Risk Probability Functionals and Probability Metrics Applied to Portfolio Theory

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Abstract:
In this paper, we investigate the impact of several portfolio selection models based on different tracking error measures, performance measures, and risk measures. In particular, mimicking the theory of ideal probability metrics, we examine ideal financial risk measures in order to solve portfolio choice problems. Thus we discuss the properties of several tracking error measures and risk measures and their consistency with the choices of risk-averse investors. Furthermore, we propose several linearizable allocation problems consistent with a given ordering and we show that most of Gini's measures, at less of linear transformations, are linearizable and coherent risk measures. Finally, assuming elliptical distributed returns, we describe the mean-risk efficient frontier using different parametric risk measures previously introduced.
Keywords: Probability metrics, tracking error measures, stochastic orderings, coherent measures, linearizable optimization problems.

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1. Introduction

The purpose of this paper is to present a unifying framework for understanding the connection between portfolio theory, ordering theory, and the theory of probability metrics. To do so, we discuss the use of different tracking error measures, performance measures, and risk measures consistent with investors' preferences and we examine the computational complexity in deriving optimization problems. More specifically, we begin by analyzing investors' optimal choices coherently with their preferences. Then we discuss the parallelism between probability metric theory and the benchmark tracking problem. Finally, we analyze the use of different measures consistent with stochastic orderings when portfolio returns are elliptically distributed.. It is within his context that we characterize the efficient choices using different parametric risk measures.

In recent years, several papers have proffered alternative definitions of the best risk measure to employ in selecting optimal portfolios. Many studies have observed the non-monotonicity feature of the traditional mean-variance (MV) criterion and proposed alternative formulations to correct the limitations of MV analysis. Maccheroni et al (2005), for example, have demonstrated recently that the non-monotonicity of MV preferences can be alleviated using the relative Gini concentration index. The most notable work on this topic is probably due to Artzner et al. (1999) who suggest a minimal set of properties that a risk measure has to satisfy in order to be a coherent risk measure. These are the axioms of coherency meaning that a coherent measure is a positively homogenous, translation invariant, subadditive, and monotone risk measure.

For evaluating expposure to market risks, Artzner et al. (1999), Martin et al. (2003), and Acerbi (2002) propose, among several coherent risk measures, the conditional Value-at-Risk (CVaR). This measure, also called expected shortfall or expected tail loss, is a linear transformation of the Lorenz curve from which it derives its properties. The Lorenz (1905) curve, first used to rank income inequality, is now used in its absolute form to capture the essential descriptive features of risk and stochastic dominance (see Shorrocks (1983) and, Shalit and Yitzhaki (1994)).

In this paper we demonstrate the links between stochastic dominance (SD) orders and dual SD rules based on Lorenz orders (see Aaberge (2005) and Ogryczak and Ruszczynski (2002a.

2000b)). In this context, we introduce different linearizable portfolio selection models that are consistent with stochastic orders. In particular, we apply a method referred to as FORS probability functionals introduced by Ortobelli et al (2006) to portfolio selection problems.

FORS probability functionals and orderings are strictly linked to the theory of probability metrics. In particular, as shown by Rachev et al (2005) and Stoyanov et al (2006b), a tracking error measure can be associated with each probability metric. Thus we examine and develop ideal financial risk measures that mimick the theory of ideal probability metrics, a theory that started with the fundamental work of Kolmogorov and Kantorovich (see Kalashnikov and Rachev (1988), Kakosyan et al (1987), Rachev and Ruschendorf (1998, 1999), and Rachev (1991)).. Many recent results on risk measures and portfolio literature can be seen as particular cases of the results presented in this paper. For example, a large subclass of spectral measures proposed by Acerbi (2002)) is simply derived from the Lorenz curve. Thus, these measures are coherent and linearizable and can be easily represented with respect to their consistency with dual orders. Among several examples proposed in the theory of probability metrics, we consider Gini-type measures that are derived from the fundamental studies of Gini and his students (see among others, Gini (1912, 1914, 1965). Salvemini (1943, 1957), and Dall'Aglio (1956)). Typically, the extended Gini mean difference (see Shalit and Yitzhaki (1984) and Yitzhaki (1983)) minus the mean is a coherent risk measure because it is a simple derivation from the absolute Lorenz curve. Accordingly, we propose linear portfolio selection models based on these measures. Moreover, in order to account for return distributional tails, that represent the future admissible losses, we generalize the tail extension of Gini mean difference proposed by Ogryczak and Ruszczynski (2002a, 2000b) with the extended Gini mean difference minus the mean being a linearizable coherent risk measure. We also demonstrate that the Gini index of dissimilarity and indexes based on the Lorenz curve define a class of FORS tracking error measures.

We also show how to utilize the FORS type risk measures when the returns are elliptical distributed. In particular, we present a characterization of the efficient frontier as function of parametric FORS type risk measures when unlimited short sales are allowed and returns are elliptically distributed. Finally, we discuss the use of different risk measures in terms of reward-risk performance ratios (see also Rachev et al (2005) and Stoyanov et al (2006a)).

We have organized the paper as follows. In Section 2, we summarize continua and inverse stochastic dominance rules and derive portfolio selection problems based on risk measures consistent with these orders. In Section 3, we describe tracking error measures based on probability metrics and propose tracking error portfolio selection problems based on concentration curves. Portfolio selection problems based on Gini-type risk measures are analyzed in Section 4. In

Section 5, we study elliptical models with FORS risk measures. In Section 6, we discuss reward-risk models based on different risk/reward measures. In the last section of the paper we briefly summarize our results.

2 Continua and Inverse Stochastic Dominance Rules in Portfolio Theory

In this section, we look at the fundamental concepts of continua and inverse stochastic dominance rules (see, among others, Ortobelli et al. (2006) and Levy (1992)) and study the linearizable optimization problems that are consistent with these stochastic orders.

In financial economics, the main stochastic orders used are (1) the first-degree stochastic dominance (FSD) for non-satiable agents, (2) the so-called Rothschild-Stiglitz stochastic dominance concave order (RSD) for risk-averse investors (Rothschild and Stiglitz (1970)), and (3) the second-degree stochastic dominance (SSD) for non-satiable risk-averse investors.

Given two risky assets, *X* strictly dominates *Y* with respect to FSD if and only if for all and every increasing utility function *u*, E(*u*(*X*))≥E(*u*(*Y*)) and a strict inequality holds for some *u*. Stated in terms of distribution functions, *X* FSD *Y* if and only if $F_X(t) = \Pr(X \le t) \le F_Y(t) = \Pr(Y \le t)$ and a strict inequality holds for at least a real *t*.

For non-satiable risk-averters, *X* strictly dominates *Y* with respect SSD, if and only if for all and every increasing, concave utility functions *u*, E(*u*(*X*))≥E(*u*(*Y*)) and a strict inequality holds for some *u*. Expressed in terms of distributions, *X* SSD *Y* if and only if $F_X^{(2)}(t) = \int_{-\infty}^{t} F_X(u)du \le F_Y^{(2)}(t) = \int_{-\infty}^{t} F_Y(u)du$ for all *t* and a strict inequality holds for at least a real *t*.

For the Rothschild-Stiglitz dominance, *X* RSD *Y* if and only if for all and every concave utility functions *u*, E(*u*(*X*))≥E(*u*(*Y*)) and the inequality is strict for some *u*, or equivalently if and only if E(*X*)=E(*Y*) and *X* SSD *Y*.

Fishburn (1976, 1980) pointed out that stochastic dominance rules can be expressed in continuous terms by using the definition of fractional integrals. Thus, we say that *X* dominates *Y* with respect to α stochastic dominance order $X \underset{\alpha}{\ge} Y$ (with $\alpha \ge 1$) if and only if $E\left(u(X)\right) \ge E(u(Y))$ for all utility functions such that

$$u \in U_\alpha = \left\{ u(x) = c - \int_{x^+}^{+\infty} (y-x)^{\alpha-1} dv(y) \middle| c, x \in R;\ \text{where } v \text{ is positive } \sigma-\text{finite measure } \int_{-\infty}^{+\infty} |y|^{\alpha-1} dv(y) < \infty \right\},$$

if and only if for every real t $F_X^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t^-}(t-y)^{\alpha-1}dF_X(y)\le F_Y^{(\alpha)}(t)$. In particular, the derivatives of $u\in U_\alpha$ satisfy the inequalities $(-1)^{k+1}u^{(k)}\ge 0$ where k=1,…, n-1 for the integer n that satisfies $n-1\le\alpha<n$.

Portfolio theory is linked to stochastic dominance theory. Indeed, to select the set of admissible choices that are coherent to a given category of investors, we can consider the direct risk measures $\rho(X)$ associated to random wealth X that are *consistent* with the order relation (i.e., $\rho(X)\le\rho(Y)$ if X dominates Y). Similarly, we can consider reward measures $v(X)$ *isotonic* with an order relation (i.e., $v(X)\ge v(Y)$ if X dominates Y). In particular, the risk measure consistent with FSD is generally called a s*afety-risk measure*. Hence a non-satiable or non-satiable risk-averse investor chooses a portfolio that minimizes the risk measure that is consistent with the FSD or SSD order. Considering that for every $\alpha>1$, $\Gamma(\alpha)F_X^{(\alpha)}(t)=E\left(\left(t-X\right)_+^{\alpha-1}\right)$, then we can easily define portfolio selection models that are consistent with α stochastic dominance order $X\underset{\alpha}{\ge}Y$.

Consider the portfolio problem of optimal allocation $x=[x_1,x_2,...,x_n]'$, between n assets with returns $r=[r_1,...,r_n]'$. No short selling is allowed, (i.e. $x_i\ge 0$). To find the optimal portfolios for investors with a utility function that belongs to U_α and $\alpha>1$, one solves the following optimization problem:

$$\begin{aligned}&\min_x\ \frac{1}{T}\sum_{k=1}^{T}v_k^{\alpha-1}\\&\textit{subject to}\\&E(x'r)\ge m;\ \sum_{j=1}^{n}x_j=1;\ \ x_j\ge 0;\ \ j=1,...,n\\&v_k\ge 0;\ v_k\ge t-\sum_{i=1}^{n}x_ir_{i,k},\ \ k=1,...,T\end{aligned}\tag{1}$$

for a mean greater than m and a given risk-aversion parameter $t\ge\max_{x_i\ge 0}\min_{1\le k\le T}\sum_{i=1}^{n}x_ir_{i,k}$.

In order to get choices consistent with α-bounded RSD, one solves a similar optimization problem as (1) adding the further constraints $v_k\ge\sum_{i=1}^{n}x_ir_{i,k}-t,\ \ k=1,...,T$. In particular, in order to get optimal choices for non-satiable risk-averse investors, we have to solve the previous linear programming (LP) problem corresponding to $\alpha=2$. Furthermore, for $\alpha>2$ the problem (1) is a convex optimization problem and thus it is linearizable.

As an alternative to classic stochastic orders, we can use the dual (also called inverse) representations of stochastic dominance rules that we now present (Shorrocks (1983), Dybvig (1988), Rachev (1991), and Ogryczak and Ruszczynski (2002a, 2002b)):

1) $X\ FSD\ Y \Leftrightarrow F_X^{-1}(p) \geq F_Y^{-1}(p) \quad 0 \leq p \leq 1$

2) $X\ SSD\ Y \Leftrightarrow L_X(p) \geq L_Y(p) \quad 0 \leq p \leq 1$

where $F_X^{-1}(0) = \lim_{p\to 0} F_X^{-1}(p)$ and $\forall p \in (0,1]$, $F_X^{-1}(p) = \inf\{x : \Pr(X \leq x) = F_X(x) \geq p\}$, is the left inverse of F_X. Furthermore, $L_X(p) = \int_0^p F_X^{-1}(t)dt = \sup_u \left\{up - \int_{-\infty}^u F_X(t)dt\right\}$ is the absolute Lorenz curve (or absolute concentration curve) of asset X with respect to distribution function F_X.

Muliere and Scarsini (1989), Yaari (1987), and Ortobelli et al (2006) show how inverse stochastic dominance rules can be extended to continuous terms. Thus, X dominates Y with respect to α inverse stochastic dominance order $X \underset{-\alpha}{\geq} Y$ (with $\alpha \geq 1$) if and only if for every $p \in [0,1]$, $F_X^{(-\alpha)}(p) = \frac{1}{\Gamma(\alpha)}\int_0^p (p-u)^{\alpha-1} dF_X^{-1}(u) \leq F_Y^{(-\alpha)}(p)$ if and only if $\int_0^1 \phi(x)dF_X^{-1}(x) \leq \int_0^1 \phi(x)dF_Y^{-1}(x)$ all utility functions $\phi \in V^\alpha$ where

$$V^\alpha = \Big\{\phi(x) = -\int_{x^+}^1 (s-x)^{\alpha-1} d\tau(s) - k(1-x)^{\alpha-1} \,\Big|\, k \geq 0; \tau \text{ is a } \sigma-\text{finite positive measure s.t. } \forall X \text{ and } \forall p \in (0,1): \left|F_X^{(-\alpha)}(p)\right| < \infty \text{ the function } \left|s-x\right|^{\alpha-1} \text{ is } d\tau(s) \times dF_X^{-1}(x) \text{ integrable in } [0,1]\times[0,1] \Big\}.$$

In the finance literature, the negative p-quantile $F_X^{-1}(p)$ is also called Value-at-Risk (VaR) of X or ($VaR_p(X) = -F_X^{-1}(p)$). It expresses the maximum loss among the best p percentage cases that could occur for a given horizon. In contrast, the absolute concentration curve $L_X(p)$ valued at p shows the mean return accumulated up to the lowest p percentage of the distribution. Both measures $F_X^{-1}(p)$ and $L_X(p)$ have important financial and economic interpretations and are widely used in the recent risk literature. In particular, the negative absolute Lorenz curve divided by probability p is the conditional Value-at-Risk (CVaR) or expected shortfall, expressed as $CVaR_p(X) = \frac{-L_X(p)}{p}$.

This risk measure is coherent in the sense of Artzner et al (1999) because it has the following properties:

1) *Subadditivity* ($CVaR_p(X+Y) \leq CVaR_p(X) + CVaR_p(Y)$);

2) *Positive homogeneity* ($\forall \alpha \geq 0 \;\; CVaR_p(\alpha X) = \alpha CVaR_p(X)$);

3) *Monotonicity* ($\forall X \leq Y \;\;\; CVaR_p(Y) \leq CVaR_p(X)$)

4) *Translation invariance* ($\forall t \in R \;\; CVaR_p(X+t) = CVaR_p(X) - t$).

CVaR is consistent with FSD and SSD stochastic orders. Furthermore, if one uses $L_X(p) = \sup_u \left\{ up - \int_{-\infty}^{u} F_X(t)dt \right\}$ then,

$$CVaR_p(X) = \frac{-L_X(p)}{p} = \inf_u \left\{ u + \frac{1}{p} E\left((-X-u)_+ \right) \right\},$$

where the optimal value *u* is $VaR_p(X) = -F_X^{-1}(p)$ (Pflug (2000)). For a given probability loss *p*, the set of optimal portfolios for non-satiable and risk-averse investors is found by solving the following LP problem:

$$\begin{aligned} & \min_{x,b} \; b + \frac{1}{Tp} \sum_{t=1}^{T} v_t \\ & subject \;\; to \\ & E(x'r) \geq m; \;\; \sum_{j=1}^{n} x_j = 1; \;\; x_j \geq 0; \;\; j = 1,...,n \\ & v_t \geq 0; \;\; v_t \geq -\sum_{i=1}^{n} x_i r_{i,t} - b, \;\; t = 1,...,T \end{aligned} \tag{2}$$

for some given *m* (see, among others, Pflug (2000)). Coherent risk measures using specific functions for the Lorenz curve are easily obtained. In particular, we observe that some classic Gini type measures are coherent measures.

Remark 1 *The following holds:*

1) *For every* $v \geq 1$ *and for every* $\beta \in (0,1)$ *the measure* $\frac{-\Gamma(v+1)}{\beta^v} F_X^{(-(v+1))}(\beta) = \frac{-(v-1)v}{\beta^v} \int_0^\beta (\beta-u)^{v-2} L_X(u)du$ *is a coherent risk measure consistent with* $\underset{-(v+1)}{\geq}$ *order that is linearizable*

2) *For every* $v \geq 1$ *and for every* $\beta \in (0,1)$ *the measure* $\Gamma_{X,\beta}(v) = E(X) - \frac{\Gamma(v+1)}{\beta^v} F_X^{(-(v+1))}(\beta)$ *is consistent with RSD order.*

The proof is given in the appendix.

A typical application of Remark 1 is provided by Acerbi's spectral measures (see Acerbi (2002)). According to his definition, any spectral measure

$$M_{\phi}(X) = -\int_0^1 \phi(u) F_X^{-1}(u) du$$

is a coherent risk measure identified by its risk spectrum ϕ that is an a. e. non-negative decreasing and integrable function such that $\int_0^1 \phi(u) du = 1$. From this definition, it follows that any spectral measure is consistent with FSD. In particular, when $F^{-1}(0) = 0$, $M_{\phi}(X) = \int_0^1 v(x) dF_X^{-1}(x)$ where $v(x) = -\int_x^1 d\tau(s)$ is a specific function that belongs to the set V^1 previously defined and $\tau(s)$ is a probability measure on [0,1] that is absolutely continuous with respect to the Lebesgue measure whose density is given by the decreasing risk spectrum function ϕ. In addition, Acerbi (2002) shows that for any a. e. non-negative, decreasing function $\phi(.)$ and for any *N* i.i.d. realizations $X_1,...,X_N$ of the integrable random variable *X*, the spectral measures $M_{\phi}(X)$ associated to the standardized risk spectrum can be estimated by the consistent statistic:

$$M_{\phi}^{N}(X) = \sum_{i=1}^{N} \frac{VaR_{i/N}(X)\phi(i/N)}{\sum_{k=1}^{N}\phi(k/N)}$$

where $VaR_{i/N}(X)$ denotes the opposite of the *i/N* percentile of *X* (i.e., the opposite of the *i*-th observation of the ordered $X_1,...,X_N$). Furthermore, when the risk spectrum $\phi(.)$ is itself absolutely continuous with respect to the Lebesgue measure on [0,1], we can define the linearizable spectral risk measure as:

$$GM_{\phi}(X) = -\int_0^1 \phi(u) F_X^{-1}(u) du = \int_0^1 \phi'(u) L_X(u) du - \phi(1) E(X).$$

Hence, when all optimal choices are uniquely determined by the mean and the risk measure $GM_{\phi}(X)$ any non-satiable risk-averse investor chooses a portfolio solution by solving the following LP problem:

$$\min_{x, a_1,...,a_{T-1}} \; -\sum_{i=1}^{T-1} \phi'\left(\frac{i}{T}\right)\left(\frac{i}{T} a_i + \frac{1}{T}\sum_{k=1}^{T} v_{k,i}\right) - \phi(1) E(x'r)$$

$$subject \quad to$$

$$v_{t,i} \geq -\sum_{j=1}^{n} x_j r_{j,t} - a_i; \; v_{t,i} \geq 0; \; t = 1,...,T; i = 1,...,T-1$$

$$E(x'r) \geq m; \; \sum_{i=1}^{n} x_j = 1; \; x_j \geq 0; \; j = 1,...,n.$$

for some given mean *m*.

Ortobelli et al (2006) show that all stochastic dominance and inverse stochastic dominance orders are specific FORS orderings. The spectral measures are FORS measures induced by FSD. Let us recall the basic concept of FORS measures and orderings. We call *FORS measure induced by* *order* $\succ$ any probability functional $\mu:\Lambda\times\Lambda\to R$ (where Λ is a space of real-valued random variables defined on the probability space $(\Omega,\Im,P)$) that is consistent with respect to a given order of preferences. For example, that *X* dominates *Y* with respect to a given order of preferences $\succ$ on Λ implies that $\mu(X,Z)\le\mu(Y,Z)$ for a fixed and arbitrary benchmark *Z*. We say that a probability functional μ is a *FORS uncertainty measure* if it is consistent with RSD orders. Hence a probability functional μ is a *FORS risk measure* if it is consistent with risk type orders (for example, $\underset{-\alpha}{>}$, $\underset{\alpha}{>}$). Example of a *FORS risk measure is* $-F_X^{(-\alpha)}(p)$, for a fixed benchmark $p\in(0,1)$ that is induced by $\underset{-\alpha}{>}$ order. Example of *FORS uncertainty measure* $\tilde{\rho}_{t,\alpha}(X)=E\left(|t-X|^{\alpha-1}\right)$ for a fixed benchmark $t\in R$, is induced by α*-RSD* order. Moreover, suppose $\rho_X:[a,b]\to\bar{R}$ is a bounded variation function, for every random variable *X* belonging to a given class Λ and assume that the functional ρ_X is simple (i.e., for every $X,Y\in\Lambda$, $\rho_X=\rho_Y\Leftrightarrow F_X=F_Y$). If, for any fixed $\lambda\in[a,b]$, $\rho_X(\lambda)$ is a FORS risk measure induced by a risk ordering $\succ$, then, we call FORS risk orderings induced by $\succ$ the following new class of orderings defined for every $\alpha>1$, $\forall X,Y\in\Lambda_{(\alpha)}=\left\{X\in\Lambda\,\middle|\,\left|\int_a^b|t|^{\alpha-1}\,d\rho_X(t)\right|<\infty\right\}$

$$X\underset{\succ,\alpha}{FORS}Y\ \text{ iff }\ \rho_{X,\alpha}(u)\le\rho_{Y,\alpha}(u)\quad\forall u\in[a,b]$$

where $\rho_{X,\alpha}(u)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^u(u-t)^{\alpha-1}\,d\rho_X(t) & if\ \alpha>1\\ \rho_X(u) & if\ \alpha=1\end{cases}$.

We call ρ_X the FORS risk measure associated with the FORS ordering of random variables belonging to class Λ. In addition, we say that *X* dominates *Y* in the sense of α FORS uncertainty ordering induced by $\succ$ (written as $X\underset{\succ,unc\ \alpha}{FORS}Y$) if and only if $\int_a^x(x-s)_+^{\alpha-1}\,d\rho_{\pm X}(s)\le\int_a^x(x-s)_+^{\alpha-1}\,d\rho_{\pm Y}(s)$ $\forall x\in[a,b]$ (i.e. when $X\underset{\succ,\alpha}{FORS}Y$ and $-X\underset{\succ,\alpha}{FORS}-Y$).

In the following definition we distinguish classes of FORS measures consistent with a given ordering of preference that satisfy only some of the coherency axioms.

Definition 1 *When the simple probability FORS risk measure* $\rho_X(\lambda)$ *associated with a FORS ordering is positively homogeneous and translation invariant for any given* $\lambda\in[a,b]$, ρ_X *is called a characteristic FORS functional of the associated ordering. If in addition,* $\rho_X(\lambda)$ *is subadditive*

$\forall\lambda \in [a,b]$ *then* $\rho_X(\lambda)$ *is called a coherent FORS functional associated with the underlining ordering.*

In particular, we observe that a spectral measure could itself generate parametrically a coherent FORS functional. As a further example, consider the characteristic functional $\rho_X(\lambda)$ =VaR$_\lambda$(X) that is not coherent (being not subadditive). Instead, the characteristic functionals $\rho_X(\lambda)$ =CVaR$_\lambda$(X) and $g_X(\beta) = \frac{-\Gamma(v+1)}{\beta^v} F_X^{(-(v+1))}(\beta)$ are coherent FORS functional for every v>1 and $\beta \in (0,1)$.

Moreover, some characteristic FORS functionals identify the underlining portfolio distributions only if all risk-returns belong to a particular class of distribution functions. For example, measures $\rho_X(\lambda) = \lambda \sqrt[q]{E\left(\left|X - E(X)\right|^q\right)} - E(X)$ and $\rho_X(\lambda) = \sqrt[\lambda]{E\left(\left|X - E(X)\right|^\lambda\right)} - E(X)$ are simple probability functionals consistent with SSD, assuming that all the admissible choices depend on the mean and a dispersion measure.

3. Probability Distances and Tracking Error Measures

Any probability functional μ is called a *probability distance* with parameter K if it is *positive* and it satisfies the following additional properties:

1) *Identity* $f(X) = f(Y) \Leftrightarrow \mu(X,Y) = 0$;

2) *Symmetry* $\mu(X,Y) = \mu(Y,X)$

3) *Triangular inequality* $\mu(X,Z) \le K\left(\mu(X,Y) + \mu(Y,Z)\right)$ for all admissible random variables *X, Y,* and *Z*

where $f(X)$ identifies some characteristics of the random variable X. If the parameter K equals 1, we have a *probability metric*. We can always define the alternative finite distance $\mu_H(X,Y) = H\left(\mu(X,Y)\right)$, where $H:[0,+\infty) \to [0,+\infty)$ is a non-decreasing positive continuous function such that H(0)=0 and $K_H = \sup_{t>0} \frac{H(2t)}{H(t)} < +\infty$ (see Rachev (1991) for further generalization).

Thus, for any probability metric μ, μ_H is a probability distance with parameter K_H. In this case we distinguish between primary, simple, and compound probability distances/metrics that depend on certain modifications of the identity property (see Rachev (1991)). Compound probability functionals identify the random variable almost surely i.e.: $\mu(X,Y) = 0 \Leftrightarrow P(X = Y) = 1$. Simple probability functionals identify the distribution (i.e., $\mu(X,Y)=0 \Leftrightarrow F_X = F_Y$). Primary probability functionals determine only some random variable characteristics.

In essence, probability metrics can be used as tracking error measures. In solving the portfolio problem with a probability distance we intend to "approach" the benchmark and change the perspective for different types of probability distances. Hence, if the goal is only to control the uncertainty of our portfolio or to limit its possible losses, mimicking the uncertainty or the losses of the benchmark can be done using a primary probability distance. When the objective for our portfolio is to mimic entirely the benchmark, a simple or compound probability distance should be used. On the other hand, using a compound distance can be twofold because in addition to their role as tracking error, we can use them as measures of uncertainty. As a matter of fact, if we apply any compound distance $\mu(X,Y)$ to X and $Y = X_1$ that are i.i.d., then we get:

$$\mu(X,X_1)=0 \Leftrightarrow P(X = X_1)=1 \Leftrightarrow X \textit{ is a constant almost surely.}$$

Thus, we call $\mu(X,X_1)=\mu_I(X)$ an index *concentration measure* derived by the compound distance μ. Similarly, if we apply any compound distance $\mu(X,Y)$ to X and $Y=E(X)$ (either $Y=M(X)$, i.e. the median or a percentile of X, if the first moment is not finite), we get:

$$\mu(X,E(X))=0 \Leftrightarrow P(X = E(X))=1 \Leftrightarrow X \textit{ is a constant almost surely.}$$

Thus, we call $\mu(X,E(X))=\mu_{E(X)}(X)$ a *dispersion measure* derived by the compound distance μ.

Let's consider the following examples of compound distances and the associated concentration/dispersion measures.

***Examples of Probability Compound Metrics*:**

As observed earlier, for each probability compound metric we can always generate a probability compound distance $\mu_H(X,Y)=H(\mu(X,Y))$ with parameter K_H.

L^p ***-metrics***: For every $p \geq 0$ we recall the L^p-metrics: $\mu_p(X,Y)=E\left(|X-Y|^p\right)^{\min(1,1/p)}$; $\mu_\infty(X,Y)=ess\sup|X-Y|=\inf\left\{\varepsilon>0: P\left(|X-Y|>\varepsilon\right)=0\right\}$ $\mu_0(X,Y)=E\left(I_{(X\neq Y)}\right)=P\left(X\neq Y\right)$; the associated concentration measures are $\mu_{I,\infty}(X)=ess\sup|X-X_1|$; $\mu_{I,0}(X)=P\left(X\neq X_1\right)$ $\mu_{I,p}(X)=E\left(|X-X_1|^p\right)^{\min(1,1/p)}$, where X_1 is an i.i.d. copy of *X*; and the associated dispersion measures are the central moments $\mu_{E(X),\infty}(X)=ess\sup|X-E(X)|$; $\mu_{E(X),0}(X)=P\left(X\neq E(X)\right)$, $\mu_{E(X),p}(X)=E\left(|X-E(X)|^p\right)^{\min(1,1/p)}$.

Ky Fan metrics: $k_1(X,Y)=\inf\{\varepsilon>0/P(|X-Y|>\varepsilon)<\varepsilon\}$ and $k_2(X,Y)=E\left(\frac{|X-Y|}{1+|X-Y|}\right)$, the respective concentration measures are $k_{1,I}(X)=\inf\{\varepsilon>0/P(|X-X_1|>\varepsilon)<\varepsilon\}$, $k_{2,I}(X)=E\left(\frac{|X-X_1|}{1+|X-X_1|}\right)$, while the associated dispersion measures are $k_{1,E(X)}(X)=\inf\{\varepsilon>0/P(|X-E(X)|>\varepsilon)<\varepsilon\}$, $k_{2,E(X)}(X)=E\left(\frac{|X-E(X)|}{1+|X-E(X)|}\right)$.

Birnbaum-Orlicz metrics: For every $p\geq 0$ $\Theta_p(X,Y)=\left(\int_{-\infty}^{+\infty}\left(P(X\leq t<Y)+P(Y\leq t<X)\right)^p dt\right)^{\min(1,1/p)}$; $\Theta_0(X,Y)=\int_{-\infty}^{+\infty}I(t)_{[P(X\leq t<Y)+P(Y\leq t<X)\neq 0]}dt$ $\Theta_\infty(X,Y)=\sup_{t\in R}P(X\leq t<Y)+P(Y\leq t<X)$; the associated concentration measures are $\Theta_{I,p}(X)=\left(\int_{-\infty}^{+\infty}\left(2F_X(t)(1-F_X(t))\right)^p dt\right)^{\min(1,1/p)}$, $\Theta_{I,0}(X)=\int_{-\infty}^{+\infty}I(t)_{[F_X(t)\neq 0\wedge F_X(t)\neq 1]}dt$ $\Theta_{I,\infty}(X)=\sup_{t\in R}2F_X(t)(1-F_X(t))$; while the associated dispersion measures are: $\Theta_{E(X),p}(X)=\left(\int_{-\infty}^{E(X)}\left(F_X(t)\right)^p dt+\int_{E(X)}^{+\infty}\left(1-F_X(t)\right)^p dt+\right)^{\min(1,1/p)}$, $\Theta_{E(X),\infty}(X)=\max\left(\sup_{t\geq E(X)}(1-F_X(t));\ \sup_{t<E(X)}F_X(t)\right)$, $\Theta_{E(X),0}(X)=\int_{-\infty}^{+\infty}I(t)_{[F_X(t)\neq 0;t<E(X)]}+I(t)_{[F_X(t)\neq 1;t\geq E(X)]}dt$. □

Generally, any compound probability metric or distance $\mu(X,Y)$ is a particular tracking error measure between *X* and the benchmark *Y*. Even if we consider the compound metric/distance as dispersion/concentration measure $\mu(X,g(X))$ (where *g*(*X*) is either a functional of *X* or an independent copy of *X*), then we should obtain a tracking error measure between *X* and *Y* using $\mu(X-Y,g(X-Y))$. In particular, some of these tracking error type measures (i.e., $\mu(X-Y,g(X-Y))$) recently have been used in the portfolio literature (see Roll (1992), Rachev et al (2005), and Barro and Canestrelli (2004)).

Moreover, even simple probability distances can be used as dispersion measures and tracking error measures, but, generally, not as concentration measures. As a matter of fact, when we apply any simple distance $\mu(X,Y)$ to *X* and $Y=E(X)$ (either $Y=M(X)$, i.e., median or a percentile of *X*, if the first moment is not finite), we get:

$$\mu(X,E(X))=0\Leftrightarrow F_X=F_{E(X)}\Leftrightarrow X \textit{ is a constant almost surely.}$$

Thus, we call $\mu(X,E(X))=\mu_{E(X)}(X)$ a *dispersion measure* derived by the simple distance μ. As for compound metrics, we can generate a simple probability distance $\mu_H(X,Y)=H\left(\mu(X,Y)\right)$ with parameter K_H for any simple probability metric $\mu(X,Y)$. Let's consider the following examples of simple metrics and the associated dispersion measures.

***Examples of Simple Probability Metrics*:**

Kolmogorov metric: One of the most used metric in the literature (also called *uniform metric*) is the Kolmogorov metric given by:

$$KS(X,Y)=\sup_{t\in R}\left|F_X(t)-F_Y(t)\right| \text{ and } KS_{E(X)}(X)=\Theta_{E(X),\infty}(X)=\max\left(\sup_{t<E(X)}F_X(t);\ \sup_{t\geq E(X)}(1-F_X(t))\right).$$

Prokhorov metric: $\pi(X,Y)=\inf\left\{\varepsilon>0/P(X\in A)\leq P(Y\in A^{\varepsilon})+\varepsilon\quad \forall A\in \mathrm{B}_R\right\}$ where A is open Borel measurable set and $A^{\varepsilon}=\left\{x\in R/\exists y\in A:\left|x-y\right|<\varepsilon\right\}$. Its associated dispersion measure is $\pi_{E(X)}(X)=\inf\left\{\varepsilon>0/P(X\in A)\leq P(E(X)\in A^{\varepsilon})+\varepsilon\quad \forall A\in \mathrm{B}_R\right\}$.

Gini-Kantorovich metric: For every $p\geq 0$, we consider

$$GK_p\left(X,Y\right)=\left[\int_{-\infty}^{+\infty}\left|F_X\left(t\right)-F_Y\left(t\right)\right|^p dt\right]^{\min(1/p,1)},\qquad GK_0(X,Y)=\left(\int_{-\infty}^{+\infty}I_{[F_X(t)\neq F_Y(t)]}dt\right)$$ and

$GK_{\infty}(X,Y)=ess\sup\left|F_X-F_Y\right|$ and the associated dispersion measures are

$$GK_{E(X),q}(X)=\Theta_{E(X),q}(X)=\left(\int_{-\infty}^{E(X)}\left(F_X(t)\right)^q dt+\int_{E(X)}^{+\infty}\left(1-F_X(t)\right)^q dt\right)^{\min(1/q,1)},$$

$$GK_{E(X),0}(X)=\Theta_{E(X),0}(X)=\int_{-\infty}^{+\infty}I(t)_{[F_X(t)\neq 0;t<E(X)]}+I(t)_{[F_X(t)\neq 1;t\geq E(X)]}dt,$$

$GK_{E(X)\infty}(X)=KS_{E(X)}(X)=\Theta_{E(X),\infty}(X)=\max\left(\sup_{t<E(X)}F_X(t);\ \sup_{t\geq E(X)}\left(1-F_X(t)\right)\right)$. A generalization of Gini-Kantorovich metric is the following Generalized Zolotarev metric (see Rachev (1991)).

Generalized Zolotarev metric: For every $q\geq 0$, we consider the Generalized Zolotarev's metric***:***

$$GZM(X,Y,q,\alpha)=\left(\int_a^b\left|F_X^{(\alpha)}(t)-F_Y^{(\alpha)}(t)\right|^q dt\right)^{\min(1/q,1)},\qquad GZM(X,Y,0,\alpha)=\left(\int_a^b I_{[F_X^{(\alpha)}(t)\neq F_Y^{(\alpha)}(t)]}dt\right),$$

$GZM(X,Y,\infty,\alpha)=ess\sup\left|F_X^{(\alpha)}-F_Y^{(\alpha)}\right|$ and the associated dispersion measures are

$$GZM(X,E(X),q,\alpha)=\left(\int_a^{E(X)}\left|F_X^{(\alpha)}(t)\right|^q dt+\int_{E(X)}^b\left|F_X^{(\alpha)}(t)-\frac{(t-E(X))^{\alpha-1}}{\Gamma(\alpha)}\right|^q dt\right)^{\min(1/q,1)},$$

$$GZM(X,E(X),0,\alpha)=\int_a^{E(X)} I_{[F_X^{(\alpha)}(t)\neq 0]}dt+\int_{E(X)}^b I_{\left[F_X^{(\alpha)}(t)\neq\frac{(t-E(X))^{\alpha-1}}{\Gamma(\alpha)}\right]}dt,$$

$GZM(X,E(X),\infty,\alpha)=\max\left(\sup_{t<E(X)} F_X^{(\alpha)}(t);\ \sup_{t\geq E(X)}\left(\left|\frac{(t-E(X))^{\alpha-1}}{\Gamma(\alpha)}-F_X^{(\alpha)}(t)\right|\right)\right)$. Note that any time $X\underset{\alpha}{\overset{b}{\geq}}Y$ then $\int_a^b\left|F_X^{(\alpha)}(t)-F_Y^{(\alpha)}(t)\right|^q \operatorname{sgn}\left(F_X^{(\alpha)}(t)-F_Y^{(\alpha)}(t)\right)dt\leq 0$. Therefore the intuition suggests to extend the generalized Zolotarev's metrics introducing an analogous metric that we call the *FORS tracking error metric*.

FORS tracking error metrics: Let us consider the functional $\rho_{X,\alpha}$ associated with an α FORS order. For every $q\geq 0$ and $\alpha>0$

$FORS_{q,\alpha}(X,Y)=\left(\int_a^b\left|\rho_{X,\alpha}(t)-\rho_{Y,\alpha}(t)\right|^q dt\right)^{\min(1/q,1)}$, $FORS_{0,\alpha}(X,Y)=\int_a^b I_{[\rho_{X,\alpha}(t)\neq\rho_{Y,\alpha}(t)]}dt$, and $FORS_{\infty,\alpha}(X,Y)=ess\sup\left|\rho_{X,\alpha}-\rho_{Y,\alpha}\right|$. Similarly, we describe the associated dispersion measures whose definition depend on the definition of the functional $\rho_{X,\alpha}$. Clearly, any time $X\underset{\succ,\alpha}{FORS}Y$, then $\int_a^b\left|\rho_{X,\alpha}(t)-\rho_{Y,\alpha}(t)\right|^q \operatorname{sgn}\left(\rho_{X,\alpha}(t)-\rho_{Y,\alpha}(t)\right)dt\leq 0$. □

In particular, to the portfolio problem we get the following definition**.**

Definition 2 *Consider a frictionless economy where a benchmark asset with return* r_Y *and* $n\geq 2$ *risky assets with returns* $r=[r_1,...,r_n]'$ *are traded. Let* $\rho_X:[a,b]\to R$ *be a FORS measure associated with a FORS risk ordering defined over any admissible portfolio of returns X=x'r and over the return Y=*r_Y*. Then we define for any v the tracking error measures*:

$$\rho_{x'r,Y}(v)=\left(\int_a^b\left|\rho_{x'r}(\lambda)-\rho_{r_Y}(\lambda)\right|^v d\lambda\right)^{\min(1/v,1)},$$

$$\rho_{x'r,Y}^{dsr}(v)=\left(\int_a^b\left(\max\left(\rho_{x'r}(\lambda)-\rho_{r_Y}(\lambda),0\right)\right)^v d\lambda\right)^{\min(1/v,1)},$$

that we call portfolio FORS tracking error measures.

In order to limit the computational complexity of the portfolio problem, one uses mostly primary metrics. These are considered choices consistent with metrics limiting dispersion or losses while maximizing expected wealth. Typically, we can think of some metrics valued exclusively on the distributional tail. In particular, mimicking the previous consideration we propose the following definition.

Definition 3 *Consider a frictionless economy where a benchmark asset with return* r_Y *and* $n \geq 2$ *risky assets with returns* $r = [r_1, ..., r_n]'$ *are traded. Let* $\rho_X : [a,b] \to R$ *be a primary FORS risk measure associated to a given FORS risk ordering that identify univocally the distributional tail i.e.*

$$\rho_X(z) = \rho_Y(z) \ \forall z \in [a,b] \Leftrightarrow F_X(x) = F_Y(x) \ \forall x \leq t \text{ for a given } t \Leftrightarrow F_X^{-1}(u) = F_Y^{-1}(u) \ \forall u \leq F_X(t) = p$$

and it is defined over any admissible portfolio of returns $X=x'r$ *and over the return* $Y= r_Y$ *.*

Then, we call $\rho_X(\lambda)$ *p-tail FORS risk measure of the portfolio X. In addition, we define*

$$\rho_{x'r,Y}(v) = \left(\int_a^b \left| \rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda) \right|^v d\lambda \right)^{\min(1/v,1)}, \ \rho_{x'r,Y}^{dsr}(v) = \left(\int_a^b \left(\max\left(\rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda), 0 \right) \right)^v d\lambda \right)^{\min(1/v,1)},$$

that we call portfolio tail FORS tracking error measures.

Typical examples of *p*-tail FORS risk measures will be considered in the following sections. Next we propose some possible portfolio problems based on linearizable tracking error measures.

3.1 Traking Error Measures Based on Concentration Curves

Two alternative ways of using the absolute Lorenz curve consist either in measuring the distance between two random portfolios or alternatively in minimizing the downside risk relative to a given benchmark *Y*.

Consider the following two FORS tracking error measures for a given weight *q*:

$$L_{X,Y}(q) = \left(\int_0^1 \left| L_X(p) - L_Y(p) \right|^q dp \right)^{\min(1/q,1)} = \left(\int_0^1 p^q \left| CVaR_p(Y) - CVaR_p(X) \right|^q dp \right)^{\min(1/q,1)},$$

and

$$L_{X,Y}^{dsr}(q) = \left(\int_0^1 \left(\max\left(L_Y(p) - L_X(p), 0 \right) \right)^q dp \right)^{\min(1/q,1)} = \left(\int_0^1 p^q \left(\max\left(CVaR_p(X) - CVaR_p(Y), 0 \right) \right)^q dp \right)^{\min(1/q,1)}.$$

The absolute Lorenz curve can be determined by solving a LP problem. If one assumes equi-probable scenarios for random portfolios *X* and *Y*, minimizing the two measures, i.e. $L_{X,Y}(1)$ and

$L_{X,Y}^{dsr}(1)$, for q=1 leads to a LP problem To minimize $L_{X,Y}(q)$, we minimize the function

$$\frac{1}{T}\sum_{t=1}^{T} z_t^q \text{ subject to}$$

$$z_t \pm \sum_{i=1}^{t}\left(\frac{i}{T}(a_i - b_i) + \frac{1}{T}\sum_{k=1}^{T}\left(v_{k,i} + u_{k,i}\right)\right) \geq 0 \text{ for } t = 1,...,T$$

$$v_{t,i} \geq 0,\ u_{t,i} \geq 0,\quad v_{t,i} \geq -\sum_{j=1}^{n} x_j r_{j,t} - a_i,\ \ u_{t,i} \geq -y_t + b_i,\ \ t,i = 1,...,T;$$

where $r_t = [r_{1,t},...,r_{n,t}]'$ is the vector of returns at time *t*, y_t is the observation at time *t* of the random variable *Y*, and the optimal values -a_i and b_i are the $\frac{i}{T}$-*th* percentiles of the portfolio *x'r* and *Y*, respectively.

Similarly, minimizing the function $L_{X,Y}^{dsr}(q)$ leads to minimizing the function $\frac{1}{T}\sum_{t=1}^{T} z_t^q$ subject to $z_t - \sum_{i=1}^{t}\left(\frac{i}{T}(b_i - a_i) + \frac{1}{T}\sum_{k=1}^{T}\left(u_{k,i} + v_{k,i}\right)\right) \geq 0$ $z_t \geq 0$, for $t = 1,...,T$ $v_{t,i} \geq 0,\ u_{t,i} \geq 0,$ $v_{t,i} \geq -\sum_{j=1}^{n} x_j r_{j,t} + a_i,\ \ u_{t,i} \geq -y_t - b_i,\ \ t,i = 1,...,T$.

By choosing a specific benchmark, these measures could satisfy different properties that could have a different impact on the portfolio selection problem (see, among others, Szego (2004) and Biglova et al (2004)). Other examples of FORS tracking error measures and risk measures consistent with FORS orderings are those based on Gini type measures which are discussed in the next section.

4. Gini-Type Measures and the Portfolio Selection Problem

In this section we propose some new risk measures related to the fundamental work of Gini (1912, 1914). For this reason we will call all these measures *Gini-type risk measures*.

4.1 Gini Mean Difference and Extensions

Gini's mean difference (GMD) is twice the area between the absolute Lorenz curve and the line joining the origin with the mean located on the right boundary vertical. Many representations for GMD exist. We report here the most used ones[1]:

$$\Gamma_X(2) = 2L_{X,E(X)}(1) = E(X) - 2\int_0^1 L_X(u)du = E(X) - 2\int_0^1 (1-u)F_X^{-1}(u)du\,, \qquad (3)$$

$$\Gamma_X(2) = E(X) - 2E\left(X(1-F_X(X))\right) = -2\operatorname{cov}\left(X,\left(1-F_X(X)\right)\right), \qquad (4)$$

$$\Gamma_X(2) = E\left(\left|X_1 - X_2\right|\right) = E(X) - E(\min(X_1, X_2))\,, \qquad (5)$$

where X_1 and X_2 are two independent copies of X. GMD depends on the spread of the observations among themselves and not on the deviations from some central value. Consequently, this measure links location with variability, two properties that Gini (1912) himself argue are distinct and do not depend on each other.

While the Gini index, i.e. the ratio GMD/E(X),[2] has been used for the past 80 years as a measure of income inequality, the interest in GMD as a measure of risk in portfolio selection is relatively recent (Yitzhaki (1982) and Shalit and Yitzhaki (1984)). On the other hand, the estimator of GMD that presents the best characteristics from a computational point of view, is obtained from formula (5)

$$\hat{\Gamma}_X(2) = \frac{1}{T(T-1)} \sum_{k=1}^{T} \sum_{t>k}^{T} \left|X_t - X_k\right|,$$

where X_t is the t-th observation of the random wealth X. (See Rao Jammalamadaka and Janson (1986) and Rachev (1993) for the asymptotic properties of all the above U-statistic deriving by concentration measures). Therefore, when the returns are uniquely determined by the mean and GMD=$\Gamma_X(2)$, all risk averters will choose a solution for some real m of the following linear optimization problem:

$$\begin{aligned}
&\min_x \sum_{k=1}^{T} \sum_{t>k}^{T} y_{t,k} \\
&s.t. \\
&y_{t,k} + \sum_{i=1}^{n} x_i (r_{i,t} - r_{i,k}) \geq 0 \quad for\ t > k = 1,\ldots,T \qquad (6) \\
&y_{t,k} - \sum_{i=1}^{n} x_i (r_{i,t} - r_{i,k}) \geq 0 \quad for\ t > k = 1,\ldots,T \\
&E(x'r) \geq m;\ \sum_{j=1}^{n} x_j = 1;\ \ x_j \geq 0;\ \ j = 1,\ldots,n
\end{aligned}$$

[1] See Yitzhaki (1999) for a complete list.

[2] In the income inequality literature, the Gini index is the area between the relative Lorenz curve and the 45° line expressing complete equality

where $r_{j,k}$ is the *k*-th observation of *j*-th asset. Introduced by Donalson and Weymark (1980, 1983), the extended GMD takes into account degree of risk aversion as reflected by the parameter *v*. Yitzhaki (1983) showed that this index can be expressed as a function of the Lorenz curve. We present here the most used of the many representations for the extended GMD:

$$\Gamma_X(v) = E(X) - v\int_0^1 (1-u)^{v-1} F_X^{-1}(u)du = E(X) - v(v-1)\int_0^1 (1-u)^{v-2} L_X(u)du\,, \tag{7}$$

$$\begin{aligned}\Gamma_X(v) &= E(X) - v\int_{-\infty}^{+\infty} (1-F_X(x))^{v-1} x dF_X(x) = \\ &= E(X) - vE\left(X(1-F_X(X))^{v-1}\right) = -v\operatorname{cov}\left(X, (1-F_X(X))^{v-1}\right)\end{aligned}, \tag{8}$$

From this definition, it follows that the measures $\Gamma_X(v) - E(X) = -\Gamma(v+1)F_X^{(-(v+1))}(1)$ which characterize the previous Gini FORS orderings are particular spectral measures. Interest in the potential applications to portfolio theory of GMD and its extension has been fostered by Yitzhaki (1983, 1998) and Shalit and Yitzhaki (1984, 2005), who have explained the financial insights of these measures. Moreover, as observed previously, all risk measures $\Gamma_X(v) - E(X)$ are coherent for every *v*>1. In addition, when assuming equally probable scenarios, one can easily linearize the associated portfolio problems. Therefore, if all optimal choices are uniquely determined by the mean and the dispersion $\Gamma_X(v)$, all risk-averse investors will choose a portfolio with a mean equal or greater than *m* that solves the LP problem:

$$\begin{aligned}&\min_{x,\alpha_1,\dots,a_{T-1}} \sum_{i=1}^{T-1}\left(1-\frac{i}{T}\right)^{v-2}\left(\frac{i}{T}a_i + \frac{1}{T}\sum_{k=1}^{T} v_{k,i}\right)\\ &\qquad \textit{subject to}\\ &v_{t,i} \geq -\sum_{j=1}^{n} x_j r_{j,t} - a_i;\ v_{t,i} \geq 0;\ t=1,\dots,T; i=1,\dots,T-1\\ &E(x'r) \geq m;\ \sum_{i=1}^{n} x_j = 1;\ \ x_j \geq 0;\ \ j=1,\dots,n.\end{aligned} \tag{9}$$

Thus, when *v*=2 the optimization problem (9) is an alternative LP optimization problem that can determine the optimal choices of all risk-averse investors.

4.2 Gini's Index of Dissimilarity

To measure the degree of difference between two random variables, Gini (1914) introduced the *index of dissimilarity*. The index properly measures the distance between two variates and has been intensively used in mass transportation problems (see, among others, Rachev (1991) and Rachev and Ruschendorf (1998, 1999)). Many researchers (see, among others, Gini (1951, 1965), Salvemini (1943, 1957), and Dall'Aglio (1956)) devoted considerable effort in obtaining the explicit expressions for this measure of discrepancy, its generalizations, and properties.

We present here some of the many representations of Gini's index of dissimilarity:

$$G_{X,Y}(1) = \int_{-\infty}^{+\infty} \left|F_X(x) - F_Y(x)\right| dx = \int_0^1 \left|F_X^{-1}(u) - F_Y^{-1}(u)\right| du \tag{10}$$

$$G_{X,Y}(1) = \inf\left\{E_F\left(\left|\tilde{X} - \tilde{Y}\right|\right) / F \in \Im(F_X, F_Y)\right\} = E_{\tilde{F}}\left(\left|\tilde{X} - \tilde{Y}\right|\right) \tag{11}$$

where $\tilde{X} = F_X^{-1}(U)$, $\tilde{Y} = F_Y^{-1}(U)$ and U is an uniform (0,1), $\tilde{F}(x,y) = \min\left(F_X(x), F_Y(y)\right)$ is the Hoeffding-Frechet bound of the class of all bivariate distribution functions $\Im(F_X, F_Y)$ with marginals F_X and F_Y (see Rachev (1991)). In portfolio theory, this risk measure changes with respect to the chosen benchmark. For example, when mean E(X) is used as benchmark Y, the index of dissimilarity is the mean absolute deviation of X that is a dispersion measure consistent with Rothschild-Stiglitz stochastic order. However, when we use the upper stochastic bound of the market as benchmark Y, Gini's distance is a safety-risk measure consistent with first stochastic dominance order (see Ortobelli and Rachev (2001)). On the other hand, Gini index of dissimilarity can be used to measure the degree of difference between the portfolio and a market index. This is the classical tracking error problem focused on minimizing the portfolio deviation from a benchmark. The index of dissimilarity can also be extended for a given weight v to provide the *extended Gini index* of dissimilarity:

$$G_{X,Y}(v) = \left(\int_0^1 \left|F_X^{-1}(u) - F_Y^{-1}(u)\right|^v du\right)^{\min(1/v,1)} \tag{12}$$

Therefore, when we consider N equally probable scenarios X and Y, an estimator of the extended Gini index of dissimilarity is obtained as:

$$\hat{G}_{X,Y}(v) = \left(\frac{1}{N}\sum_{k=1}^{N} \left|VaR_{k/N}(Y) - VaR_{k/N}(X)\right|^v\right)^{\min(1/v,1)}. \tag{13}$$

If, on the other hand, we are interested in minimizing the downside risk with respect to a given benchmark Y, the following tracking error measure can be used:

$$G_{X,Y}^{dsr}(v) = \left(\int_0^1 \left(\max(F_Y^{-1}(u) - F_X^{-1}(u), 0)\right)^v du\right)^{\min(1/v,1)} \tag{19}$$

Unfortunately, portfolio optimization problems with these tracking error measures $G_{X,Y}(v)$, $G_{X,Y}^{dsr}(v)$ are not easily linearizable in most cases. Further extensions to the Gini index of dissimilarity can be found by minimizing the expected value of convex positive functions (or quasi-antitone functions) of $\tilde{X}-\tilde{Y}$ with respect to all admissible joint bivariate distributions (see Cambanis et al (1976), Kalashnikov and Rachev (1988), Rachev (1991), and Rachev et al (2005)). However, among many FORS type tracking error measures, we can use those based on the spectral measures $M_{\phi}(X)=-\int_0^1 \phi(u)F_X^{-1}(u)du$. Thus, we identify the class of spectral tracking error measures:

$$G_{X,Y}(\phi,v)=\left(\int_0^1 \phi^v(u)\left|F_X^{-1}(u)-F_Y^{-1}(u)\right|^v du\right)^{\min(1/v,1)} \quad \text{and}$$

$$G_{X,Y}^{dsr}(\phi,v)=\left(\int_0^1 \phi^v(u)\left(\max(F_Y^{-1}(u)-F_X^{-1}(u),0)\right)^v du\right)^{\min(1/v,1)}.$$

In particular, by using spectral FORS type measures $GM_{\phi}(X)=\int_0^1 \phi'(u)L_X(u)du-\phi(1)E(X)$, we obtain the following tracking error measures based on Lorenz curves:

$$G_{X,Y}(\phi,1)=\int_0^1 \left|\phi'(u)\left(L_X(u)-L_Y(u)\right)-\phi(1)\left(E(X)-E(Y)\right)\right|du$$

$$G_{X,Y}^{dsr}(\phi,1)=\int_0^1 \left(\max\left(\phi'(u)\left(L_X(u)-L_Y(u)\right)-\phi(1)\left(E(X)-E(Y)\right),0\right)\right)du,$$

that are easily linearizable.

4.3 Tail Gini Measures

To capture the downside risk of portfolios, Biglova et al (2004), among others, propose several tail risk measures. Typically, the Lorenz curve is a tail measure as it is a linear function of CVaR. Alternatively, we can define tail measures using the Lorenz type measures for some $p\in[0,1]$:

$$L_{X,Y}(q,p)=\left(\int_0^p \left|L_X(u)-L_Y(u)\right|^q du\right)^{\min(1/q,1)} \tag{14}$$

$$L_{X,Y}^{dsr}(q,p)=\left(\int_0^p \left(\max\left(L_Y(u)-L_X(u),0\right)\right)^q du\right)^{\min(1/q,1)} \tag{15}$$

By minimizing these tail measures, we obtain LP problems when q=1 and equally probable scenarios are considered. Using Gini measures, Ogryczak and Ruszczynski (2002a, 2000b) propose the tail Gini measure for a given β:

$$\Gamma_{X,\beta}(2)=\frac{2}{\beta^2}\int_0^{\beta}\left(E(X)u-L_X(u)\right)du=E(X)-\frac{2}{\beta^2}\int_0^{\beta}\left(\beta-u\right)F_X^{-1}(u)du=E(X)-\frac{2}{\beta^2}\int_0^{\beta}L_X(u)du$$

Ogryczak and Ruszczynski's analysis can also be developed to the extended GMD by using the tail measure:

$$\Gamma_{X,\beta}(v) = E(X) - \frac{v}{\beta^v}\int_0^\beta (\beta - u)^{v-1} F_X^{-1}(u)du = E(X) - \frac{v(v-1)}{\beta^v}\int_0^\beta (\beta - u)^{v-2} L_X(u)du \quad (16)$$

$$\Gamma_X(v) = E(X) - v\int_0^1 (1-u)^{v-1} F_X^{-1}(u)du = E(X) - v(v-1)\int_0^1 (1-u)^{v-2} L_X(u)du \quad (17)$$

$$\begin{aligned}\Gamma_X(v) &= E(X) - v\int_{-\infty}^{+\infty} (1-F_X(x))^{v-1} x dF_X(x) = \\ &= E(X) - vE\left(X(1-F_X(X))^{v-1}\right) = -v\operatorname{cov}\left(X,(1-F_X(X))^{v-1}\right)\end{aligned} \quad (18)$$

for some $\beta \in [0,1]$. As a result of Remark 1, all measures $\Gamma_{X,\beta}(v) - E(X) = \frac{-\Gamma(v+1)}{\beta^v} F_X^{(-(v+1))}(\beta)$ are coherent that can be linearized by considering equally probable scenarios.

Assuming equally probable T scenarios and $\beta = \frac{s}{T}$ minimizing the risk $\Gamma_{x'r,\beta}(v) - E(x'r)$ of portfolio $x'r$ with a mean equal or greater than m is equivalent to solving the LP problem:

$$\begin{gathered}\min_{x,b_1,\dots,b_{s-1}} \frac{v(v-1)}{\beta^v T}\sum_{i=1}^{s-1}\left(\frac{s-i}{T}\right)^{v-2}\left(\frac{i}{T}b_i + \frac{1}{T}\sum_{t=1}^{T} v_{t,i}\right) \\ \textit{subject to} \\ x'E(r) \ge m;\ \sum_{j=1}^{n} x_j = 1; x_j \ge 0;\ j = 1,\dots,n \\ v_{t,i} \ge 0;\ v_{t,i} \ge -\sum_{j=1}^{n} x_j r_{j,t} - b_i;\ t = 1,\dots,T; i = 1,\dots,s-1\end{gathered} \quad (19)$$

Moreover, we can consider the following tail extensions to linearizable FORS tracking error measures for some $p \in [0,1]$:

$$G_{X,Y}(\phi, p, 1) = \int_0^p \left|\phi'(u)\left(L_X(u) - L_Y(u)\right) - \frac{\phi(p)}{p}\left(L_X(p) - L_Y(p)\right)\right| du, \quad (20)$$

$$G_{X,Y}^{dsr}(\phi, p, 1) = \int_0^p \left(\max\left(\phi'(u)\left(L_X(u) - L_Y(u)\right) - \frac{\phi(p)}{p}\left(L_X(p) - L_Y(p)\right), 0\right)\right) du. \quad (21)$$

All these measures take into account only the tail behavior of portfolio distributions.

5 Efficient Frontier with Elliptical Distributions and FORS-type Measures

We now study the risk measures for jointly elliptical distributed returns $r = [r_1,\dots,r_n]'$. The portfolio return $x'r$ belongs to a family of *elliptical distributions* with finite mean as determined by the non-negative integrable function *g*(*y*) if asset returns have the following distribution:

$$F_{x'r}(\lambda) = \Pr(x'r \le \lambda) = \int_{-\infty}^{\lambda} \frac{1}{|K| U \sqrt{x'Qx}} g\left(\frac{(t - x'\mu)^2}{K^2 x'Qx} \right) dt$$

where $\mu = E(r)$ is the vector of expected returns; Q is a positive-definite dispersion matrix that differs by a positive constant factor from the covariance matrix when all assets returns have finite variance; $U = \int_{-\infty}^{+\infty} \frac{1}{|K| \left(x'Qx \right)^{1/2}} g\left(\frac{(t - x'\mu)^2}{K^2 x'Qx} \right) dt$; and K is a constant.

Chamberlain (1983) has shown that the elliptical families with finite variance are all possible families of distributions deemed necessary and sufficient for the expected utility of final wealth to be a function of only the mean and variance. Bawa (1975, 1978), Owen and Rabinovitch (1983), Ingersoll (1987), and Ortobelli (2001) have shown that the mean-dispersion dominance rule is equivalent to the SSD rule if portfolio returns belong to the same elliptical family of unbounded random variables.

Even if elliptical families are the natural best candidate to study the mean-risk portfolio problem, any risk measure proposed for the portfolio selection problem has to be properly used taking into account its specific characteristics (see Ortobelli et al (2005a)). In particular, when we assume that returns are elliptical distributed, the extended Gini mean difference and its tail extensions $\Gamma_{x'r}(v)$, $\Gamma_{x'r,\beta}(v)$ for a given portfolio $x'r$ are proportional to the respective Gini-type measures applied to the standard elliptical distribution Ell(0,1), the constant of proportionality is the dispersion $\sqrt{x'Qx}$, i.e. $\Gamma_{x'r}(v) = \sqrt{x'Qx}\,\Gamma_{Ell(0,1)}(v)$, and $\Gamma_{x'r,\beta}(v) = \sqrt{x'Qx}\,\Gamma_{Ell(0,1),\beta}(v)$.

Thus, with elliptical distributions, we have exchangeability among assets (see Shalit and Yitzhaki (2005)). These distribution families are scalar and translation invariant. Any element of the family $Ell(\mu_i, \sigma_i)$ with mean μ_i and dispersion σ_i has the same distribution of $\sigma_i Ell(0,1) + \mu_i$. This property of the elliptical families implies that one can characterize the portfolio efficient frontier with respect to the risk measure one uses. We now state the following proposition and corollaries.

Proposition 1 *Suppose there are* $n \ge 2$ *risky assets with returns* $r = [r_1, \ldots, r_n]'$ *traded in a frictionless economy with unlimited short selling. If returns belong to an elliptical multivariate family* Ell *with finite mean* $\boldsymbol{\mu} = E(r)$ *and non-singular dispersion matrix* **Q**, *then, for every characteristic FORS risk measure* ρ *associated with a FORS ordering induced either by* $\underset{\alpha}{\ge}$ *or by*

$\underset{-\alpha}{\geq}$ *order (with* $\alpha \geq 1$*), all portfolios satisfying the first-order conditions of the constrained problem:*

$$\begin{cases} \min_x \rho_{x'r} \\ x'e = 1 \end{cases} \tag{22}$$

are portfolios of the mean-dispersion frontier

$$\sigma^2(AC - B^2) - m^2 C + 2mB - A = 0 \tag{23}$$

whose portfolio weights are given by

$$x(\rho_{Ell(0,1)}) = \frac{\operatorname{sgn}\left(\rho_{Ell(0,1)}\right)(CQ^{-1}\mu - BQ^{-1}e)}{C\sqrt{C\left(\rho_{Ell(0,1)}\right)^2 - AC + B^2}} + \frac{Q^{-1}e}{C} \tag{24}$$

where $e' = [1,1,...,1]$ *;* $m = x'\mu$*;* $\sigma = \sqrt{x'Qx}$ *;* $A = \mu' Q^{-1}\mu$*;* $B = e'Q^{-1}\mu$*; and* $C = e'Q^{-1}e$.

Corollary 1 *Assuming Proposition 1, we express the mean* $m = x'\mu$*, the dispersion* $\sigma = \sqrt{x'Qx}$ *, and the risk measure* $\rho_{x'r}$ *of the optimal portfolios satisfying the first-order conditions of the constrained problem* (22) *as:*

$$m = x'\mu = \operatorname{sgn}\left(\rho_{Ell(0,1)}\right)\frac{AC - B^2}{C\sqrt{C\left(\rho_{Ell(0,1)}\right)^2 - AC + B^2}} + \frac{B}{C} \tag{25}$$

$$\sigma = \sqrt{x'Qx} = \frac{\left|\rho_{Ell(0,1)}\right|}{\sqrt{C\left(\rho_{Ell(0,1)}\right)^2 - AC + B^2}} \tag{26}$$

$$\rho_{x'r} = \sigma\rho_{Ell(0,1)} - m = \frac{\operatorname{sgn}\left(\rho_{Ell(0,1)}\right)\sqrt{C\left(\rho_{Ell(0,1)}\right)^2 - AC + B^2} - B}{C} \tag{27}$$

Consequently, we characterize the optimal portfolios satisfying the first-order conditions of constrained problem (22) by varying the parameter λ. More specifically, we can find the optimal portfolios as function of the parameter λ that minimize particular safety-risk measures such as: $\rho_X(\lambda) = \text{VaR}_\lambda(X)$ $\rho_X(\lambda) = \text{CVaR}_\lambda(X)$ or those obtained by positively homogeneous and Gaivoronsky-Pflug (G-P) translation invariant measures minus the expected mean (such as $\rho_X(\lambda) = \sqrt[\lambda]{E\left(\left|X - E(X)\right|^\lambda\right)} - E(X)$, $\rho_X(\lambda) = \Gamma_X(\lambda) - E(X)$, $\rho_X(\lambda) = \Gamma_{X,\beta}(\lambda) - E(X)$,

$\rho_X(\lambda) = \lambda \sqrt[q]{E\left(|X - E(X)|^q\right)} - E(X)$ and many others). This feature is important because often the parameter λ points out the investor's aversion to risk. In particular, let's look at the following examples.

Assume that returns follow one of the three joint multivariate elliptical distribution: (1) a Gaussian distribution, (2) a Student's *t* distribution with *u*>1 degrees of freedom, and (3) stable sub-Gaussian distribution with index of stability $\alpha \in (1,2)$. Under these distributional assumptions we have the following $\rho_{Ell(0,1)}(\lambda)$ =CVaR$_\lambda$(Ell(0,1)) functions (see Rachev and Mittnik (2000) and Stoyanov et al. (2006a)):

1) $$\text{CVaR}_\lambda(\text{N}(0,1)) = \frac{1}{\lambda\sqrt{2\pi}} \exp\left(-\frac{\left(VaR_\lambda\left(N(0,1)\right)\right)^2}{2}\right);$$

2) $$\text{CVaR}_\lambda(t_u(0,1)) = \frac{\Gamma\left(\frac{u+1}{2}\right)\sqrt{u}}{\lambda(u-1)\Gamma\left(\frac{u}{2}\right)\sqrt{\pi}}\left(1+\frac{\left(VaR_\lambda\left(t_u(0,1)\right)\right)^2}{u}\right)^{\frac{1-u}{2}};$$

3) $$\text{CVaR}_\lambda(\text{S}_\alpha(1,0,0)) = \frac{\alpha\left|VaR_\lambda\left(\text{S}_\alpha(1,0,0)\right)\right|}{\lambda(1-\alpha)\pi}\int_0^{\pi/2} g(\theta)\exp\left(-v(\theta)\left|VaR_\lambda\left(\text{S}_\alpha(1,0,0)\right)\right|^{\alpha/(\alpha-1)}\right)d\theta$$

where $g(\theta) = \frac{\sin\left((\alpha-2)\theta\right)}{\sin\left(\alpha\theta\right)} - \frac{\alpha\cos^2(\theta)}{\sin^2\left(\alpha\theta\right)}$ and $v(\theta) = \frac{\cos\left((\alpha-1)\theta\right)}{\cos\left(\theta\right)}\left(\frac{\cos(\theta)}{\sin\left(\alpha\theta\right)}\right)^{\frac{\alpha}{\alpha-1}}$.

Similarly, the safety-first functions $\rho_X(\lambda) = \sqrt[\lambda]{E\left(|X - E(X)|^\lambda\right)} - E(X)$ univocally determine the elliptical distributions. Thus, for *X* stable sub-Gaussian distributed and $\alpha \in (1,2]$ ($\alpha = 2$ represents the Gaussian case), we obtain the following set of parametric positively homogeneous and translation invariant safety-risk measures:

$$\rho_{Ell(0,1)}(\lambda) = \begin{cases} \left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\lambda+1}{2}\right)\right)^{\frac{1}{\lambda}}\sqrt{2} & if \quad Ell(0,1) = N(0,1) \\ 2\left(\frac{1}{\Gamma\left(1-\frac{\lambda}{2}\right)\sqrt{\pi}}\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(1-\frac{\lambda}{\alpha}\right)\right)^{\frac{1}{\lambda}} & if \quad Ell(0,1) = S_\alpha(1,0,0) \end{cases},$$

for $\lambda \in (0,s] \wedge s \in (0,\alpha)$ (see Ortobelli et al (2005b)). As a consequence of the previous corollaries, the optimal portfolios solutions for (22) are characterized by the functions (25), (26), and (27).

[Insert Figure 1 about here]

In Figure 1, different efficient frontiers are been used to distinguish between the different risk aversions of investors. The figure shows the extended Gini mean difference $\Gamma_{Ell(0,1)}(v)$, the *v-mean* (formula (25)), and the *v-variance* (formula (26)) efficient frontiers assuming either Gaussian returns or Student's *t* returns with 5 degrees of freedom. On the other hand, Figure 1 confirms the intuitive result that optimal portfolios minimizing $\Gamma_X(v) - E(X)$ with the lowest risk aversion *v* present the highest mean and variance. However, investors who assume Student's *t* distributed returns are more conservative because they account for the risk in the heavier tails. As a matter of fact, for the same *v* (for example, *v* = 2) "Student's *t* investors" choose portfolios with lower mean and variance than "Gaussian investors".

The mean-tracking error variance frontier has been analyzed by Roll (1992). Consequently we can define the mean- tracking error dispersion frontier with respect to different risk measures when returns belong to an elliptical family.

Corollary 2 *Suppose there are* $n \geq 2$ *risky assets with returns* $r = [r_1,...,r_n]'$ *traded in a frictionless economy where unlimited short selling is allowed. Let us assume as benchmark return* $r_Y = \overline{x}'r$ *a particular portfolio* (i.e. $\overline{x}'e = 1$) *of these returns and assume that the returns belong to an elliptical multivariate family* Ell *with finite mean* $\boldsymbol{\mu} = E(r)$ *and non-singular dispersion matrix* **Q.** *Then, for every positively homogeneous measure* ρ *consistent either with* $\underset{\alpha}{\geq}$ *or with* $\underset{-\alpha}{\geq}$ *order* ($\alpha \geq 1$), *that is either translation invariant (i.e.* $\rho_{X+t} = \rho_X - t \; \forall t \in R$ *) or translation invariant in the sense of G-P (i.e.* $\rho_{X+t} = \rho_X \; \forall t \in R$ *), all the portfolios satisfying the first-order conditions of the following constrained tracking error problem:*

$$\begin{cases} \min_x \rho_{(x-\overline{x})'r} \\ (x-\overline{x})'e = 0; \;\; (x-\overline{x})'\mu = g \end{cases} \tag{28}$$

are portfolios of the mean-tracking error dispersion frontier whose portfolio weights are given by

$$x = \overline{x} + \frac{g}{A/B - B/C} Q^{-1}\left(\frac{\mu}{B} - \frac{e}{C}\right) \tag{29}$$

where $e' = [1,1,...,1]$ *;* $A = \mu' Q^{-1}\mu$; $B = e'Q^{-1}\mu$ *and* $C = e'Q^{-1}e$.

Moreover, we can also characterize the FORS tracking error measures as we prove in the following proposition.

Proposition 2 *Suppose, in a frictionless economy, are traded* $n \geq 2$ *risky assets with returns* $r = [r_1,...,r_n]'$ *and a benchmark asset with return* r_Y *that is not a portfolio of the other returns. Assume that all admissible portfolios of returns* $x'r$ *belong to an elliptical family* Ell *with finite mean and non-singular dispersion matrix* **Q** *and even* r_Y *belongs to the same family of elliptical distributions. Let* $\rho_{Ell(0,1)}(\lambda)$ *be a continuous and monotone function that defines a* (*p-tail*) *FORS characteristic risk measure on a compact real interval* [a,b]*, then:*

$$\rho_{x'r,Y}(1) = \int_a^b \left|\rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda)\right| d\lambda = \\ = \left|\left(\sigma_{x'r} - \sigma_{r_Y}\right)\left(\int_a^t \rho_{Ell(0,1)}(\lambda)d\lambda - \int_t^b \rho_{Ell(0,1)}(\lambda)d\lambda\right) + (2t-a-b)\left(m_{r_Y} - m_{x'r}\right)\right| \tag{30}$$

$$\rho^{dsr}_{x'r,Y}(1) = \int_a^b \max\left(\rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda), 0\right) d\lambda = \\ = \begin{cases} \left(\sigma_{x'r} - \sigma_{r_Y}\right)\int_a^t \rho_{Ell(0,1)}(\lambda)d\lambda + (t-a)\left(m_{r_Y} - m_{x'r}\right) & if \;\; s > 0 \\ \left(\sigma_{x'r} - \sigma_{r_Y}\right)\int_t^b \rho_{Ell(0,1)}(\lambda)d\lambda + (b-t)\left(m_{r_Y} - m_{x'r}\right) & if \;\; s < 0 \\ 0 & otherwise \end{cases} \tag{31}$$

where m_X, σ_X *are respectively the mean and the dispersion of X,* $\rho^{-1}_{Ell(0,1)}$ *is the inverse function of* $\rho_{Ell(0,1)}$, $s = \left(\sigma_{x'r} - \sigma_{r_Y}\right)\rho_{Ell(0,1)}(\lambda) + \left(m_{r_Y} - m_{x'r}\right)$ *for a given* $\lambda \in (a,t)$, *and*

$$t = \begin{cases} \rho^{-1}_{Ell(0,1)}\left(\dfrac{m_{r_Y} - m_{x'r}}{\sigma_{x'r} - \sigma_{r_Y}}\right) & if \;\; \exists \lambda \in (a,b) : \rho_{Ell(0,1)}(\lambda) = \dfrac{m_{r_Y} - m_{x'r}}{\sigma_{x'r} - \sigma_{r_Y}} \\ b & otherwise \end{cases}.$$

Observe that if *b* is opportunely small (say *b*=*t*), s=s(λ,x)>0 for every $\lambda \in (a,t)$ and for every portfolio *x*, then when we minimize $\rho^{dsr}_{x'r,Y}(1)$ subject to $x'e = 1; \;\; x'\mu = m$, we get portfolios of the mean-dispersion frontier. Typical applications of Proposition 2 are the Gini-type tracking error measures $L_{X,Y}(q)$, $L^{dsr}_{X,Y}(q)$, $G_{X,Y}(q)$, $G^{dsr}_{X,Y}(q)$ and their tail extensions. However, we can apply Proposition 2 to many other FORS tracking error measures. For example, if we consider a *p*-tail characteristic risk measure $\rho_X(\lambda) = \Gamma_{X,p}(\lambda) - E(X)$, then we can apply Proposition 2 to the new FORS tracking error measures (see Ortobelli et al (2006)). In particular, it is not difficult to prove that for any unbounded elliptical family the index of dissimilarity is given by:

$$G_{X,Y}(1)=2\left|\left(\sigma_{x'r}-\sigma_{r_Y}\right)L_{Ell(0,1)}\left(F_{Ell(0,1)}\left(\frac{m_{r_Y}-m_{x'r}}{\sigma_{x'r}-\sigma_{r_Y}}\right)\right)\right|+\left|\left(1-2F_{Ell(0,1)}\left(\frac{m_{r_Y}-m_{x'r}}{\sigma_{x'r}-\sigma_{r_Y}}\right)\right)\left(m_{r_Y}-m_{x'r}\right)\right| \text{ if}$$

$$\sigma_{x'r}\neq\sigma_{r_Y} \text{ and } G_{X,Y}(1)=\left|m_{r_Y}-m_{x'r}\right| \text{ if } \sigma_{x'r}=\sigma_{r_Y}.$$

This formula generalizes the analogous one obtained by Salvemini (1957) with Gaussian distributions.

6. Reward-Risk Analysis with FORS-Type Risk Measures

The importance of including the investor's preference toward reward in portfolio analysis is well founded. To consider both risk and reward, the so-called *performance measures* use reward/risk ratios. In particular, when we maximize a given reward/risk ratio as v/ρ, we could determine non-dominated choices that are consistent with expected utility maximization. When the performance ratio v/ρ is maximized and we get choices that are non-dominated with respect to a given stochastic dominance law, we say that the ratio v/ρ is *coherent* with the investors' choices ordered with respect to the underlined dominance law. The following proposition proves conditions that guarantee the coherency with more than one ordering.

Proposition 3 *Consider a frictionless economy where a benchmark asset with return* r_Y *and* $n\geq 2$ *risky assets with returns* $r=[r_1,...,r_n]'$ *are traded. Let* v,ρ *be two probability functionals defined on a space of random portfolios with weights that belong to*

$$V=\left\{x\in R^n\,/\,x'e=1; Lb\leq Ax\leq Ub; A\in R^{n\times k}; Lb,Ub\in R^k\right\}, \tag{32}$$

where we assume they are strictly positive. Suppose that v is a positively homogeneous concave FORS reward measure induced by a risk ordering and ρ *is positively homogeneous convex FORS measure that is consistent with another stochastic order. If we maximize ratio* $\frac{v(x'r)}{\rho(x'r)}$, *or* $\frac{v(x'r-r_Y)}{\rho(x'r)}$ *or* $\frac{v(x'r-r_Y)}{\rho(x'r-r_Y)}$ *subject to the portfolio weights that belong to the space* V, *we obtain non-dominated portfolios* $x'r$ (*or* $x'r-r_Y$) *with respect to both the previous stochastic orders* (*of v and* ρ).

Proposition 3 justifies the use of performance measures that enable an investor to determine optimal non-dominated choices. Furthermore, as a consequence of the analysis of Stoyanov et al (2005), we get the following corollary.

Corollary 3 *Under the assumption of Proposition 3, the maximization of ratio* $\frac{v(x'r)}{\rho(x'r)}$ *or* $\frac{v(x'r-r_Y)}{\rho(x'r)}$, *or* $\frac{v(x'r-r_Y)}{\rho(x'r-r_Y)}$), *subject to the portfolio weights that belong to the convex closed space V, is a quasi-concave problem.*

A typical example of the performance ratio that satisfies the previous corollary is $\frac{E\left(\left(x'r-t\right)^p\right)}{E\left(\left(a-x'r\right)_+^q\right)}$ where $p\in(0,1)$, $q>1$, $t\le\min_{x\in V}\min_{1\le k\le T}x'r_{(k)}$, $a\ge\max_{x\in V}\min_{1\le k\le T}x'r_{(k)}$ ($r_{(k)}$ points out the *k*-th observation of return vector). By maximizing this ratio, we obtain non-dominated choices with respect to $\underset{p+1}{\ge}$ and $\underset{q+1}{\ge}$ orders. In addition, the resulting optimization problem is easily linearizable. There exist many possible generalizations to these results and there are many performance measures that do not fit the previous classification even if they present very good performance (see, for example, the Rachev Ratio and generalized Rachev Ratio in Stoyanov et al (2005)). Therefore this analysis cannot be exhaustive. On the other hand, these considerations and the computational simplicity of the optimization problems suggest using optimization of the performance measures $\frac{v(x'r)}{\rho(x'r)}$ as an alternative to classical portfolio selection models.

7. Concluding remarks

The theory of portfolio choice is based on the assumption that investors allocate their wealth across the available assets in order to maximize their utility. For this reason, most of portfolio theory is based on minimizing a probability functional consistent with an ordering of preferences, keeping constant some portfolio characteristics. Therefore, portfolio optimization problems can be reformulated from the probability metric theory point of view.

The first contribution of this paper is that it links probability metric theory and stochastic orderings and thereby offering a new perspective to portfolio theory. Thus, we classified probability metrics for their use, i.e., metrics used either as uncertainty measures (concentration measures and dispersion measures) or as tracking error measures. Then we proposed new coherent risk measures, performance measures and tracking error measures based on probability functionals consistent with some stochastic orderings. A second contribution is that we discussed the computational applicability of portfolio problems arising by optimizing or a risk measure or a probability metric or a performance measure. Thus, we propose linearizable portfolio selection problems for each category of these measures. This aspect is fundamental to solve large-scale portfolio problems. As a matter of fact, even the Markowitz's portfolio optimization model has not been widely used in its original form because the deriving large-scale quadratic problem with a dense covariance matrix is still too computationally difficult. Our last, not less important, contribution is that we suggested to parameterize the portfolio measures in order to characterize opportunely the investor's attitude toward risk. In particular, we demonstrated how to parameterize differently the efficient frontier, providing other financial interpretations of the optimal portfolios when the returns are elliptically distributed.

Several new perspectives and problems arise from this analysis. Since we can better specify the problem of portfolio optimization by taking into account the investor's attitude toward risk, we have to consider the ideal characteristics of the associated statistics and their asymptotic behavior. Thus, using the theory of probability metrics, we can explain and argue why a given metric must be used for a particular optimization problem. However, further theoretical and empirical analyses are still necessary. We suggest that future research should (1) extend the findings presented in this paper to a multi-period context, (2) test empirically the financial impact of different probability functionals, and (3) investigate the possible use of different parameterizations of the efficient choices.

APPENDIX: Proofs

Proof of Remark 1: By definition $\frac{-\Gamma(v+1)}{\beta^{v}}F_X^{(-(v+1))}(\beta)$ is consistent with $\underset{-(v+1)}{\geq}$. The coherency of CVaR implies the coherency of $\frac{-\Gamma(v+1)}{\beta^{v}}F_X^{(-(v+1))}(\beta)$. Moreover, if X *RSD* Y then E(X)=E(Y) and $X \underset{-2}{\geq} Y$. Then $\frac{-\Gamma(v+1)}{\beta^{v}}F_X^{(-(v+1))}(\beta) \leq \frac{-\Gamma(v+1)}{\beta^{v}}F_Y^{(-(v+1))}(\beta)$ for every $v \geq 1$ and for every $\beta \in [0,1]$. Thus $\Gamma_{X,\beta}(v)$ is consistent with *RSD* order. □

Proof of Proposition 1: The solutions of problem (22) are not FSD dominated because ρ is consistent measure and hence it is consistent either with $\underset{\alpha}{\geq}$ or with $\underset{-\alpha}{\geq}$ order with $\alpha \geq 1$. Thus, these solutions belong to the mean-dispersion frontier (23) (see Ortobelli and Rachev (2001)). Considering that ρ is positively homogeneous and translation invariant, then for any portfolio $x'r$ with mean $x'\mu = m$ and dispersion $\sigma = \sqrt{x'Qx}$ we obtain $\rho_{x'r} = \rho_{Ell(m,\sigma)} = \sigma\rho_{Ell(0,1)} - m$. Hence the optimization problem (22) is equivalent to $\min_{x} \sqrt{x'Qx}\rho_{Ell(0,1)} - x'\mu$ subject to $x'e = 1$. Considering the first order conditions of the Lagrangian $L(x,\lambda)$, we have $x'e = 1$ and $\frac{Qx}{\sqrt{x'Qx}}\rho_{Ell(0,1)} - \mu - \lambda e = 0$ i.e. $x = \frac{Q^{-1}(\mu + \lambda e)\sqrt{x'Qx}}{\rho_{Ell(0,1)}}$. Therefore, imposing the condition $x'e = 1$ we get $\lambda = \frac{\rho_{Ell(0,1)} - B\sqrt{x'Qx}}{C\sqrt{x'Qx}}$. Thus, the portfolio $x = \frac{Q^{-1}\left(\mu C\sqrt{x'Qx} - eB\sqrt{x'Qx} + e\rho_{Ell(0,1)}\right)}{C\rho_{Ell(0,1)}}$ and considering that the square dispersion is given by $\sigma^2 = x'Qx = \frac{\left(\rho_{Ell(0,1)}\right)^2}{C\left(\rho_{Ell(0,1)}\right)^2 - AC + B^2}$ then we get formula (24). □

Proof of Corollaries 1 and 2: We get the results as a consequence of the previous Proposition and of the definition of FORS characteristic functionals.

Proof of Proposition 2: Considering that $\rho_X(\lambda)$ is positively homogeneous and translation invariant, then $\rho_X(\lambda) = \sigma_X \rho_{Ell(0,1)}(\lambda) - m_X$. Since $\rho_{Ell(0,1)}(\lambda)$ is continuous and monotone, then there exist at most a point *t* such that

$$0 = \rho_{x'r}(t) - \rho_{r_Y}(t) = \left(\sigma_{x'r} - \sigma_{r_Y}\right)\rho_{Ell(0,1)}(t) + \left(m_{r_Y} - m_{x'r}\right), \text{ that is } t = \rho_{Ell(0,1)}^{-1}\left(\frac{m_{r_Y} - m_{x'r}}{\sigma_{x'r} - \sigma_{r_Y}}\right).$$

Thus $\rho_{x'r,Y}(1) = \left|\int_a^t \rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda)\, d\lambda - \int_t^b \rho_{x'r}(\lambda) - \rho_{r_Y}(\lambda)\, d\lambda\right|$ and (28) holds. Similarly we get formula (29). □

Proof of Proposition 3: Using the same arguments of Proposition 4 in Stoyanov et al (2005), we know that maximizing the ratio $\frac{v(x'r - r_Y)}{\rho(x'r - r_Y)}$ (even with r_Y=0) is equivalent to maximize the measure v maintaining the risk ρ opportunely lower than a fixed risk and it is also equivalent to minimize the risk ρ maintaining the measure v opportunely higher than a fixed reward. Then, we obtain the thesis. □

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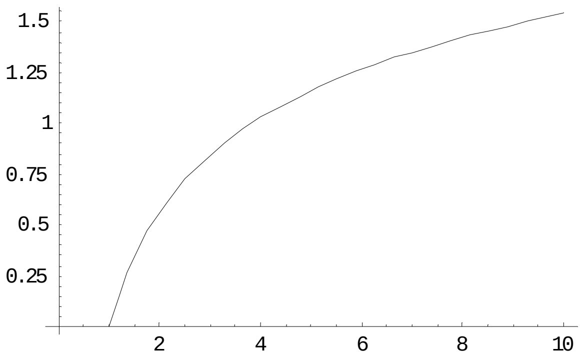
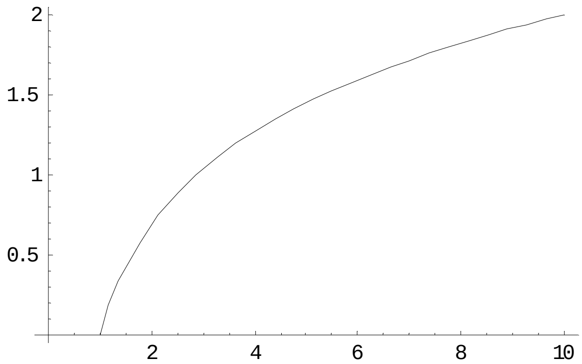
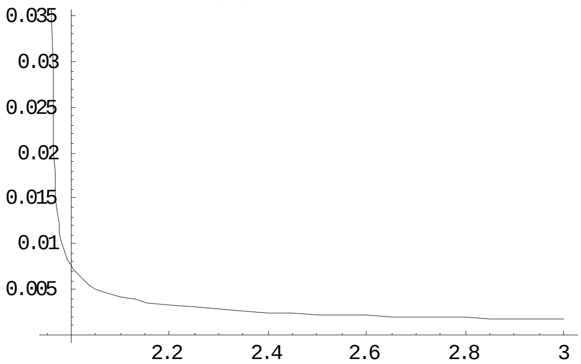
Figure 1: *Extended Gini mean difference and efficient frontiers with Gaussian and Student t_5 distributions.*



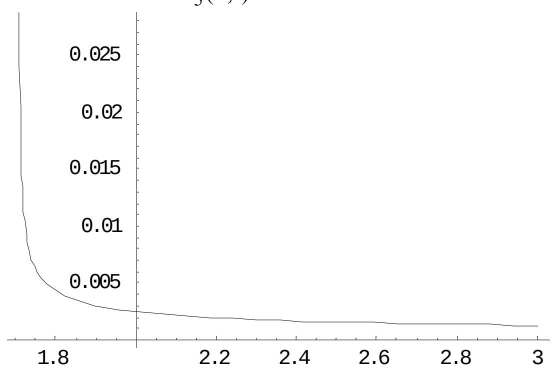
Plane v-$\Gamma_{N(0,1)}(v)$



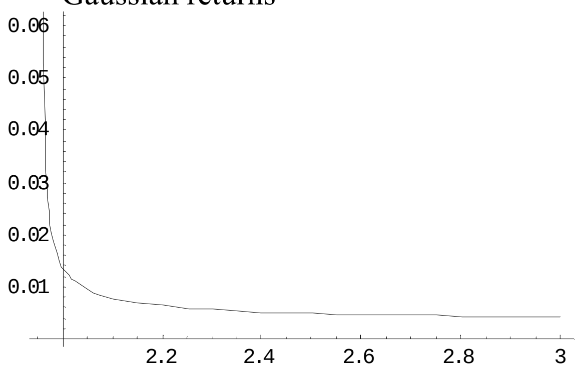
Plane v-$\Gamma_{t_5(0,1)}(v)$ i



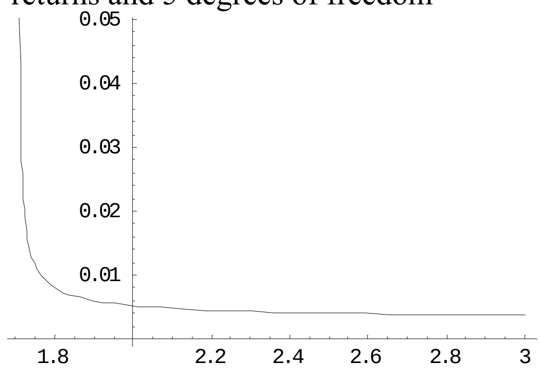
Optimal v-*mean* portfolios with Gaussian returns



Optimal v-*mean* portfolios with Student t returns and 5 degrees of freedom



Optimal v-*variance* portfolios with Gaussian returns



Optimal v-*variance* portfolios with Student t returns and 5 degrees of freedom