A Modified Tempered Stable Distribution with Volatility Clustering

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Abstract

We first introduce a new variant of the tempered stable distribution, named the modified tempered stable(MTS) distribution and we use it to develop the GARCH option pricing model with MTS innovations. This model allows one to describe some stylized phenomena observed in financial markets such as volatility clustering, skewness, and heavy tails of the return distribution.

1 Introduction

Since Black and Scholes (1973) introduced the pricing and hedging theory for the option market, their model has been the most popular model for option pricing. However, the model which assumes homoskedasticity and lognormality, cannot explain stylized phenomena such as skewness, heavy tails, and volatility clustering of the stock returns, which are observed in stock prices.

To explain the stylized phenomena, Mandelbrot (1963a, 1963b) was the first to use a non-normal Lévy process as asset price process. Hurst, Platen and Rachev (1999) used a model based on stable processes to price options. However, stable distributions have infinite moments of the second or higher order because of the heavy distributional tails. To have more adaptability, a new class of Lévy processes called the tempered stable (TS) process has been introduced under different names including: "truncated Lévy flight" (Koponen (1995)), "KoBoL" process (Boyarchenko and Levendorskiǐ (2000)), and "CGMY" process (Carr et al. (2002)). Subsequently, the Normal Tempered Stable (NTS) distribution has been introduced and applied in finance (Barndorff-Nielsen and Levendorskiǐ (2001), and Barndorff-Nielsen and Shephard (2001)). The NTS distribution is obtained by a time changed Brownian motion with a tempered stable subordinator. In order to obtain a closed form solution of the European option price, the FFT option price method (Carr and Madan (1999) and Lewis (2001)) is used under the assumption of the Markov property. However, the Markov property is often rejected by the empirical evidence as in the case of volatility clustering.

*Svetlozar Rachev's research was supported by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, and the Deutschen Forschungsgemeinschaft.

†Michele Leonardo Bianchi's research was supported by a Ph.D. scholarship of the University of Bergamo.

Autoregressive conditional heteroskedastic (ARCH) models introduced by Engle (1982) and generalized ARCH (GARCH) models by Bollerslev (1986) have become standard tools in empirical finance. The GARCH option pricing models have been developed to price options under the assumption of volatility clustering. GARCH models of Duan (1995) and Heston and Nandi (2000) are remarkable works on the non-Markovian structure of asset returns even though they did not take into account conditional leptokurtosis and skewness. Duan et al. (2004) modified the classical GARCH model by adding jumps to the innovation processes. Furthermore, Menn and Rachev (2005a,2005b) introduced an enhanced GARCH model with innovations which follow the smoothly truncated stable (STS) distribution, which has a finite variance and at the same time allows for conditional leptokurtosis and skewness.

In this paper, we introduce a variant of the tempered stable distributions, called a modified tempered stable (MTS) distribution, and we apply it to the GARCH option pricing model.

The MTS distribution is obtained by taking an α-stable law and multiplying the Lévy measure by a modified Bessel function of the second kind onto each half of the real axis. It is infinitely divisible, has a closed form characteristic function, finite moments of all orders, and its Lévy measure behaves asymptotically like the α-stable distribution near zero and like the $\alpha/2$-TS distribution on the tails.

The GARCH option pricing model presented in this paper follows the method introduced by Menn and Rachev (2005a,2005b). However, instead of STS innovations, we assume that the innovations of the classical GARCH model follow the MTS distribution with zero mean and unit variance, and we are able to describe both leptokurtosis and skewness.. In contrast to the STS distribution, the Laplace transform of a MTS distribution is analytic, therefore it is more tractable. Moreover, it is infinitely divisible and its characteristic function provides a concrete method to find an equivalent martingale measure by applying a general result on density transformations for Levy processes, presented by Sato (1999).

The remainder of this paper is organized as follows: In Section 2 we summarize some features of the TS distributions. Section 3 introduces the MTS distribution. The properties of the MTS distribution, the relation to the NTS distribution, and measure changes of the MTS distributions will be discussed in this section. The GARCH model with MTS innovations is reported in the forth section. Section 5 is a summary of our conclusions.

2 Preliminary

2.1 Tempered Stable Distributions

Before introducing the MTS distribution and the MTS-GARCH model, let us review the tempered stable distribution. It is well known that α-stable distributions have infinite p-th moments for all $p \geq \alpha$. This is due to the fact that its Lévy density decays polynomially. Tempering of the tails with the exponential rate is one choice to ensure finite moments. The Tempered Stable (TS) distribution is obtained by taking a symmetric α-stable distribution and multiplying the Lévy measure with exponential functions on each half of the real axis. Indeed, it is defined in the following:

Definition 2.1. *An infinitely divisible distribution is called a* tempered stable (TS) distribution *with parameter* $(C_1, C_2, \lambda_+, \lambda_-, \alpha)$, *or* α-tempered stable (α-TS), *if its Lévy triplet* (σ^2, ν, γ) *is given by* $\sigma = 0$, $\gamma \in \mathbb{R}$ *and*

$$\nu(dx) = \left(\frac{C_1 e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{C_2 e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0} \right) dx, \tag{2.1}$$

where $C_1, C_2, \lambda_+, \lambda_- > 0$ *and* $\alpha < 2$.

This process was first constructed by Koponen (1995) which he named truncated Lévy flights. In particular, if $C_1 = C_2 = C > 0$, then this distribution is called the CGMY distribution which has been used in Carr et al. (2002) for financial modeling.

In the above definition, λ_+ and λ_- give the tail decay rates, α describes the jumps near zero, and C_1 and C_2 determine the arrival rate of jumps for a given size.

The characteristic function $\phi_{TS}(u)$ for a tempered stable distribution is given by

$$\begin{aligned} \phi_{TS}(u) = \exp(iu\mu &+ C_1\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) \\ &+ C_2\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha)), \end{aligned} \tag{2.2}$$

for some $\mu \in \mathbb{R}$. Moreover, ϕ_{TS} can be extended to the region $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\lambda_-, \lambda_+)\}$.The proof can be found in Carr et al. (2002), Cont and Tankov (2004), and Kim (2005). Using the characteristic function, we obtain cumulants

$$c_m(X) = \frac{d^m}{du^m} \log \phi_{TS}(u)\Big|_{u=0}$$

of all orders.

Proposition 2.2. *Let* X *be a tempered stable distributed random variable whose characteristic function is given by* (2.2). *The cumulant* $c_n(X)$ *of* X *is given by*

$$c_n(X) = \Gamma(n-\alpha)C_1\lambda_+^{\alpha-n} + (-1)^n\Gamma(n-\alpha)C_2\lambda_-^{\alpha-n}, \quad \textit{for} \quad n \in \mathbb{N}, \quad n \geq 2,$$

and $c_1(X) = \mu + \Gamma(1-\alpha)C_1\lambda_+^{\alpha-1} - \Gamma(1-\alpha)C_2\lambda_-^{\alpha-1}$.

The infinitely divisibility of this distribution allows one to construct the corresponding Lévy process.

Definition 2.3. The Lévy process $X = (X_t)_{t\geq 0}$ is said to be a *tempered stable process* if X_1 follows a tempered stable distribution.

The properties of infinite activity and infinite variation are determined by the value of α.

Proposition 2.4. *The tempered stable process is*

1. *of finite activity if* $\alpha < 0$ *and of infinite activity if* $0 < \alpha < 2$.
2. *of finite variation if* $0 < \alpha < 1$ *and of infinite variation if* $1 < \alpha < 2$.

As a generalization of the tempered stable process, the Regular Lévy Process of Exponential type (RLPE) has been introduced by Barndorff-Nielsen and Levendorskiĭ (2001).

Definition 2.5. A Lévy process X with Lévy measure ν is called a *Lévy process of exponential type* $[-\lambda_-, \lambda_+]$ if $\lambda_+, \lambda_- > 0$ and ν satisfies

$$\int_{-\infty}^{-1} e^{\lambda_-|x|}\nu(dx) < \infty \quad \text{and} \quad \int_1^{\infty} e^{\lambda_+ x}\nu(dx) < \infty. \tag{2.3}$$

The Lévy process with Lévy triplet $(0, \nu, \gamma)$ is called a *pure jump* Lévy process.

Definition 2.6. Let $\lambda_+, \lambda_- > 0$ and $\beta \in (0, 2)$. A pure jump Lévy process X with Lévy triplet $(0, \nu, \gamma)$ is called a *RLPE* of type $[-\lambda_-, \lambda_+]$ and order β if its Lévy measure ν satisfies the following two conditions:

1. X is a Lévy process of exponential type $[-\lambda_-, \lambda_+]$.
2. In a neighborhood of zero, a representation $\nu(dx) = f(x)dx$ where f satisfies the condition: there exist $\beta' < \beta$, k and $\mathcal{K}$ such that

$$|f(x) - k|x|^{-\beta-1}| \le \mathcal{K}|x|^{-\beta'-1}, \quad \forall |x| \le 1.$$

Example 2.7. *From* (2.1) *we can show that a tempered stable process is the RLPE of type* $[-\lambda_2, \lambda_1]$ *and order* α.

3 The Model

3.1 The Modified Tempered Stable Distributions

In this section, we introduce a variant of the tempered stable distribution named modified tempered stable (MTS) distribution. The proofs can be found in Kim (2005). The MTS distribution is defined as follows:

Definition 3.1. *An infinitely divisible distribution is said to be an* α-modified tempered stable (α-MTS) or modified tempered stable (MTS) distribution *if its Lévy triplet is given by*

$$\begin{aligned}
\sigma^2 &= 0 \\
\nu(dx) &= C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_-|x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0}\right) dx \\
\gamma &= \mu + \frac{C\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}}\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right) \\
&\qquad - C\lambda_+^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_+) + C\lambda_-^{\frac{\alpha-1}{2}} K_{\frac{\alpha-1}{2}}(\lambda_-),
\end{aligned}$$

where $C > 0$, $\lambda_+, \lambda_- > 0$, $\mu \in \mathbb{R}$, $\alpha \in (-\infty, 2) \setminus \{1\}$ *and* $K_p(x)$ *is the modified Bessel function of the second kind. We denote an MTS distributed random variable* X *by* $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. *The Lévy measure* $\nu(dx)$ *is called the* MTS Lévy measure with parameter $(\alpha, C, \lambda_+, \lambda_-)$.

The definition and properties for the modified Bessel function of the second kind $K_p(x)$ can be found in Andrews (1998). The MTS distribution is obtained by taking a symmetric α-stable distribution with $\alpha \in (0, 2)$ and multiplying the Lévy measure with $(\lambda|x|)^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda|x|)$ on each half of the real axis. The measure can be extended to the case of $\alpha \le 0$. If $\alpha = 1$, then γ may not defined. Hence, we remove it. The following result shows that $\nu(dx)$ is a Lévy measure.

Proposition 3.2. *Let ν be a Borel measure on $\mathbb{R}$ such that $\nu(0) = 0$ and*

$$\nu(dx) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}}1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_-|x|)}{|x|^{\frac{\alpha+1}{2}}}1_{x<0}\right)dx, \tag{3.1}$$

where $C > 0$, $\lambda_+, \lambda_- > 0$, and $\alpha < 2$. Then the measure ν is a Lévy measure on $\mathbb{R}$.

The following result obtained from the asymptotic behavior of the modified Bessel function of the second kind.

Proposition 3.3. *Let*

$$f(x) = C\left(\frac{\lambda_+^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}}1_{x>0} + \frac{\lambda_-^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda_-|x|)}{|x|^{\frac{\alpha+1}{2}}}1_{x<0}\right),$$

where $C > 0$, $\lambda_+, \lambda_- > 0$ and $\alpha \in (0, 2) \setminus \{1\}$. Then

$$f(x) \sim 2^{\frac{\alpha-1}{2}}C\Gamma\left(\frac{\alpha+1}{2}\right)\frac{1}{x^{\alpha+1}}, \quad \text{as } x \to 0, \tag{3.2}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}}C\lambda_+^{\frac{\alpha}{2}}\frac{e^{-\lambda_+ x}}{x^{\frac{\alpha}{2}+1}}, \quad \text{as } x \to \infty, \tag{3.3}$$

$$f(x) \sim \sqrt{\frac{\pi}{2}}C\lambda_-^{\frac{\alpha}{2}}\frac{e^{-\lambda_-|x|}}{|x|^{\frac{\alpha}{2}+1}}, \quad \text{as } x \to -\infty. \tag{3.4}$$

Remark 3.4. *If $\alpha \in (0, 2) \setminus \{1\}$, the Lévy measures of the α-stable, the α-TS and the α-MTS distribution have the same asymptotic behavior at the zero neighborhood. However, the tails of the Lévy measures for the α-MTS distribution are thinner than those of the α-stable and fatter than those of the α-TS distribution.*

The characteristic function of the MTS distribution is given in the following result.

Theorem 3.5. *Let $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. Then the characteristic function of X is given by*

$$\begin{aligned}&\phi_X(u; \alpha, C, \lambda_+, \lambda_-, \mu)\\ &= \exp(iu\mu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-)),\end{aligned}$$

where for $u \in \mathbb{R}$,

$$\begin{aligned}&G_R(u; \alpha, C, \lambda_+, \lambda_-)\\ &= \sqrt{\pi}2^{-\frac{\alpha}{2}-\frac{3}{2}}C\Gamma\left(-\frac{\alpha}{2}\right)\left((\lambda_+^2 + u^2)^{\frac{\alpha}{2}} - \lambda_+^\alpha + (\lambda_-^2 + u^2)^{\frac{\alpha}{2}} - \lambda_-^\alpha\right)\end{aligned}$$

and

$$\begin{aligned}&G_I(u;\alpha,C,\lambda_+,\lambda_-)\\ =&\frac{iuC\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}}\left[\lambda_+^{\alpha-1}F\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda_+^2}\right)-\lambda_-^{\alpha-1}F\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda_-^2}\right)\right],\end{aligned}$$

where F is the hypergeometric function (See Andrews (1998)). Moreover, ϕ_X can be extended to the region $\{z\in\mathbb{C}:|\mathrm{Im}(z)|<\lambda_+\wedge\lambda_-\}$.

Corollary 3.6. *Let $X\sim MTS(\alpha,C,\lambda_+,\lambda_-;\mu)$. Then the Laplace transform of X is given by*

$$E[\exp(uX)]=\exp\left(u\mu+G_R(-iu;\alpha,C,\lambda_+,\lambda_-)+G_I(-iu;\alpha,C,\lambda_+,\lambda_-)\right) \tag{3.5}$$

for $u\in\mathbb{C}$ with $|\mathrm{Re}(u)|<\lambda_+\wedge\lambda_-$.

Using the characteristic function, we obtain the cumulants of all orders.

Proposition 3.7. *Let $X\sim MTS(\alpha,C,\lambda_+,\lambda_-,\mu)$. The cumulants $c_m(X)$ of X are given as follows :*

(3.6)

$$c_m(X)=\begin{cases}\mu & \text{if } m=1\\ 2^{m-\frac{\alpha+3}{2}}\left(\frac{m-1}{2}\right)!C\Gamma\left(\frac{m-\alpha}{2}\right)\left(\lambda_+^{\alpha-m}-\lambda_-^{\alpha-m}\right) & \text{if } m=3,5,7,\cdots\\ 2^{-\frac{\alpha+3}{2}}\sqrt{\pi}\frac{m!}{\left(\frac{m}{2}\right)!}C\Gamma\left(\frac{m-\alpha}{2}\right)\left(\lambda_+^{\alpha-m}+\lambda_-^{\alpha-m}\right) & \text{if } m=2,4,6\cdots\end{cases}$$

Remark 3.8. *Let $X\sim MTS(\alpha,C,\lambda_+,\lambda_-,\mu)$.*

1. *By Proposition 3.7, we obtain the mean, variance, skewness and excess kurtosis of X which are given as follows :*

$$\begin{aligned}E[X]&=c_1(X)=\mu+2^{-\frac{\alpha+1}{2}}C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1}-\lambda_-^{\alpha-1}\right)\\ \mathrm{Var}(X)&=c_2(X)=2^{-\frac{\alpha+1}{2}}\sqrt{\pi}C\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)\\ s(X)&=\frac{c_3(X)}{c_2(X)^{\frac{3}{2}}}=\frac{2^{\frac{\alpha+9}{4}}\Gamma\left(\frac{3-\alpha}{2}\right)\left(\lambda_+^{\alpha-3}-\lambda_-^{\alpha-3}\right)}{\pi^{\frac{3}{4}}C^{\frac{1}{2}}\left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)\right)^{\frac{3}{2}}}\\ k(X)&=\frac{c_4(X)}{c_2(X)^2}=\frac{3\cdot 2^{\frac{\alpha+3}{2}}\Gamma\left(2-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-4}+\lambda_-^{\alpha-4}\right)}{\sqrt{\pi}C\left(\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2}\right)\right)^2}.\end{aligned}$$

2. *Figure 1 illustrates the dependence of skewness $s(X)$ and excess kurtosis $k(X)$ on λ_+ and λ_- when α and C are fixed.*
3. *λ_+ and λ_- control the rate of decay on the positive and negative part, respectively. If $\lambda_+>\lambda_-$ ($\lambda_+<\lambda_-$), then the distribution is skewed to the left (right). Moreover, if $\lambda_+=\lambda_-$, then it is symmetric. Figure 2 illustrates this fact.*
4. *C controls the kurtosis of the distribution. If C increases, then the peakness of the distribution decreases. Figure 3 shows the effect of C.*
5. *Figure 4 shows that as α decreases, the distribution has fatter tails and increased peakness.*

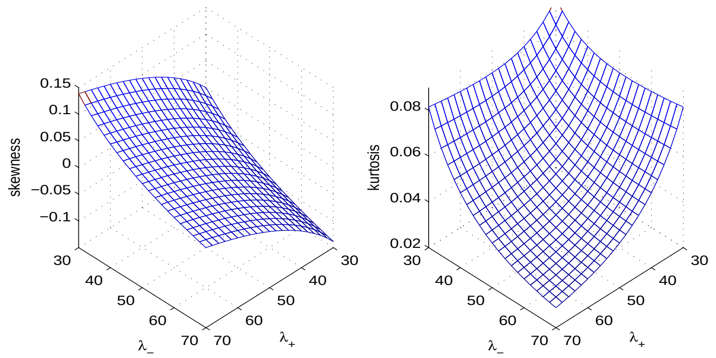

Figure 1: Skewness and Excess Kurtosis of MTS distributions : dependence on λ_+ and λ_-.
Parameters : $\alpha = 1.4$, $C = 0.02$, $\mu = 0$, $t = 1$.

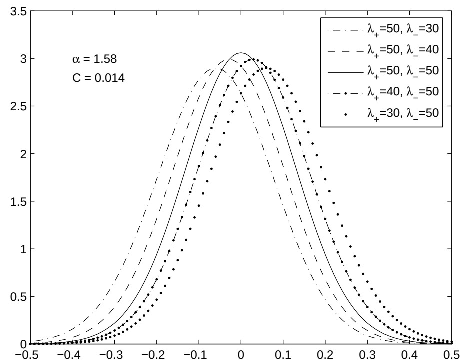

Figure 2: Probability density of the MTS distributions: dependence on λ_+ and λ_-. Parameters : $\lambda_+ = 50$, $\lambda_- \in \{30, 40, 50, 60, 70\}$, $\alpha = 1.58$, $C = 0.02$, $\mu = 0$.

If we put

$$C = 2^{\frac{\alpha+1}{2}} \left(\sqrt{\pi}\Gamma\left(1 - \frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{-1}$$

and

$$\mu = -2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right),$$

then $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$ has zero mean and unit variance. In this case, we say that the random variable X has the *standard MTS distribution*, and denote $X \sim stdMTS(\alpha, \lambda_+, \lambda_-)$.

Since the MTS distribution is infinitely divisible, we can generate a Lévy process called the MTS process.

Definition 3.9. A Lévy process $X = (X_t)_{t \geq 0}$ is said to be a *modified tempered stable(MTS) Lévy process* with parameter $(\alpha, C, \lambda_+, \lambda_-, \mu)$ if $X_1 \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$.

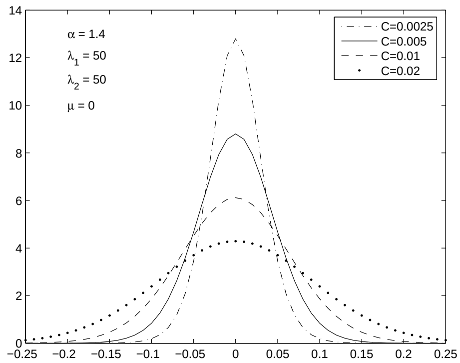

Figure 3: Probability density of the MTS distributions: dependence on C. Parameters : $C \in \{0.0025, 0.005, 0.01, 0.02\}$, $\alpha = 1.4$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

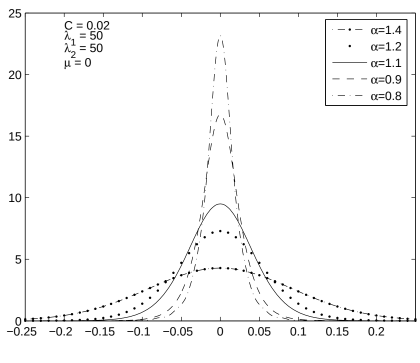

Figure 4: Fat Tail Probability density of the MTS distributions: dependence on α. Parameters : $\alpha \in \{0.8, 0.9, 1.1, 1.2, 1.4\}$, $C = 0.02$, $\lambda_+ = 50$, $\lambda_- = 50$, $\mu = 0$.

Proposition 3.10. *An MTS Lévy process $X = (X_t)_{t\geq 0}$ with parameter $(\alpha, C, \lambda_+, \lambda_-, \mu)$ is an RLPE of type $[-\lambda_-, \lambda_+]$ and order α.*

The path behavior of the MTS process is determined by the parameter α.

Proposition 3.11. *The MTS Lévy process is*

1. *of finite activity if $\alpha \in (-\infty, 0)$, and of infinite activity if $\alpha \in (0, 2)$.*
2. *of finite variation if $\alpha \in (0, 1)$, and of infinite variation if $\alpha \in (1, 2)$.*

3.2 The Exponential Tilting and The Normal Tempered Stable Distribution

Let ν be a Lévy measure. If there exist $\theta \in \mathbb{R}$ such that

$$\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty$$

then the measure $\tilde{\nu}$ defined by $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$ is also a Lévy measure. This transform is called *exponential tilting* of the Lévy measure. The procedure of tilting is also related to the *Esscher transform* (Gerber and Shiu (1994, 1996)) where it can be viewed as tilting but on the level of stochastic processes. Let $\phi(u)$ and $\tilde{\phi}(u)$ be the characteristic functions for the infinitely divisible distributions with Lévy triplets $(0, \nu, \gamma)$ and $(0, \tilde{\nu}, \tilde{\gamma})$ respectively, then we can show that

$$\log\tilde{\phi}(u) = \log\phi(u - i\theta) - \log\phi(-i\theta) + iu\left(\tilde{\gamma} - \gamma - \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx)\right). \tag{3.7}$$

If we set $\lambda = \lambda_+ = \lambda_-$ then we obtain a symmetric MTS distribution. The value of G_I for the symmetric MTS process is always zero, and hence we obtain the characteristic function as

$$\phi(u) = \exp\left(i\mu u + \frac{\sqrt{\pi}C\Gamma\left(-\frac{\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}}\left((\lambda^2 + u^2)^{\alpha/2} - \lambda^\alpha\right)\right). \tag{3.8}$$

If $\beta \in \mathbb{R}$ satisfied $-\lambda < \beta < \lambda$, then $\int_{|x|\geq 1} e^{\beta x}\nu(dx) < \infty$, and hence we can apply exponential tilting to the symmetric MTS distribution. Indeed, the measure

$$\tilde{\nu}(dx) = e^{\beta x}\nu(dx) = Ce^{\beta x}\frac{(\lambda|x|)^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda|x|)}{|x|^{\alpha+1}}dx, \quad -\lambda < \beta < \lambda$$

is a Lévy measure. By (3.7) and (3.8), we obtain the characteristic function $\tilde{\phi}(u)$ of the infinitely divisible distribution with Lévy triplet $(0, \tilde{\nu}, \tilde{\gamma})$ as

$$\tilde{\phi}(u) = \exp\left(i\tilde{\mu}u + \frac{C\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}}\left((\lambda^2 - (\beta + iu)^2)^{\alpha/2} - (\lambda^2 - \beta^2)^{\alpha/2}\right)\right),$$

which is the Normal Tempered Stable (NTS) distribution introduced by Barndoroff-Nielsen and Levendorskii (2001), where $\tilde{\mu}$ is a real number.

3.3 Measure Change On Modified Tempered Stable Distributions

To apply the MTS distributions to no-arbitrage option pricing, we would need to determine an equivalent martingale measure (EMM). In this section, we review a general result of equivalence of measures presented by Sato (1999) and apply it to the MTS distribution.

The following Theorem is a particular case of Theorem 33.1 in Sato (1999).

Theorem 3.12. *Let $(X, \mathbf{P})$ and $(X, \mathbf{Q})$ be two infinitely divisible random variables on $\mathbb{R}$ with Lévy triplet (σ^2, ν, γ) and $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ respectively. Then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if and only if the Lévy triplet satisfies*

$$\sigma^2 = \tilde{\sigma}^2, \tag{3.9}$$

$$\int_{-\infty}^{\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty, \tag{3.10}$$

where $\psi(x) = \log\left(\frac{\tilde{\nu}(dx)}{\nu(dx)}\right)$. If $\sigma^2 = 0$ then

$$\tilde{\gamma} - \gamma = \int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx). \tag{3.11}$$

When $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, the Radon-Nikodym derivative is

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^U \tag{3.12}$$

where $(U, \mathbf{P})$ is an infinitely divisible random variable with Lévy triplet $(\sigma_U^2, \nu_U, \gamma_U)$ given by

$$\begin{cases} \sigma_U^2 = \sigma^2\eta^2 \\ \nu_U = \nu \circ \psi^{-1} \\ \gamma_U = -\dfrac{\sigma^2\eta^2}{2} - \displaystyle\int_{-\infty}^{\infty} (e^y - 1 - y1_{|y|\leq 1})\nu_U(dy) \end{cases}. \tag{3.13}$$

Here η is such that

$$\tilde{\gamma} - \gamma - \int_{|x|\leq 1} x(\tilde{\nu} - \nu)(dx) = \sigma^2\eta$$

if $\sigma > 0$ and zero if $\sigma = 0$.

Since MTS distributions are infinitely divisible, we can apply Theorem 3.12 to obtain the change of measure.

Proposition 3.13. *Let $(X, \mathbf{P})$ and $(X, \mathbf{Q})$ be two MTS distributed random variables on $\mathbb{R}$ with parameters $(\alpha,\ C,\ \lambda_+,\ \lambda_-,\ \mu)$ and $(\tilde{\alpha},\ \tilde{C},\ \tilde{\lambda}_+,\ \tilde{\lambda}_-,\ \tilde{\mu})$, respectively. Then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if and only if $C = \tilde{C}$, $\alpha = \tilde{\alpha}$ and $\tilde{\mu} = \mu$.*

When $\mathbf{P}$ and $\mathbf{Q}$ are equivalent, the Radon-Nikodym derivative is

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = e^U \tag{3.14}$$

where $(U, \mathbf{P})$ *is an infinitely divisible random variable with Lévy triplet* $(0, \nu_U, \gamma_U)$ *given by*

$$\left\{ \begin{array}{l} \nu_U = \nu \circ \psi^{-1} \\ \gamma_U = -\int_{-\infty}^{\infty} (e^y - 1 - y 1_{|y|\le 1})(\nu \circ \psi^{-1})(dy) \end{array} \right. , \tag{3.15}$$

where

$$\psi(x) = \log \left(\frac{\tilde{\lambda}_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_+ x)}{\lambda_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_+ x)} \right) 1_{x>0} - \log \left(\frac{\tilde{\lambda}_-^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\tilde{\lambda}_- x)}{\lambda_-^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda_- x)} \right) 1_{x<0}.$$

4 The MTS-GARCH Model

The MTS-GARCH stock price model is defined over a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ which is constructed as follows: Consider a sequence $(\varepsilon_t)_{t\in\mathbb{N}}$ of iid real random variables on a sequence of probability spaces $(\Omega_t, \mathbf{P}_t)_{t\in\mathbb{N}}$, such that $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ on the space $(\Omega_t, \mathbf{P}_t)$. Next, we define $\Omega := \prod_{t\in\mathbb{N}} \Omega_t$, $\mathfrak{F}_t := \otimes_{k=1}^t \sigma(\varepsilon_k) \otimes \mathfrak{F}_0 \otimes \mathfrak{F}_0 \cdots$, $\mathfrak{F} := \sigma\left(\cup_{t\in\mathbb{N}} \mathfrak{F}_t\right)$, and $\mathbb{P} := \otimes_{t\in\mathbb{N}} \mathbf{P}_t$, where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\varepsilon_k)$ means the σ-algebra generated by ε_k on Ω_k.

We propose the following model for the stock price dynamics:

$$\log \frac{S_t}{S_{t-1}} = r_t - d_t + \lambda_t \sigma_t - g(\sigma_t; \alpha, \lambda_+, \lambda_-) + \sigma_t \varepsilon_t, \quad t \in \mathbb{N}, \tag{4.1}$$

where S_t denotes the price of the underlying asset at time t, r_t and d_t denote the risk free rate and dividend rate for the period $[t-1, t]$, and λ_t is a $\mathfrak{F}_{t-1}$ measurable random variable. S_0 is the present observed price. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is the characteristic exponent of the Laplace transform for the distribution $stdMTS(\alpha, \lambda_+, \lambda_-)$, i.e. $g(x; \alpha, \lambda_+, \lambda_-) = \log(E_{\mathbf{P}_t}[\exp(x\varepsilon_t)])$. The function $g(x; \alpha, \lambda_+, \lambda_-)$ is defined if $x \in (-\lambda_-, \lambda_+)$ and its value can be obtained from (3.5) if $|x| < \lambda_+ \wedge \lambda_-$, and by numerical calculation if $x \in \{x \in (-\lambda_-, \lambda_+) \mid |x| \ge \lambda_+ \wedge \lambda_-\}$. The one period ahead conditional variance σ_t^2 follows a GARCH(1,1) process with a restriction $0 < \sigma_t < \lambda_+$, i.e.

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho, \quad t \in \mathbb{N}, \quad \varepsilon_0 = 0, \tag{4.2}$$

where the coefficients α_0, α_1, and β_1 are non-negative, $\alpha_1 + \beta_1 < 1$, $\alpha_0 > 0$, and $0 < \rho < \lambda_+^2$. Clearly σ_t is $\mathfrak{F}_{t-1}$-measurable and hence the process $(\sigma_t)_{t\in\mathbb{N}}$ is predictable. Moreover, the conditional expectation $E[\hat{S}_t/\hat{S}_{t-1}|\mathfrak{F}_{t-1}]$ equals $\exp(r_t + \lambda_t \sigma_t)$ where $\hat{S}_t = S_t \exp\left(\sum_{k=1}^t d_k\right)$ is the stock price considering reinvestment of the dividends, thus λ_t can be interpreted as the market price of risk.

Remark 4.1. *If* ε_t *equals the standard normal distributed random variable for all* $t \in \mathbb{N}$, g *is to be the Laplace transform of* ε_t *and we ignore the restriction* $\sigma_t < \lambda_+$, *then the model becomes 'the normal GARCH model' introduced by Duan (1995).*

Proposition 4.2. *Let* $t \in \mathbb{N}$ *be fixed and* $\varepsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$ *under* $\mathbf{P}_t$. *Suppose positive real numbers* $\tilde{\lambda}_+$ *and* $\tilde{\lambda}_-$ *satisfy the equation*

$$\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} = \tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}. \tag{4.3}$$

Let

$$k = 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1} - \tilde{\lambda}_+^{\alpha-1} + \tilde{\lambda}_-^{\alpha-1}\right), \tag{4.4}$$

where

$$C = 2^{\frac{\alpha+1}{2}}\left(\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})\right)^{-1}.$$

Then, there is a probability measure $\mathbf{Q}_t$ *equivalent to* $\mathbf{P}_t$*, such that* $(\varepsilon_t + k) \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$*.*

Assumption (A) (i) There exist $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ satisfying equations (4.3) and $\tilde{\lambda}_+ \geq \lambda_+$. (ii) The market price of risk λ_t is given by $\lambda_t = k - (g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) - g(\sigma_t; \alpha, \lambda_+, \lambda_-))/\sigma_t$, for each $0 \leq t \leq T$, where k is defined as (4.4).

Under Assumption (A), let $\mathbf{Q}_t$ be the measure described in Proposition 4.2.

Definition 4.3. *Let* $T \in \mathbb{N}$ *be the time horizon. Define a new measure* $\mathbb{Q}$ *on* $\mathfrak{F}_T$ *equivalent to measure* $\mathbb{P}$*, with Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ *where the density process* $(Z_t)_{0\leq t\leq T}$ *is defined according to*

$$\begin{aligned} &Z_0 \equiv 1, \\ &Z_t := \frac{d(\mathbf{P}_1 \otimes \cdots \otimes \mathbf{P}_{t-1} \otimes \mathbf{Q}_t \otimes \mathbf{P}_{t+1} \otimes \cdots \otimes \mathbf{P}_T)}{d\mathbb{P}} Z_{t-1}, \\ &\quad \text{where} \quad t = 1, 2, \cdots, T. \end{aligned}$$

Lemma 4.4. *The measure* $\mathbb{Q}$ *satisfies the following requirements:*

(a) The discount asset price process $(e^{-r_t}\hat{S}_t)_{1\leq t\leq T}$ *is a* $\mathbb{Q}$*-martingale w.r.t. the filtration* $(\mathfrak{F}_t)_{1\leq t\leq T}$*.*

(b) We have

$$\mathrm{Var}_{\mathbb{Q}}\left(\log\frac{S_t}{S_{t-1}}\bigg|\mathfrak{F}_{t-1}\right) \overset{\text{a.s.}}{=} \mathrm{Var}_{\mathbb{P}}\left(\log\frac{S_t}{S_{t-1}}\bigg|\mathfrak{F}_{t-1}\right), \quad 1 \leq t \leq T$$

(c) The stock price dynamics under $\mathbb{Q}$ *can be written as*

$$\log\frac{S_t}{S_{t-1}} = r_t - d_t - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_t\xi_t, \quad 1 \leq t \leq T$$

where $(\xi_t)_{1\leq t\leq T}$ *is a sequence of real random variables on* Ω_t *satisfying* $\xi_t \sim stdMTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ *under* $\mathbf{Q}_t$ *for* $1 \leq t \leq T$*. The variance process under* $\mathbb{Q}$ *has the form*

$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - k)^2 + \beta_1\sigma_{t-1}^2) \wedge (\lambda_+^2(1-\epsilon)), \; t \in \mathbb{N}, \; \xi_0 = 0.$$

The stock price dynamics under $\mathbb{Q}$ which is stated in Lemma 4.4 (c) is called *the MTS-GARCH risk neutral price process.* The arbitrage free price of a call option with strike price K and maturity T is given by

$$C_t = \exp\left(-\sum_{k=t+1}^{T} r_k\right) E_{\mathbb{Q}}[(S_T - K)^+|\mathfrak{F}_t] \tag{4.5}$$

where the stock price S_T at time T is given by

$$S_T = S_t \exp\left(\sum_{k=t+1}^{T}\left((r_k - d_k) - g(\sigma_t; \alpha, \tilde{\lambda}_+, \tilde{\lambda}_-) + \sigma_k\xi_k\right)\right).$$

5 Conclusion

This paper introduces an alternative class of tempered stable distributions which we call Modified Tempered Stable (MTS) distribution model. It is sufficiently flexible in describing the skewness and kurtosis of asset returns and has all moments finite. The Lévy process derived from the MTS distribution is included in the class of RLPE. Furthermore, we obtain the NTS distribution applying the exponential tilting to the symmetric MTS distribution. Next, we introduced an enhanced GARCH-model, namely the MTS-GARCH model, by applying MTS innovations to the classical GARCH model. As a result, the MTS-GARCH time series model for stock returns explains the volatility clustering phenomenon, the leverage effect, and both conditional skewness and leptokurtosis. The risk neutral measure is obtained by applying a change of measure to the MTS distribution. The MTS-GARCH model can be a more realistic model than the normal-GARCH model.

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