. *Let $\lambda>0$, $\alpha\in(0,1)$. Then we have*

$$\begin{aligned}
(1)\quad &(\lambda x)^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda x)\\
&=2^{\alpha-\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)+\frac{2^{\alpha-\frac{1}{2}}\pi}{\cos(\alpha\pi)}\left(\sum_{k=1}^{\infty}\frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right)\\
(2)\quad &\int_0^1 x^{-2\alpha}\left(\sum_{k=1}^{\infty}\frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right)dx\\
&=\frac{-\cos(\alpha\pi)}{2^{\alpha-\frac{1}{2}}\pi}\lambda^{\alpha-\frac{1}{2}}K_{\alpha-\frac{1}{2}}(\lambda)-\frac{1}{2\Gamma\left(\frac{3}{2}-\alpha\right)}+\frac{\lambda^{2\alpha-1}}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}
\end{aligned}$$

Proof. (1) The series form of the modified Bessel function of the second kind is given by

$$K_p(x)=\frac{\pi}{2\sin p\pi}\left(\sum_{k=0}^{\infty}\frac{(x/2)^{2k-p}}{k!\Gamma(k-p+1)}-\sum_{k=0}^{\infty}\frac{(x/2)^{2k+p}}{k!\Gamma(k+p+1)}\right),\quad x\in\mathbb{R},\quad p\in\mathbb{R}.$$

Hence we have

$$\begin{aligned}
&(\lambda x)^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(\lambda x)\\
&=(\lambda x)^{\alpha+\frac{1}{2}}\frac{\pi}{2\sin\left(\alpha+\frac{1}{2}\right)\pi}\left(\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k-\alpha+\frac{1}{2}}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k+\alpha+\frac{1}{2}}}{k!\Gamma\left(k+\left(\alpha+\frac{1}{2}\right)+1\right)}\right)\\
&=\frac{2^{\alpha-\frac{1}{2}}\pi}{\sin\left(\alpha+\frac{1}{2}\right)\pi}\left(\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\left(\alpha+\frac{1}{2}\right)+1\right)}\right)\\
&=\frac{2^{\alpha-\frac{1}{2}}\pi}{\sin\left(\alpha+\frac{1}{2}\right)\pi}\left(\frac{1}{\Gamma\left(1-\alpha-\frac{1}{2}\right)}+\sum_{k=1}^{\infty}\frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\left(\alpha+\frac{1}{2}\right)+1\right)}\right)
\end{aligned}$$

Since $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$ and $\sin\left(\alpha+\frac{1}{2}\right)\pi=\cos(\alpha\pi)$, we obtain the result.

(2) We have

$$\begin{aligned}
&\int_0^1 x^{-2\alpha}\left(\sum_{k=1}^{\infty}\frac{(\lambda x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{(\lambda x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right)dx\\
&=\left(\frac{\lambda}{2}\right)^{2\alpha}\left(\sum_{k=1}^{\infty}\frac{\int_0^1(\lambda x/2)^{2k-2\alpha}\,dx}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)}-\sum_{k=0}^{\infty}\frac{\int_0^1(\lambda x/2)^{2k+1}\,dx}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right)
\end{aligned}$$

$$= \left(\frac{\lambda}{2}\right)^{2\alpha} \left(\sum_{k=1}^{\infty} \frac{\frac{1}{\lambda(k-\alpha+\frac{1}{2})}(\lambda/2)^{2k-2\alpha+1}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^{\infty} \frac{\frac{1}{\lambda(k+1)}(\lambda/2)^{2k+2}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)} \right)$$
$$= \left(\frac{\lambda^{2\alpha-1}}{2^{2\alpha}}\right) \left(\sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k-2\alpha+1}}{k!\Gamma\left(k-\alpha+\frac{1}{2}+1\right)} - \sum_{k=0}^{\infty} \frac{(\lambda/2)^{2k+2}}{(k+1)!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)} \right)$$
$$= \left(\frac{\lambda^{2\alpha-1}}{2^{2\alpha}}\right) \left(\sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k-2\alpha+1}}{k!\Gamma\left(k-\alpha+\frac{1}{2}+1\right)} - \sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k}}{k!\Gamma\left(k+\alpha+\frac{1}{2}\right)} \right)$$
$$= \left(\frac{\lambda^{2\alpha-1}}{2^{2\alpha}}\right) \left(\sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k-(\alpha-\frac{1}{2})-(\alpha-\frac{1}{2})}}{k!\Gamma\left(k-\left(\alpha-\frac{1}{2}\right)+1\right)} - \sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k+(\alpha-\frac{1}{2})-(\alpha-\frac{1}{2})}}{k!\Gamma\left(k+\left(\alpha-\frac{1}{2}\right)+1\right)} \right)$$
$$= \left(\frac{\lambda^{\alpha-\frac{1}{2}}}{2^{\alpha+\frac{1}{2}}}\right) \left(\sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k-(\alpha-\frac{1}{2})}}{k!\Gamma\left(k-\left(\alpha-\frac{1}{2}\right)+1\right)} - \sum_{k=1}^{\infty} \frac{(\lambda/2)^{2k+(\alpha-\frac{1}{2})}}{k!\Gamma\left(k+\left(\alpha-\frac{1}{2}\right)+1\right)} \right),$$

By the series form of $K_{\alpha-\frac{1}{2}}(\lambda)$, we have

$$= \left(\frac{\lambda^{\alpha-\frac{1}{2}}}{2^{\alpha+\frac{1}{2}}}\right) \left(\frac{2\sin\left(\alpha-\frac{1}{2}\right)\pi}{\pi} K_{\alpha-\frac{1}{2}}(\lambda) - \frac{(\lambda/2)^{-\alpha+\frac{1}{2}}}{\Gamma\left(\frac{3}{2}-\alpha\right)} + \frac{(\lambda/2)^{\alpha-\frac{1}{2}}}{\Gamma\left(\alpha+\frac{1}{2}\right)} \right).$$

Hence, we obtain the result. □

proof of Proposition 2.12. Let (σ, ν, γ) and $(\tilde{\sigma}, \tilde{\nu}, \tilde{\gamma})$ be Lévy triplets of $X_{\mathbf{P}}$ and $X_{\mathbf{Q}}$, respectively. Since $\sigma = \tilde{\sigma} = 0$, (2.9) is satisfied. Let

$$k(\alpha, \lambda, x) = \frac{\lambda^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda x)}{x^{\alpha+\frac{1}{2}}}.$$

Then the function $\psi(x) = \ln \frac{d\tilde{\nu}(x)}{d\nu(x)}$ is given by

$$\psi(x) = \ln\left(\frac{\tilde{C}k(\tilde{\alpha}, \tilde{\lambda}_+, x)}{Ck(\alpha, \lambda_+, x)}\right) 1_{x>0} + \ln\left(\frac{\tilde{C}k(\tilde{\alpha}, \tilde{\lambda}_-, x)}{Ck(\alpha, \lambda_-, x)}\right) 1_{x<0},$$

and so

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(e^{\frac{\psi(x)}{2}} - 1\right)^2 \nu(dx) \\ &= \int_0^{\infty} \left(\sqrt{\tilde{C}} k(\tilde{\alpha}, \tilde{\lambda}_+, x)^{1/2} - \sqrt{C} k(\alpha, \lambda_+, x)^{1/2}\right)^2 dx \\ &\quad + \int_{-\infty}^{0} \left(\sqrt{\tilde{C}} k(\tilde{\alpha}, \tilde{\lambda}_-, x)^{1/2} - \sqrt{C} k(\alpha, \lambda_-, x)^{1/2}\right)^2 dx. \end{aligned}$$

If $\alpha < \tilde{\alpha}$, then for $j = 1, 2$, we have

$$\lim_{x\to 0} \frac{\sqrt{\tilde{C}} k(\tilde{\alpha}, \tilde{\lambda}_j, x)^{1/2} - \sqrt{C} k(\alpha, \lambda_j, x)^{1/2}}{x^{-\tilde{\alpha}-\frac{1}{2}}} = \sqrt{\tilde{C} 2^{\tilde{\alpha}-\frac{1}{2}} \Gamma\left(\tilde{\alpha} + \frac{1}{2}\right)}.$$

If $\alpha = \tilde{\alpha}$ but $C < \tilde{C}$, then for $j = 1, 2$, we have

$$\lim_{x\to 0} \frac{\sqrt{\tilde{C}} k(\tilde{\alpha}, \tilde{\lambda}_j, x)^{1/2} - \sqrt{C} k(\alpha, \lambda_j, x)^{1/2}}{x^{-\tilde{\alpha}-\frac{1}{2}}} = (\sqrt{\tilde{C}} - \sqrt{C}) \sqrt{2^{\tilde{\alpha}-\frac{1}{2}} \Gamma\left(\tilde{\alpha} + \frac{1}{2}\right)}.$$

Hence if $\alpha < \tilde{\alpha}$ or $\alpha = \tilde{\alpha}$ and $C < \tilde{C}$, then $\left(e^{\frac{\psi(x)}{2}} - 1\right)^2$ is equivalent to $x^{-2\tilde{\alpha}-1}$ near zero, so it is not integrable.

Suppose $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$. Then we have

$$\psi(x) = \ln\left(\frac{k(\alpha, \tilde{\lambda}_+, x)}{k(\alpha, \lambda_+, x)}\right) 1_{x>0} + \ln\left(\frac{k(\alpha, \tilde{\lambda}_-, x)}{k(\alpha, \lambda_-, x)}\right) 1_{x<0}.$$

We can show that $\lim_{x\to 0} \psi(x) = 0$ and $\lim_{x\to 0} \psi'(x) = 0$. So there is a θ such that $\psi(x) < \theta|x|$ for $x \in [-1, 1]$. Thus

$$\int_{|x|\le 1} \left(e^{\frac{\psi(x)}{2}} - 1\right)^2 \nu(dx) \le \int_{|x|\le 1} \left(e^{\frac{\theta|x|}{2}} - 1\right)^2 \nu(dx) < \infty,$$

and

$$\int_1^\infty \left(e^{\frac{\psi(x)}{2}} - 1\right)^2 \nu(dx) \le \int_1^\infty \tilde{\nu}(dx) + \int_1^\infty \nu(dx) < \infty,$$

and similarly, we can show that

$$\int_{-\infty}^{-1} \left(e^{\frac{\psi(x)}{2}} - 1\right)^2 \nu(dx) < \infty.$$

Therefore, the condition (2.10) holds if and only if $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$.

We have, by Lemma A.5 (1),

$$\begin{aligned}
&\int_0^1 x^{-2\alpha}\left((\tilde{\lambda}_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_\pm x) - (\lambda_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_\pm x)\right) dx \\
&= \int_0^1 x^{-2\alpha}\left((\tilde{\lambda}_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_\pm x) - 2^{\alpha-\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)\right) dx \\
&\quad - \int_0^1 x^{-2\alpha}\left((\lambda_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_\pm x) - 2^{\alpha-\frac{1}{2}}\Gamma\left(\alpha+\frac{1}{2}\right)\right) dx \\
&= \frac{2^{\alpha-\frac{1}{2}}\pi}{\cos(\alpha\pi)} \int_0^1 \left(\sum_{k=1}^\infty \frac{\left(\tilde{\lambda}_\pm x/2\right)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^\infty \frac{\left(\tilde{\lambda}_\pm x/2\right)^{2k+2\alpha+1}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right) x^{-2\alpha} dx \\
&\quad - \frac{2^{\alpha-\frac{1}{2}}\pi}{\cos(\alpha\pi)} \int_0^1 \left(\sum_{k=1}^\infty \frac{(\lambda_\pm x/2)^{2k}}{k!\Gamma\left(k-\left(\alpha+\frac{1}{2}\right)+1\right)} - \sum_{k=0}^\infty \frac{(\lambda_\pm x/2)^{2k+2\alpha+1}}{k!\Gamma\left(k+\alpha+\frac{1}{2}+1\right)}\right) x^{-2\alpha} dx.
\end{aligned}$$

By the Lemma A.5 (2) and the fact $\frac{\pi}{\cos(\alpha\pi)} = \frac{\pi}{\sin(\alpha+\frac{1}{2})\pi} = \Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\alpha\right)$, we obtain

(A.18)

$$\begin{aligned}
&\int_0^1 x^{-2\alpha}\left((\tilde{\lambda}_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_\pm x) - (\lambda_\pm x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_\pm x)\right) dx \\
&= \frac{2^{\alpha-\frac{1}{2}}\pi}{\cos(\alpha\pi)}\left[\frac{-\cos(\alpha\pi)}{2^{\alpha-\frac{1}{2}}\pi}\lambda_\pm^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\tilde{\lambda}_\pm) + \frac{\tilde{\lambda}_\pm^{2\alpha-1}}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)} + \frac{\cos(\alpha\pi)}{2^{\alpha-\frac{1}{2}}\pi}\lambda_\pm^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_\pm) - \frac{\lambda_\pm^{2\alpha-1}}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\right] \\
&= -\left(\tilde{\lambda}_\pm^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\tilde{\lambda}_\pm) - \lambda_p m^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_\pm)\right) + \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}(\tilde{\lambda}_\pm^{2\alpha-1} - \lambda_\pm^{2\alpha-1})
\end{aligned}$$

Providing that the $\alpha = \tilde{\alpha}$ and $C = \tilde{C}$, the condition (2.11) is equal to

$$\begin{aligned}
&\tilde{\mu} + C\left(\frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}\left(\tilde{\lambda}_+^{2\alpha-1} - \tilde{\lambda}_-^{2\alpha-1}\right) - \tilde{\lambda}_+^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\tilde{\lambda}_+) + \tilde{\lambda}_-^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\tilde{\lambda}_-)\right) \\
&\quad - \mu - C\left(\frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{\alpha+\frac{1}{2}}}\left(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1}\right) - \lambda_+^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_+) + \lambda_-^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_-)\right) \\
&= C\int_0^1 x^{-2\alpha}\left((\tilde{\lambda}_+ x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_+ x) - (\lambda_+ x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_+ x)\right) dx \\
&\quad - C\int_0^1 x^{-2\alpha}\left((\tilde{\lambda}_- x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_- x) - (\lambda_- x)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_- x)\right) dx.
\end{aligned}$$

Hence, by the equation (A.18), the condition holds if and only if $\tilde{\mu} = \mu$.

At last, the Lévy triplet (2.15) can be obtained from equations (2.13) in Theorem 2.11 with $\eta = 0$. □

A.4 Proofs of Proposition 2.14

proof of Proposition 2.14. Let $t \in \mathbb{N}$ be fixed, $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfy equations (2.18) and $\xi_t = \varepsilon_t + k$ where k is defined as (2.19). Then $\xi_t \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu + k)$, where

$$C = 2^{\alpha+\frac{1}{2}}\left(\sqrt{\pi}\Gamma(1-\alpha)\left(\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}\right)\right)^{-1}$$

and

$$\mu = -2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\alpha\right)(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1}).$$

For any $\lambda_+, \lambda_- > 0$, put

$$\tilde{\mu}_{\tilde{\lambda}_+,\tilde{\lambda}_-} = \mu + k$$

then $\xi_t \sim MTS(\alpha, C, \tilde{\lambda}_+, \tilde{\lambda}_-, \tilde{\mu}_{\tilde{\lambda}_+,\tilde{\lambda}_-})$ under the probability measure $\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$ whose Radon-Nikodym derivative is given as (2.14) in Proposition 2.12. Under measure $\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$, the variance equals

$$\begin{aligned}
\mathrm{Var}_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) &= \frac{\sqrt{\pi}\Gamma(1-\alpha)}{2^{\alpha+\frac{1}{2}}} C(\tilde{\lambda}_+^{2\alpha-2} + \tilde{\lambda}_-^{2\alpha-2}) \\
&= \frac{\tilde{\lambda}_+^{2\alpha-2} + \tilde{\lambda}_-^{2\alpha-2}}{\lambda_+^{2\alpha-2} + \lambda_-^{2\alpha-2}}
\end{aligned}$$

and the mean equals

$$\begin{aligned}
E_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) &= \tilde{\mu}_{\tilde{\lambda}_+,\tilde{\lambda}_-} + 2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\alpha\right)(\tilde{\lambda}_+^{2\alpha-1} - \tilde{\lambda}_-^{2\alpha-1}) \\
&= k - 2^{-\alpha-\frac{1}{2}} C\Gamma\left(\frac{1}{2}-\alpha\right)(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1} - \tilde{\lambda}_+^{2\alpha-1} + \tilde{\lambda}_-^{2\alpha-1})
\end{aligned}$$

By (2.18) and (2.19), we have $E_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) = 0$ and $\mathrm{Var}_{\mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}}(\xi_t) = 1$. Hence, let $\mathbf{Q}_t = \mathbf{Q}_{\tilde{\lambda}_+,\tilde{\lambda}_-}$. Then $\mathbf{Q}_t$ and $\mathbf{P}_t$ are equivalent and $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$. $\square$