Estimation of α-Stable Sub-Gaussian Distributions for Asset Returns

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Abstract

Fitting multivariate α-stable distributions to data is still not feasible in higher dimensions since the (non-parametric) spectral measure of the characteristic function is extremely difficult to estimate in dimensions higher than 2. This was shown by Chen and Rachev (1995) and Nolan, Panorska and McCulloch (1996). α-stable sub-Gaussian distributions are a particular (parametric) subclass of the multivariate α-stable distributions. We present and extend a method based on Nolan (2005) to estimate the dispersion matrix of an α-stable sub-Gaussian distribution and estimate the tail index α of the distribution. In particular, we develop an estimator for the off-diagonal entries of the dispersion matrix that has statistical properties superior to the normal off-diagonal estimator based on the covariation. Furthermore, this approach allows estimation of the dispersion matrix of any normal variance mixture distribution up to a scale parameter. We demonstrate the behaviour of these estimators by fitting an α-stable sub-Gaussian distribution to the DAX30 components. Finally, we conduct a stable principal component analysis and calculate the coefficient of tail dependence of the prinipal components.

Keywords and Phrases:α-Stable Sub-Gaussian Distributions, Elliptical Distributions, Estimation of Dispersion Matrix, Coefficient of Tail Dependence, Risk Management, DAX30.

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1 Introduction

Classical models in financial risk management and portfolio optimization such as the Markowitz portfolio optimization approach are based on the assumption that risk factor returns and stock returns are normally distributed. Since the seminal work of Mandelbrot (1963) and further investigations by Fama (1965), Chen and Rachev (1995), McCulloch (1996), and Rachev and Mittnik (2000) there has been overwhelming empirical evidences that the normal distribution must be rejected. These investigations led to the conclusion that marginal distributions of risk factors and stock returns exhibit skewness and leptokurtosis, i.e., a phenomena that cannot be explained by the normal distribution.

Stable or α-stable distributions have been suggested by the authors above for modeling these pecularities of financial time series. Beside the fact that α-stable distributions capture these phenomena very well, they have further attractive features which allow them to generalize Gaussian-based financial theory. First, they have the property of stability meaning, that a finite sum of independent and identically distributed (i.i.d.) α-stable distributions is a stable distribution. Second, this class of distribution allows for the generalized Central Limit Theorem: A normalized sum of i.i.d. random variables converges in distribution to an α-stable random vector.

A drawback of stable distributions is that, with a few exceptions, they do not know any analytic expressions for their densities. In the univariate case, this obstacle could be negotiated by numerical approximation based on new computational possibilities. These new possibilities make the α-stable distribution also accessible for practitioners in the financial sector, at least, in the univariate case. The multivariate α-stable case is even much more complex, allowing for a very rich dependence structure, which is represented by the so-called spectral measure. In general, the spectral measure is very difficult to estimate even in low dimensions. This is certainly one of the main reasons why multivariate α-stable distributions have not been used in many financial applications.

In financial risk management as well as in portfolio optimization, all the models are inherently multivariate as stressed by McNeil, Frey and Embrechts (2005). The multivariate normal distribution is not appropriate to capture the complex dependence structure between assets, since it does not allow for modeling tail dependencies between the assets and leptokurtosis as well as heavy tails of the marginal return distributions. In many models for market risk management multivariate elliptical distributions, e.g. t-distribution or symmetric generalized hyperbolic distributions, are applied. They model better than the multivariate normal distributions (MNDs) the dependence structure of assets and offer an efficient estimation procedure. In general, elliptical distributions (EDs) are an extension of MNDs since they are also elliptically contoured and characterized by the so-called dispersion matrix. The dispersion matrix equals the variance covariance matrix up to a scaling constants if second moments of the distributions exist, and has a similar interpretation as the variance-covariance matrix for MNDs. In empirical studies [1] it is shown that especially data of multivariate asset returns are roughly elliptically contoured.

In this paper, we focus on multivariate α-stable sub-Gaussian distributions (MSSDs).

[1] For further information, see McNeil, Frey and Embrechts (2005)

In two aspects they are a very natural extension of the MNDs. First, they have the stability property and allow for the generalized Central Limit Theorem, important features making them attractive for financial theory. Second, they belong to the class of EDs implying that any linear combination of an α-stable sub-Gaussain random vector remains α-stable sub-Gaussian and therefore the Markowitz portfolio optimization approach is applicable to them.

We derive two methods to estimate the dispersion matrix of an α-stable sub-Gaussian random vector and analyze them emprically. The first method is based on the covariation and the second one is a moment-type estimator. We will see that the second one outperforms the first one. We conclude the paper with an empirical analysis of the DAX30 using α-stable sub-Gaussian random vectors.

In section 2 we introduce α-stable distributions and MSSDs, respectively. In section 3 we provide background information about EDs and normal variance mixture distributions, as well as outline their role in modern quantitative market risk management and modeling. In section 4 we present our main theoretical results: we derive two new moments estimators for the dispersion matrix of an MSSD and show the consistency of the estimators. In section 5 we analyze the estimators empirically using boxplots. In section 6 we fit, as far as we know, for the first time an α-stable sub-Gaussian distribution to the DAX30 and conduct a principal component analysis of the stable dispersion matrix. We compare our results with the normal distribution case. In section 7 we summarize our findings.

2 α-stable Distribution: Definitions and Properties

2.1 Univariate α-stable distribution

The applications of α-stable distributions to financial data come from the fact that they generalize the normal (Gaussian) distribution and allow for the heavy tails and skewness, frequently observed in financial data.

There are several ways to define stable distribution.

Definition 1. Let $X, X_1, X_2, ..., X_n$ be i.i.d. random variables. If the equation

$$X_1 + X_2 + ... + X_n \stackrel{d}{=} c_n X + d_n$$

holds for all $n \in \mathbf{N}$ with $c_n > 0$ and $d_n \in \mathbf{R}$, then we call X stable or α-stable distributed.

The definition justifies the term stable because the sum of i.i.d. random variables has the same distribution as X up to a scale and shift parameter. One can show that the constant c_n in definition 1 equals $n^{1/\alpha}$.

The next definition represents univariate α-stable distributions in terms of their characteristic functions and determines the parametric family which describes univariate stable distributions.

Definition 2. A random variable is α-stable if the characteristic function of X is

$$E(\exp(itX)) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left[1 - i\beta\left(\tan\frac{\pi\alpha}{2}\right)(\text{sign } t)\right] + i\mu t\right) & , \quad \alpha \neq 1 \\ \exp\left(-\sigma|t|\left[1 + i\beta\frac{\pi}{2}(\text{sign } \ln|t|)\right] + i\mu t\right) & , \quad \alpha = 1. \end{cases}$$

where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma \in (0, \infty)$ and $\mu \in \mathbf{R}$.

The probability densities of α-stable random variables exist and are continuous but, with a few exceptions, they are not known in closed forms. These exceptions are the Gaussian distribution for $\alpha = 2$, the Cauchy distribution for $\alpha = 1$, and the Lévy distribution for $\alpha = 1/2$. (For further information, see Samorodnitsky and Taqqu (1994), where the equivalence of these definitions is shown). The parameter α is called the index of the law, the index of stability or the characteristic exponent. The parameter β is called skewness of the law. If $\beta = 0$, then the law is symmetric, if $\beta > 0$, it is skewed to the right, if $\beta < 0$, it is skewed to the left. The parameter σ is the scale parameter. Finally, the parameter μ is the location parameter. The parameters α and β determine the shape of the distribution. Since the characteristic function of an α-stable random variable is determined by these four parameters, we denote stable distributions by $S_\alpha(\sigma, \beta, \mu)$. $X \sim S_\alpha(\sigma, \beta, \mu)$, indicating that the random variable X has the stable distribution $S_\alpha(\sigma, \beta, \mu)$. The next definition of an α-stable distribution which is equivalent to the previous definitions is the generalized Central Limit Theorem:

Definition 3. A random variable X is said to have a stable distribution if it has a *domain of attraction*, i.e., if there is a sequence of i.i.d. random variables $Y_1, Y_2, ...$ and sequences of positive numbers $(d_n)_{n \in \mathbf{N}}$ and real numbers $(a_n)_{n \in \mathbf{N}}$, such that

$$\frac{Y_1 + Y_2 + ... + Y_n}{d_n} + a_n \xrightarrow{d} X.$$

The notation $\xrightarrow{d}$ denotes convergence in distribution. If we assume that the sequence of random variables $(Y_i)_{i \in \mathbf{N}}$ has second moments, we obtain the ordinary Central Limit Theorem (CLT). In classical financial theory, the CLT is the theoretical justification for the Gaussian approach, i.e., it is assumed that the price process (S_t) follows a log-normal distribution. If we assume that the log-returns $\log(S_{t_i}/S_{t_{i-1}})$, $i = 1, ..., n$, are i.i.d. and have second moments, we conclude that $\log(S_t)$ is approximately normally distributed. This is a result of the ordinary CLT since the stock price can be written as the sum of independent innovations,i.e.,

$$\begin{aligned} \log(S_t) &= \sum_{i=1}^{n} \log\left(S_{t_i}\right) - \log(S_{t_{i-1}}) \\ &= \sum_{i=1}^{n} \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right), \end{aligned}$$

where $t_n = t$, $t_0 = 0$, $S_0 = 1$ and $t_i - t_{i-1} = 1/n$. If we relax the assumption that stock returns have second moments, we derive from the generalized CLT, that $\log(S_t)$ is approximately α-stable distributed. With respect to the CLT, α-stable distributions are the natural extension of the normal approach. The tail parameter α has an important meaning for α-stable distributions. First, α determines the tail behavior of a stable distribution, i.e.,

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda^\alpha P(X > \lambda) &\to C_+ \\ \lim_{\lambda \to -\infty} \lambda^\alpha P(X < \lambda) &\to C_-. \end{aligned}$$

Second, the parameter α characterizes the distributions in the domain of attraction of a stable law. If X is a random variable with $\lim_{\lambda\to\infty}\lambda^{\alpha}P(|X|>\lambda)=C>0$ for some $0<\alpha<2$, then X is in the domain of attraction of a stable law. Many authors claim that the returns of assets should follow an infinitely divisible law, i.e., for all $n\in\mathbf{N}$ there exists a sequence of i.i.d. random variable $(X_{n,k})_{k=1,\dots,n}$ satisfying

$$X \stackrel{d}{=} \sum_{k=1}^{n} X_{n,k}.$$

The property is desirable for models of asset returns in efficient markets since the dynamics of stock prices are caused from continuously arising but independent information. From definition 3, it is obvious that α-stable distribution are infinitely divisible.

The next lemma is useful for deriving an estimator for the scale parameter σ.

Lemma 1. *Let $X\sim S_{\alpha}(\sigma,\beta,\mu)$, $1<\alpha<2$ and $\beta=0$. Then for any $0<p<\alpha$ there exists a constant $c_{\alpha,\beta}(p)$ such that:*

$$E(|X-\mu|^{p})^{1/p}=c_{\alpha,\beta}(p)\sigma$$

where $c_{\alpha,\beta}(p)=(E|X_0|^{p})^{1/p}$, $X_0\sim S_{\alpha}(1,\beta,0)$.

Proof. See Samorodnitsky and Taqqu (1994). □

To get a first feeling for the sort of data we are dealing with, we display in Figure 1 the kernel density plots of the empirical returns, the Gaussian fit and the α-stable fit of some representative stocks. We can clearly discern the individual areas in the plot where the normal fit causes problems. It is around the mode where the empirical peak is too high to be captured by the Gaussian parameters. Moreover, in the mediocre parts of the tails, the empirical distribution attributes less weight than the Gaussian distribution. And finally, the tails are underestimated, again. In contrast to the Gaussian, the stable distribution appears to account for all these features of the empirical distribution quite well.

Another means of presenting the aptitude of the stable class to represent stock returns is the quantile plot. In Figure 2, we match the empirical stock return percentiles of Adidas AG with simulated percentiles for the normal and stable distributions, for the respective estimated parameter tuples. The stable distribution is liable to produce almost absurd extreme values compared to the empirical data. Hence, we need to discard the most extreme quantile pairs. However, the overall position of the line of the joint empirical-stable percentiles with respect to the interquartile line appears quite convincingly in favor of the stable distribution. [2]

2.2 Multivariate α-stable distributions

Multivariate stable distributions are the distributions of stable random vectors. They are defined by simply extending the definition of stable random variables to $\mathbf{R}^{d}$. As

[2] In Figure 2 we remove the two most extreme points in the upper and lower tails, respectively.

in the univariate case, multivariate Gaussian distribution is a particular case of multivariate stable distributions. Any linear combination of stable random vectors is a stable random variate. This is an important property in terms of portfolio modeling. Multivariate stable cumulative distribution functions or density functions are usually not known in closed form and therefore, one works with their characteristic functions. The representation of these characteristic functions include a finite measure on the unit sphere, the so-called *spectral measure*. This measure describes the dependence structure of the stable random vector. In general, stable random vectors are difficult to use for financial modeling, because the spectral measure is difficult to estimate even in low dimensions. For stable financial model building, one has to focus on certain subclasses of stable random vectors where the spectral measure has an easier representation. Such a subclass is the multivariate α-stable *sub-Gaussian* law. They are obtained by multiplying a *Gaussian* vector by $W^{1/2}$ where W is a stable random variable totally skewed to the right. Stable sub-Gaussian distributions inherit their dependence structure from the underlying Gaussian vector. In the next section we will see that the distribution of multivariate stable sub-Gaussian random vectors belongs to the class of elliptical distributions.The definition of stability in $\mathbf{R}^d$ is analogous to that in $\mathbf{R}$.

Definition 4. A random vector $X = (X_1, ..., X_d)$ is said to be a *stable random vector in* $\mathbf{R}^d$ if for any positive numbers A and B there is a positive number C and a vector $D \in \mathbf{R}^d$ such that

$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX + D$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X.

Note, that an α-stable random vector X is called symmetric stable if X satisfies

$$P(X \in A) = P(-X \in A)$$

for all Borel-sets A in $\mathbf{R}^d$.

Theorem 1. *Let X be a stable (respectively symmetric stable) vector in $\mathbf{R}^d$. Then there is a constant $\alpha \in (0, 2]$ such that in Definition 4, $C = (A^\alpha + B^\alpha)^{1/\alpha}$. Moreover, any linear combination of the components of X of the type $Y = \sum_{i=1}^d b_k X_k = b'X$ is an α-stable (respectively symmetric stable) random variable.*

Proof. A proof is given in Samorodnitsky and Taqqu (1994). □

The parameter α in theorem 1 is called the index of stability. It determines the tail behavior of a stable random vector, i.e., the α-stable random vector is regularly varying with tail index α[3]. For portfolio analysis and risk management, it is very important that stable random vectors are closed under linear combinations of the components due to theorem 1. In the next section we will see that elliptically distributed random vectors have this desirable feature as well.

The next theorem determines α-stable random vectors in terms of the characteristic function. Since there is a lack of formulas for stable densities and distribution functions, the characteristic function is the main device to fit stable random vectors to data.

[3]For further information about regularly varying random vectors, see Resnick (1987).

Theorem 2. *The random vector* $X = (X_1, ..., X_d)$ *is an* α*-stable random vector in* $\mathbf{R}^d$ *iff there exists an unique finite measure* Γ *on the unit sphere* $\mathcal{S}^{d-1}$*, the so-called* spectral measure*, and an unique vector* $\mu \in \mathbf{R}^d$ *such that:*

(i) If $\alpha \neq 1$*,*

$$E(e^{it'X}) = \exp\{-\int_{\mathcal{S}^{d-1}} |(t,s)|^\alpha (1 - i \operatorname{sign}((t,s)) \tan \frac{\pi\alpha}{2}) \Gamma(ds) + i(t,\mu)\}$$

(ii) If $\alpha = 1$*,*

$$E(e^{it'X}) = \exp\{-\int_{\mathcal{S}^{d-1}} |(t,s)|(1 + i\frac{2}{\pi} \operatorname{sign}((t,s)) \ln |(t,s)|) \Gamma(ds) + i(t,\mu)\}$$

In contrast to the univariate case, stable random vectors have not been applied frequently in financial modeling. The reason is that the spectral measure, as a measure on the unit sphere $\mathcal{S}^{d-1}$, is extremely difficult to estimate even in low dimensions. (For further information see Rachev and Mittnik (2000) and Nolan, Panorska and McCulloch (1996).)

Another way to describe stable random vectors is in terms of linear projections. We know from theorem 1 that any linear combination

$$(b, X) = \sum_{i=1}^{d} b_i X_i$$

has an α-stable distribution $S_\alpha(\sigma(b), \beta(b), \mu(b))$. By using theorem 2 we obtain for the parameters $\sigma(b), \beta(b)$ and $\mu(b)$

$$\sigma(b) = \left(\int_{\mathcal{S}^{d-1}} |(b,s)|^\alpha \Gamma(ds)\right)^{1/\alpha},$$

$$\beta(b) = \frac{\int_{\mathcal{S}^{d-1}} |(b,s)|^\alpha \operatorname{sign}(b,s) \Gamma(ds)}{\int_{\mathcal{S}^{d-1}} |(b,s)|^\alpha \Gamma(ds)}$$

and

$$\mu(b) = \begin{cases} (b,\mu) & \text{if} \quad \alpha \neq 1 \\ (b,\mu) - \frac{2}{\pi} \int_{\mathcal{S}^{d-1}} (b,s) \ln |(b,s)| \Gamma(ds) & \text{if} \quad \alpha = 1. \end{cases}$$

The parameters $\sigma(b), \beta(b)$, and $\mu(b)$ are also called the projection parameters and $\sigma(.)$, $\beta(.)$ and $\mu(.)$ are called the projection parameter functions. If one knows the values of the projection functions for several directions, one can reconstruct approximatively the dependence structure of an α-stable random vector by estimating the spectral measure. Because of the complexity of this measure, the method is still not very efficient. But for specific subclasses of stable random vectors where the spectral measure has a much simpler form, we can use this technique to fit stable random vectors to data.

Another quantity for characterizing the dependence structure between two stable random vectors is the *covariation*.

Definition 5. Let X_1 and X_2 be jointly symmetric stable random variables with $\alpha > 1$ and let Γ be the spectral measure of the random vector $(X_1, X_2)'$. The *covariation* of X_1 on X_2 is the real number

$$[X_1, X_2]_\alpha = \int_{\mathcal{S}_1} s_1 s_2^{<\alpha-1>} \Gamma(ds), \tag{1}$$

where the signed power $a^{<p>}$ equals

$$a^{<p>} = |a|^p \operatorname{sign} a.$$

The covariance between two normal random variables X and Y can be interpreted as the inner product of the space $L_2(\Omega, \mathcal{A}, \mathbf{P})$. The covariation is the analogue of two α-stable random variables X and Y in the space $L_\alpha(\Omega, \mathcal{A}, \mathbf{P})$. Unfortunately, $L_\alpha(\Omega, \mathcal{A}, \mathbf{P})$ is not a Hilbert space and this is why it lacks some of the desirable and strong properties of the covariance. It follows immediately from the definition that the covariation is linear in the first argument. Unfortunately, this statement is not true for the second argument. In the case of $\alpha = 2$, the covariation equals the covariance.

Proposition 1. Let (X, Y) be joinly symmetric stable random vectors with $\alpha > 1$. Then for all $1 < p < \alpha$,

$$\frac{EXY^{<p-1>}}{E|Y|^p} = \frac{[X, Y]_\alpha}{||Y||_\alpha^\alpha},$$

where $||Y||_\alpha$ denotes the scale parameter of Y.

Proof. For the proof, see Samorodnitsky and Taqqu (1994). □

In particular, we apply proposition 1 in section 4.1 in order to derive an estimator for the dispersion matrix of an α-stable sub-Gaussian distribution.

2.3 α-stable sub-Gaussian random vectors

In general, as pointed out in the last section, α-stable random vectors have a complex dependence structure defined by the spectral measure. Since this measure is very difficult to estimate even in low dimensions, we have to retract to certain subclasses, where the spectral measure becomes simpler. One of these special classes is the multivariate α-stable sub-Gaussian distribution.

Definition 6. Let Z be a zero mean Gaussian random vector with variance covariance matrix Σ and $W \sim S_{\alpha/2}((\cos \frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)$ a totally skewed stable random variable independent of Z. The random vector

$$X = \mu + \sqrt{W} Z$$

is said to be a sub-Gaussian α-stable random vector. The distribution of X is called multivariate α-stable sub-Gaussian distribution.

An α-stable sub-Gaussian random vector inherits its dependence structure from the underlying Gaussian random vector. The matrix Σ is also called the dispersion matrix. The following theorem and proposition show properties of α-stable sub-Gaussian random vectors. We need these properties to derive estimators for the dispersion matrix.

Theorem 3. *The sub-Gaussian α-stable random vector X with location parameter $\mu \in \mathbf{R}^d$ has the characteristic function*

$$E(e^{it'X}) \quad = \quad e^{it'\mu} e^{-(\frac{1}{2}t'\Sigma t)^{\alpha/2}},$$

where $\Sigma_{ij} = EZ_iZ_j$, $i, j = 1, ..., d$ are the covariances of the underlying Gaussian random vector $(Z_1, ..., Z_d)'$.

For α-stable sub-Gaussian random vectors, we do not need the spectral measure in the characteristic functions. This fact simplifies the calculation of the projection functions.

Proposition 2. Let $X \in \mathbf{R}^d$ be an α-stable sub-Gaussian random vector with location parameter $\mu \in \mathbf{R}^d$ and dispersion matrix Σ. Then, for all $a \in \mathbf{R}^d$, we have $a'X \sim S_\alpha(\sigma(a), \beta(a), \mu(a))$, where

(i) $\sigma(a) = \frac{1}{2}(a'\Sigma a)^{1/2}$

(ii) $\beta(a) = 0$

(iii) $\mu(a) = a'\mu$.

Proof. It is well known that the distribution of $a'X$ is determined by its characteristic function.

$$\begin{aligned} E(\exp(it(a'X))) &= E(\exp(i(ta')X))) \\ &= \exp(ita'\mu)\exp(-|\frac{1}{2}(ta)'\Sigma(ta)|^{\alpha/2}|) \\ &= \exp(ita'\mu)\exp(-|\frac{1}{2}t^2a'\Sigma a|^{\alpha/2}|) \\ &= \exp(-|t|^\alpha|(\frac{1}{2}a'\Sigma a)^{\frac{1}{2}}|^\alpha + ita'\mu) \end{aligned}$$

If we choose $\sigma(a) = \frac{1}{2}(a'\Sigma a)^{1/2}$, $\beta(a) = 0$ and $\mu(a) = a'\mu$, then for all $t \in \mathbf{R}$ we have

$$E(\exp(it(a'X))) = \exp\left(-\sigma(a)^\alpha|t|^\alpha\left[1 - i\beta(a)\left(\tan\frac{\pi\alpha}{2}\right)(\text{sign } t)\right] + i\mu(a)t\right).$$

□

In particular, we can calculate the entries of the dispersion matrix directly.

Corollary 1. *Let $X = (X_1, ..., X_n)'$ be an α-stable sub-Gaussian random vector with dispersion matrix Σ. Then we obtain*

(i) $\sigma_{ii} = 2\sigma(e_i)^2$

(ii) $\sigma_{ij} = \frac{\sigma^2(e_i+e_j)-\sigma^2(e_i-e_j)}{2}$.

Since α-stable sub-Gaussian random vectors inherit their dependence structure of the underlying Gaussian vector, we can interpret σ_{ii} as the quasi-variance of the component X_i and σ_{ij} as the quasi-covariance between X_i and X_j.

Proof. It follows from proposition 2 that $\sigma(e_i) = \frac{1}{2}\sigma_{ii}^2$. Furthermore, if we set $a = e_i + e_j$ with $i \neq j$, we yield $\sigma(e_i + e_j) = (\frac{1}{2}(\sigma_{ii} + 2\sigma_{ij} + \sigma_{jj}))^{1/2}$ and for $b = e_i - e_j$, we obtain $\sigma(e_i - e_j) = (\frac{1}{2}(\sigma_{ii} - 2\sigma_{ij} + \sigma_{jj}))^{1/2}$. Hence, we have

$$\sigma_{ij} = \frac{\sigma^2(e_i + e_j) - \sigma^2(e_i - e_j)}{2}.$$

□

Proposition 3. Let $X = (X_1, ..., X_n)'$ be a zero mean α-stable sub-Gaussian random vector with dispersion matrix Σ. Then it follows

$$[X_i, X_j]_\alpha = 2^{-\alpha/2}\sigma_{ij}\sigma_{jj}^{(\alpha-2)/2}.$$

Proof. For a proof see Samorodnitsky and Taqqu (1994). □

3 α-stable sub-Gaussian distributions as elliptical distributions

Many important properties of α-stable sub-Gaussian distributions with respect to risk management, portfolio optimization, and principal component analysis can be understood very well, if we regard them as elliptical or normal variance mixture distributions. Elliptical distributions are a natural extension of the normal distribution which is a special case of this class. They obtain their name because of the fact that, their densities are constant on ellipsoids. Furthermore, they constitute a kind of ideal environment for standard risk management, see Embrechts, McNeil and Strautmann (1999). First, correlation and covariance have a very similar interpretation as in the Gaussian world and describe the dependence structure of risk factors. Second, the Markowitz optimization approach is applicable. Third, value-at-risk is a coherent risk measure. Fourth, they are closed under linear combinations, an important property in terms for portfolio optimization. And finally, in the elliptical world minimizing risk of a portfolio with respect to any coherent risk measures leads to the same optimal portfolio.

Empirical investigations have shown that multivariate return data for groups of similar assets often look roughly elliptical and in market risk management the elliptical hypothesis can be justified. Elliptical distributions cannot be applied in credit risk or operational risk, since hypothesis of elliptical risk factors are found to be rejected.

3.1 Elliptical distributions and basic properties

Definition 7. A random vector $X = (X_1, ..., X_d)'$ has

(i) a spherical distribution iff, for every orthogonal matrix $U \in \mathbf{R}^{d\times d}$,

$$UX \stackrel{d}{=} X.$$

(ii) an elliptical distribution if

$$X \stackrel{d}{=} \mu + AY,$$

where Y is a spherical random variable and $A \in \mathbf{R}^{d\times K}$ and $\mu \in \mathbf{R}^d$ are a matrix and a vector of constants, respectively.

Elliptical distributions are obtained by multivariate affine transformations of spherical distributions. Figure 7 (a) and (b) depict a bivariate scatterplot of BMW versus Daimler Chrysler and Commerzbank versus Deutsche Bank log-returns. Both scatterplots are roughly elliptical contoured.

Theorem 4. *The following statements are equivalent*

(i) X *is spherical.*

(ii) *There exists a function* ψ *of a scalar variable such that, for all* $t \in \mathbf{R}^d$*,*

$$\phi_X(t) = E(e^{it'X}) = \psi(t't) = \psi(t_1^2 + ... + t_d^2).$$

(iii) *For all* $a \in \mathbf{R}^d$*, we have*

$$a'X \stackrel{d}{=} ||a||X_1$$

(iv) X *can be represented as*

$$X \stackrel{d}{=} RS$$

where S *is uniformly distributed on* $\mathcal{S}^{d-1} = \{x \in \mathbf{R}^d : x'x = 1\}$ *and* $R \geq 0$ *is a radial random variable independent of S.*

Proof. See McNeil, Frey and Embrechts (2005) □

ψ is called the characteristic generator of the spherical distribution and we use the notation $X \in S_d(\psi)$.

Corollary 2. *Let* X *be a d-dimensional elliptical distribution with* $X \stackrel{d}{=} \mu + AY$*, where* Y *is spherical and has the characteristic generator* ψ*. Then, the characteristic function of* X *is given by*

$$\phi_X(t) := E(e^{it'X}) = e^{it'\mu}\psi(t'\Sigma t),$$

where $\Sigma = AA'$*.*

Furthermore, X *can be represented by*

$$X = \mu + RAS,$$

where S *is the uniform distribution on* $\mathcal{S}^{d-1}$ *and* $R \geq 0$ *is a radial random variable.*

Proof. We notice that

$$\begin{aligned}\phi_X(\boldsymbol{t}) &= E(e^{it'X}) = E(e^{it'(\mu+AY)}) = e^{it'\mu}E(e^{i(A't)'Y}) = e^{it'\mu}\psi((A't)'(A't)) \\ &= e^{it'\mu}\psi(t'AA't)\end{aligned}$$

□

Since the characteristic function of a random variate determines the distribution, we denote an elliptical distribution by

$$X \sim E_d(\boldsymbol{\mu}, \Sigma, \psi).$$

Because of

$$\mu + RAS = \mu + cR\frac{A}{c}S,$$

the representation of the elliptical distribution in equation (2) is not unique. We call the vector μ the location parameter and Σ the dispersion matrix of an elliptical distribution, since first and second moments of elliptical distributions do not necessarily exist. But if they exist, the location parameter equals the mean and the dispersion matrix equals the covariance matrix up to a scale parameter. In order to have uniqueness for the dispersion matrix, we demand $\det(\Sigma) = 1$.

If we take any affine linear combination of an elliptical random vector, then, this combination remains elliptical with the same characteristic generator ψ. Let $X \sim E_d(\mu, \Sigma, \psi)$, then it can be shown with similar arguments as in corollary 2 that

$$BX + b \quad \sim \quad E_k(B\mu + b, B\Sigma B', \psi)$$

where $B \in \mathbf{R}^{k\times d}$ and $b \in \mathbf{R}^d$.

Let X be an elliptical distribution. Then the density $f(x)$, $x \in \mathbf{R}^d$, exists and is a function of the quadratic form

$$f(x) = \det(\Sigma)^{-1/2} g(Q) \text{ with } Q := (x-\mu)'\Sigma^{-1}(x-\mu).$$

g is the density of the spherical distribution Y in definition 7. We call g the density generator of X. As a consequence, since Y has an unimodal density, so is the density of X and clearly, the joint density f is constant on hypersheres $H_c = \{x \in \mathbf{R}^d : Q(x) = c\}$, $c > 0$. These hyperspheres H_c are elliptically contoured.

Example 1. *An α-stable sub-Gaussian random vector is an elliptical random vector. The random vector $\sqrt{W}Z$ is spherical, where $W \sim S_\alpha((\cos\frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)$ and $Z \sim N(0,1)$ because of*

$$\sqrt{W}Z \stackrel{d}{=} U\sqrt{W}Z$$

for any orthogonal matrix. The equation is true, since Z is rotationally symmetric. Hence any linear combination of $\sqrt{W}Z$ is an elliptical random vector. The characteristic function of an α-stable sub-Gaussian random vector is given by

$$E(e^{it'X}) \quad = \quad e^{it'\mu}e^{-(\frac{1}{2}t'\Sigma t)^{\alpha/2}}$$

due to Theorem 3. Thus, the characteristic generator of an α-stable sub-Gaussian random vector equals

$$\psi_{sub}(s,\alpha) = e^{-(\frac{1}{2}s)^{2/\alpha}}.$$

Using the characteristic generator, we can derive directly that an α-stable sub-Gaussian random vector is infinitely divisible, since we have

$$\begin{aligned}\psi_{sub}(s,\alpha) &= e^{-(\frac{1}{2}s)^{\alpha/2}} = \left(e^{-\left(\frac{1}{2}\frac{s}{n^{2/\alpha}}\right)^{\alpha/2}}\right)^n \\ &= \left(\psi_{sub}\left(\frac{s}{n^{2/\alpha}}, \alpha\right)\right)^n.\end{aligned}$$

3.2 Normal variance mixture distributions

Normal variance mixture distributions are a subclass of elliptical distributions. We will see that they inherit their dependence structure from the underlying Gaussian random vector. Important distributions in risk management such as the multivariate t-, generalized hyperbolic, or α-stable sub-Gaussian distribution belong to this class of distributions.

Definition 8. The random vector X is said to have a (multivariate) normal variance mixture distribution (NVMD) if

$$X = \mu + W^{1/2}AZ$$

where

(i) $Z \sim N_d(0, I_d)$;

(ii) $W \geq 0$ is a non-negative, scalar-valued random variable which is independent of G, and

(iii) $A \in \mathbf{R}^{d\times d}$ and $\mu \in \mathbf{R}^d$ are a matrix of constants, respectively.

We call a random variable X with NVMD a normal variance mixture (NVM). We observe that $X_w = (X|W = w) \sim N_d(\mu, w\Sigma)$, where $\Sigma = AA'$. We can interpret the distribution of X as a composite distribution. According to the law of W, we take normal random vectors X_w with mean zero and covariance matrix $w\Sigma$ randomly. In the context of modeling asset returns or risk factor returns with normal variance mixtures, the mixing variable W can be thought of as a shock that arises from new information and influences the volatility of all stocks.

Since $U\sqrt{W}Z \overset{d}{=} \sqrt{W}Z$ for all $U \in O(d)$ every normal variance mixture distribution is an elliptical distribution. The distribution F of X is called the mixing law. Normal variance mixture are closed under affine linear combinations, since they are elliptical. This can also be seen directly by

$$\begin{aligned} BX + \mu_1 &\overset{d}{=} B(\sqrt{W}AZ + \mu_0) + \mu = \sqrt{W}BAZ + (B\mu_0 + \mu_1) \\ &= \sqrt{W}\tilde{A}Z + \tilde{\mu}. \end{aligned}$$

This property makes NVMDs and, in particular, MSSDs applicable to portfolio theory. The class of NVMD has the advantage that structural information about the mixing law W can be transferred to the mixture law. This is true, for example, for the property of infinite divisibility. If the mixing law is infinitely divisible, then so is the mixture law. (For further information see Bingham, Kiesel and Schmidt (2003).) It is obvious from the definition that an α-stable sub-Gaussian random vector is also a normal variance mixture with mixing law $W \sim S_\alpha((\cos\frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)$.

3.3 Market risk management with elliptical distributions

In this section, we discuss the properties of elliptical distributions in terms of market risk management and portfolio optimization. In risk management, one is mainly interested in modeling the extreme losses which can occur. From empirical investigations, we know that an extreme loss in one asset very often occurs with high losses in many other assets. We show that this market behavior cannot be modeled by the normal distribution but, with certain elliptical distributions, e.g. α-stable sub-Gaussian distribution, we can capture this behavior.

The Markowitz's portfolio optimization approach which is originally based on the normal assumption can be extended to the class of elliptical distributions. Also, statistical dimensionality reduction methods such as the principal component analysis are applicable to them. But one must be careful, in contrast to the normal distribution, these principal components are not independent.

Let F be the distribution function of the random variable X, then we call

$$F^{\leftarrow}(\alpha) = \inf\{x \in \mathbf{R} : F(x) \geq \alpha\}$$

the quantile function. $F^{\leftarrow}$ is also the called generalized inverse, since we have

$$F(F^{\leftarrow}(\alpha)) = \alpha,$$

for any df F.

Definition 9. Let X_1 and X_2 be random variables with dfs F_1 and F_2. The coefficient of the upper tail dependence of X_1 and X_2 is

$$\lambda_u := \lambda_u(X_1, X_2) := \lim_{q \to 1^-} P(X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q)), \tag{2}$$

provided a limit $\lambda_u \in [0, 1]$ exists. If $\lambda_u \in (0, 1]$, then X_1 and X_2 are said to show upper tail dependence; if $\lambda_u = 0$, they are asymptotically independent in the upper tail. Analogously, the coefficient of the lower tail dependence is

$$\lambda_l = \lambda_l(X_1, X_2) = \lim_{q \to 0^+} P(X_2 \leq F^{\leftarrow}(q) | X_1 \leq F_1^{\leftarrow}(q)), \tag{3}$$

provided a limit $\lambda_l \in [0, 1]$ exists.

For a better understanding of tail dependence we introduce the concept of copulas.

Definition 10. A d-dimensional copula is a distribution function on $[0, 1]^d$.

It is easy to show that for $U \sim U(0, 1)$, we have $P(F^{\leftarrow}(U) \leq x) = F(x)$ and if the random variable Y has a continuous df G, then $G(Y) \sim U(0, 1)$. The concept of copulas gained its importance because of Sklar's Theorem.

Theorem 5. *Let F be a joint distribution function with margins $F_1, ..., F_d$. Then, there exists a copula $C : [0, 1]^d \to [0, 1]$ such that for all $x_1, ..., x_d$ in $\overline{\mathbf{R}} = [\infty, \infty]$,*

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{4}$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on $F_1(\overline{R}) \times F_2(\overline{R}) \times .... \times F_d(\overline{R})$. Conversely, if C is a copula and $F_1, ..., F_d$ are univariate distribution functions, the function F defined in (4) is a joint distribution function with margins $F_1, ..., F_d$.

This fundamental theorem in the field of copulas, shows that any multivariate distribution F can be decomposed in a copula C and the marginal distributions of F. Vice versa, we can use a copula C and univariate dfs to construct a multivariate distribution function.

With this short excursion in the theory of copulas we obtain a simpler expression for the upper and the lower tail dependencies, i.e.,

$$\begin{aligned}\lambda_l &= \lim_{q\to 0^+} \frac{P(X_2 \le F^{\leftarrow}(q), X_1 \le F_1^{\leftarrow}(q))}{P(X_1 \le F_1^{\leftarrow}(q))} \\ &= \lim_{q\to 0^+} \frac{C(q,q)}{q}.\end{aligned}$$

Elliptical distributions are radially symmetric, i.e., $\mu - X \stackrel{d}{=} \mu + X$, hence the coefficient of lower tail dependence λ_l equals the coefficient of upper tail dependence λ_u. We denote with λ the coefficient of tail dependence.

We call a measurable function $f : \mathbf{R}_+ \to \mathbf{R}_+$ *regularly varying* (at ∞) with index $\alpha \in \mathbf{R}$ if, for any $t > 0$, $\lim_{x\to\infty} f(tx)/f(x) = t^\alpha$. It is now important to notice that regularly varying functions with index $\alpha \in \mathbf{R}$ behave asymptotically like a power function. An elliptically distributed random vector $X = RAU$ is said to be regularly varying with tail index α, if the function $f(x) = P(R \ge x)$ is regularly varying with tail index α. (see Resnick (1987).) The following theorem shows the relation between the tail dependence coefficient and the tail index of elliptical distributions.

Theorem 6. *Let $X \sim E_d(\mu, \Sigma, \psi)$ be regularly varying with tail index $\alpha > 0$ and Σ a positive definite dispersion matrix. Then, every pair of components of X, say X_i and X_j, is tail dependent and the coefficient of tail dependence corresponds to*

$$\lambda(X_i, X_j; \alpha, \rho_{ij}) = \frac{\int_0^{f(\rho_{ij})} \frac{s^\alpha}{\sqrt{1-s^2}} ds}{\int_0^1 \frac{s^\alpha}{\sqrt{1-s^2}} ds} \tag{5}$$

where $f(\rho_{ij}) = \sqrt{\frac{1+\rho_{ij}}{2}}$ and $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$.

Proof. See Schmidt (2002). □

It is not difficult to show that an α-stable sub-Gaussian distribution is regularly varying with tail index α. The coefficient of tail dependence between two components, say X_i and X_j, is determined by equation (5) in theorem 6. In the next example, we demonstrate that the coefficient of tail dependence of a normal distribution is zero.

Example 2. *Let (X_1, X_2) be a bivariate normal random vector with correlation ρ and standard normal marginals. Let C_ρ be the corresponding Gaussian copula due to*

Sklar's theorem, then, by the L'Hôpital rule,

$$\begin{aligned}
\lambda &= \lim_{q\to 0^+} \frac{C_\rho(q,q)}{q} \stackrel{l'H}{=} \lim_{q\to 0^+} \frac{dC_\rho(q,q)}{dq} = \lim_{q\to 0^+}\lim_{h\to 0^+} \frac{C_\rho(q+h,q+h)-C_\rho(q,q)}{h} \\
&= \lim_{q\to 0^+}\lim_{h\to 0} \frac{C_\rho(q+h,q+h)-C_\rho(q+h,q)+C_\rho(q+h,q)-C_\rho(q,q)}{h} \\
&= \lim_{q\to 0^+}\lim_{h\to 0} \frac{P(U_1\le q+h, q\le U_2\le q+h))}{P(q\le U_2\le q+h)} \\
&+ \lim_{q\to 0^+}\lim_{h\to 0} \frac{P(q\le U_1\le q+h, U_2\le q)}{P(q\le U_1\le q+h)} \\
&= \lim_{q\to 0^+} P(U_2\le q|U_1=q) + \lim_{q\to 0^+} P(U_1\le q|U_2=q) \\
&= 2\lim_{q\to 0^+} P(U_2\le q|U_1=q) \\
&= 2\lim_{q\to 0^+} P(\Phi^{-1}(U_2)\le \Phi^{-1}(q)|\Phi^{-1}(U_1)=\Phi^{-1}(q)) \\
&= 2\lim_{x\to -\infty} P(X_2\le x|X_1=x)
\end{aligned}$$

Since we have $X_2|X_1 = x \sim N(\rho x, 1-\rho^2)$, we obtain

$$\lambda = 2\lim_{x\to -\infty} \Phi(x\sqrt{1-\rho}/\sqrt{1+\rho}) = 0 \qquad (6)$$

Equation (6) shows that beside the fact that a normal distribution is not heavy tailed the components are asymptotically independent. This, again, is a contradiction to empirical investigations of market behavior. Especially, in extreme market situations, when a financial market declines in value, market participants tend to behave homogeneously, i.e. they leave the market and sell their assets. This behavior causes losses in many assets simultaneously. This phenomenon can only be captured by distributions which are asymptotically dependent.

Markowitz (1952) optimizes the risk and return behavior of a portfolio based on the expected returns and the covariances of the returns in the considered asset universe. The risk of a portfolio consisting of these assets is measured by the variance of the portfolio return. In addition, he assumes that the asset returns follow a multivariate normal distribution with mean μ and covariance Σ. This approach leads to the following optimization problem

$$\min_{w\in\mathbf{R}^d} w'\Sigma w,$$

subject to

$$\begin{aligned}
w'\mu &= \mu_p \\
w'\underline{1} &= 1.
\end{aligned}$$

This approach can be extended in two ways. First, we can replace the assumption of normally distributed asset returns by elliptically distributed asset returns and second, instead of using the variance as the risk measure, we can apply any positive-homogeneous, translation-invariant measure of risk to rank risk or to determine the optimal risk-minimizing portfolio. In general, due to the work of Artzner et al. (1999), a risk measure is a real-valued function $\varrho : \mathcal{M} \to \mathbf{R}$, where $\mathcal{M} \subset L^0(\Omega, \mathcal{F}, P)$ is a convex cone. $L^0(\Omega, \mathcal{F}, P)$ is the set of all almost surely finite random variables. The risk measure ϱ is *translation invariant* if for all $L \in \mathcal{M}$ and every $l \in \mathbf{R}$, we have $\varrho(L + l) = \varrho(L) + l$. It is *positive-homogeneous* if for all $\lambda > 0$, we have $\varrho(\lambda L) = \lambda \varrho(L)$. Note, that value-at-risk (VaR) as well as conditional value-at-risk (CVaR) fulfill these two properties.

Theorem 7. *Let the random vector of asset returns X be $E_d(\mu, \Sigma, \psi)$. We denote by $\mathcal{W} = \{w \in \mathbf{R}^d : \sum_{i=1}^d w_i = 1\}$ the set of portfolio weights. Assume that the current value of the portfolio is V and let $L(w) = V \sum_{i=1}^d w_i X_i$ be the (linearized) portfolio loss. Let ϱ be a real-valued risk measure depending only on the distribution of a risk. Suppose ϱ is positive homogeneous and translation invariant and let $\mathcal{Y} = \{w \in \mathcal{W} : -w'\mu = m\}$ be the subset of portfolios giving expected return m. Then,* $\operatorname{argmin}_{w \in \mathcal{Y}} \varrho(L(w)) = \operatorname{argmin}_{w \in \mathcal{Y}} w'\Sigma w$.

Proof. See McNeil, Frey and Embrechts (2005). □

The last theorem stresses that the dispersion matrix contains all the information for the management of risk. In particular, the tail index of an elliptical random vector has no influence on optimizing risk. Of course, the index has an impact on the value of the particular risk measure like VaR or CVaR, but not on the weights of the optimal portfolio, due to the Markowitz approach.

In risk management, we have very often to deal with portfolios consisting of many different assets. In many of these cases it is important to reduce the dimensionality of the problem in order to not only understand the portfolio's risk but also to forecast the risk. A classical method to reduce the dimensionality of a portfolio whose assets are highly correlated is principal component analysis (PCA). PCA is based on the spectral decomposition theorem. Any symmetric or positive definite matrix Σ can be decomposed in

$$\Sigma = PDP',$$

where P is an orthogonal matrix consisting of the eigenvectors of Σ in its columns and D is a diagonal matrix of the eigenvalues of Σ. In addition, we demand $\lambda_i \geq \lambda_{i-1}$, $i = 1, ..., d$ for the eigenvalues of Σ in D. If we apply the spectral decomposition theorem to the dispersion matrix of an elliptical random vector X with distribution $E_d(\mu, \Sigma, \psi)$, we can interpret the principal components which are defined by

$$Y_i = P_i'(X - \mu), i = 1, ..., d, \tag{7}$$

as the main statistical risk factors of the distribution of X in the following sense

$$P_1'\Sigma P_1 = \max\{w'\Sigma w : w'w = 1\}. \tag{8}$$

More generally,

$$P_i'\Sigma P_i = \max\{w'\Sigma w : w \in \{P_1, ..., P_{i-1}\}^\perp, w'w = 1\}.$$

From equation (8), we can derive that the linear combination $Y_1 = P_1'(X - \mu)$ has the highest dispersion of all linear combinations and $P_i'X$ has the highest dispersion in the linear subspace $\{P_1, ..., P_{i-1}\}^\perp$. If we interpret $\operatorname{trace}\Sigma = \sum_{j=1}^d \sigma_{ii}$ as a measure of total variability in X and since we have

$$\sum_{i=1}^d P_i'\Sigma P_i = \sum_{i=1}^d \lambda_i = \operatorname{trace}\Sigma = \sum_{i=1}^d \sigma_{ii},$$

we can measure the ability of the first principal component to explain the variability of X by the ratio $\sum_{j=1}^k \lambda_j / \sum_{j=1}^d \lambda_j$.

Furthermore, we can use the principal components to construct a statistical factor model. Due to equation (7), we have

$$Y = P'(X - \mu),$$

which can be inverted to

$$X = \mu + PY.$$

If we partition Y due to $(Y_1, Y_2)'$, where $Y_1 \in \mathbf{R}^k$ and $Y_2 \in \mathbf{R}^{d-k}$ and also P leading to (P_1, P_2), where $P_1 \in \mathbf{R}^{d\times k}$ and $P_2 \in \mathbf{R}^{d\times(d-k)}$, we obtain the representation

$$X = \mu + P_1Y_1 + P_2Y_2 = \mu + P_1Y_1 + \epsilon.$$

But one has to be careful. In contrast to the normal distribution case, the principal components are only quasi-uncorrelated but not independent. Furthermore, we obtain for the coefficient of tail dependence between two principal components, say Y_i and Y_j,

$$\lambda(Y_i, Y_j, 0, \alpha) = \frac{\int_0^{\sqrt{1/2}} \frac{s^\alpha}{\sqrt{1-s^2}} ds}{\int_0^1 \frac{s^\alpha}{\sqrt{1-s^2}} ds}.$$

4 Estimation of an α-stable sub-Gaussian distributions

In contrast to the general case of multivariate α-stable distributions, we show that the estimation of the parameters of an α-stable sub-Gaussian distribution is feasible. As shown in the last section, α-stable sub-Gaussian distributions belong to the class of elliptical distributions. In general, one can apply a two-step estimation procedure for the elliptical class. In the first step, we estimate the dispersion matrix up to a scale parameter and the location parameter. In the second step, we estimate the parameter of the radial random variable W. We apply this idea to α-stable sub-Gaussian distributions. In sections 4.1 and 4.2 we present our main theoretical results, deriving estimators for the dispersion matrix and proving their consistency. In section 4.3 we present a new procedure to estimate the parameter α of an α-stable sub-Gaussian distribution. Note that with $\sigma_i := \sigma(e_i)$ we denote the scale parameter of the jth component of an α-stable random vector.

4.1 Estimation of the dispersion matrix with covariation

In section 2.1, we introduced the covariation of a multivariate α-stable random vector. This quantity allows us to derive a consistent estimator for an α-stable dispersion matrix.

Proposition 4. (a) Let $X \in \mathbf{R}^d$ be a zero mean α-stable sub-Gaussian random vector with dispersion matrix Σ. Then, we have

$$\sigma_{ij} = \frac{2}{c_{\alpha,0}(p)^p}\sigma(e_j)^{2-p}E(X_i X_j^{<p-1>}), p \in (1,\alpha). \tag{9}$$

(b) Let $X_1, X_2, ..., X_n$ be i.i.d. samples with the same distribution as the random vector X. Let $\hat{\sigma}_j$ be a consistent estimator for σ_j, the scale parameter of the jth component of X, then, the estimator

$$\hat{\sigma}_{ij}^{(2)} = \frac{2}{c_{\alpha,0}(p)^p}\hat{\sigma}_j^{2-p}\frac{1}{n}\sum_{t=1}^{n} X_{ti}X_{tj}^{<p-1>} \tag{10}$$

is a consistent estimator for σ_{ij}, where X_{ti} refers to the ith entries of the observation X_t, $t = 1, ..., n$.

Proof. a) Due to the proposition we have

$$\begin{aligned}
\sigma_{ij} &\overset{\text{Prop.3}}{=} 2^{\alpha/2}\sigma_{jj}^{(2-p)/2}[X_i, X_j]_\alpha \\
&\overset{\text{Prop.1}}{=} 2^{\alpha/2}\sigma_{jj}^{(2-\alpha)/2}E(X_i X_j^{<p-1>})\sigma_j^\alpha / E(|X_j|^p) \\
&\overset{\text{Lemma 1}}{=} 2^{\alpha/2}\sigma_{jj}^{(2-\alpha)/2}E(X_i X_j^{<p-1>})\sigma_j^\alpha / (c_{\alpha,0}(p)^p\sigma_j^p) \\
&\overset{\text{Cor.1 (i)}}{=} 2^{p/2}\sigma_{jj}^{(2-p)/2}E(X_i X_j^{<p-1>}) / (c_{\alpha,0}(p)^p) \\
&= \frac{2}{c_{\alpha,0}(p)^p}\hat{\sigma}_j^{2-p}\frac{1}{n}\sum_{t=1}^{n} X_{ti}X_{tj}^{<p-1>}
\end{aligned}$$

b) The estimator $\hat{\sigma}_j$ is consistent and $f(x) = x^{2-p}$ is continuous. Then, the estimator $\hat{\sigma}^{2-p}$ is consistent for σ_j^{2-p}. $\frac{1}{n}\sum_{k=1}^{n} X_{ki}X_{kj}^{<p-1>}$ is consistent for $E(X_i X_j^{<p-1>})$ due to the law of large numbers. Since the product of two consistent estimators is consistent, $\hat{\sigma}_{ij}^{(2)}$ is consistent.

□

4.2 Estimation of the dispersion matrix with moment-type estimators

In this section, we present an approach of estimating the dispersion matrix which is applicable to the class of normal variance mixtures. In general, the approach allows one to estimate the dispersion matrix up to a scaling constant. If we know the tail parameter α of an α-stable sub-Gaussian random vector, we can estimate the α-stable sub-Gaussian dispersion matrix.

Lemma 2. *Let $X = \mu + \sqrt{W}Z$ be a d-dimensional normal variance mixture with mean μ and dispersion matrix Σ. W belongs to the parametric distribution family Θ with existing first moments. Furthermore, we have*

(i) $P^W = P_{\theta_0} \in \{P_\theta : \theta \in \Theta\}$,

(ii) $P(\sqrt{W} \geq x) = L(x)x^{-\alpha}$ *and* $L : \mathbf{R} \to \mathbf{R}$ *slowly varying,*

(iii) $\det(\Sigma) = 1$.

Then, there exists a function $c : \Theta \times (0, \alpha) \to \mathbf{R}$ such that, for all $a \in \mathbf{R}^d$, we have

$$E(|a'(X - \mu)|^p) = c(\theta_0, p)^p (a'\Sigma a)^{p/2}.$$

The function c is defined by

$$c(\theta, p) = \int x^{p/2} P_\theta(dx) E(|\tilde{Z}|^p)$$

and, for all $\theta \in \Theta$, satisfies

$$\lim_{p \to 0} c(\theta, p) = 1, \tag{11}$$

where $\tilde{Z}$ is standard normally distributed.

Proof. We have

$$\begin{aligned} E(|a'(X - \mu)|^p) &= E(|a'W^{1/2}Z|^p) \\ &= E(W^{p/2})(a'\Sigma a)^{p/2} E(|a'Z/(a'\Sigma a)^{1/2}|^p). \end{aligned}$$

Note that $\tilde{Z} = a'Z/(a'\Sigma a)^{1/2}$ is standard normally distributed.

Since $f(x) = |x|$ is integrable with respect to P_θ for all $\theta \in \Theta$ and the normal distribution, we conclude by Lebesque's Theorem

$$\begin{aligned} \lim_{p \to 0} c(\theta, p) &= \lim_{p \to 0} \int x^{p/2} P_\theta(dx) E(|\tilde{Z}|^p) \\ &= \int \lim_{p \to 0} x^{p/2} P_\theta(dx) E(\lim_{p \to 0} |\tilde{Z}|^p) \\ &= \int 1 P_\theta(dx) E(1) = 1. \end{aligned}$$

□

Theorem 8. *Let $X, c : \Theta \times (0, \alpha) \to \mathbf{R}$ be as in lemma 2 and let $X_1, ..., X_n$ be i.i.d. samples with the same distribution as X. The estimator*

$$\hat{\sigma}_n(p, a) = \frac{1}{n} \sum_{i=1}^{n} \frac{|a'(X_i - \mu)|^p}{c(\theta_0, p)^p} \tag{12}$$

(i) is unbiased, i.e.,

$$E(\hat{\sigma}_n(p, a)) = (a'\Sigma a)^{p/2} \text{ for all } a \in \mathbf{R}$$

(ii) is consistent, i.e.,

$$P(|\hat{\sigma}_n(p,a) - (a'\Sigma a)^{p/2}| > \epsilon) \to 0 \; (n \to \infty).$$

Proof. (i) follows directly from lemma 2. For statement (ii), we notice that we can show

$$P(|\hat{\sigma}_n(p,a) - (a'\Sigma a)^{p/2}| > \epsilon) \quad \leq \quad \frac{1}{n\epsilon^2}\left(\left(\frac{c(\theta_0, 2/n)}{c(\theta_0, 1/n)}\right)^{2p} - 1\right)(a'\Sigma a)^p$$

with the Markov inequation. Then (ii) follows from equation (11) in lemma 2. □

$\hat{\sigma}_n(p,a)^{2/p}$, $a \in \mathbf{R}^d$, is a biased, but consistent estimator for $(a\Sigma a')$. The advantages of the method are that it is easy to implement and it is intuitive. Furthermore, we can easily extend the method by using weights w_i for every observation X_i, $i = 1, ..., n$, which enables us to implement the EWMA method suggested by Riskmetrics. The last theorem allows us to estimate the dispersion matrix of a normal variance mixture by component and up to a scaling parameter, since we can use one of the two following estimation approaches

$$\hat{\sigma}_{ij} = \frac{\hat{\sigma}_n^{2/p}(e_i + e_j) - \hat{\sigma}_n^{2/p}(e_i - e_j)}{4} \tag{13}$$

or

$$\hat{\Sigma} = \arg \min_{\Sigma \text{ pos. def.}} \sum_{i=1}^{n} (\sigma_n^{2/p}(a_i) - a_i'(\Sigma)a_i)^2. \tag{14}$$

The last approach even ensures positive definiteness of the dispersion matrix, but the approach is mathematically more involved.

In the next theorem, we present an estimator that is based on the following oberservation. Leting $X, X_1, X_2, X_3, ...$ be a sequence of i.i.d. normal variance mixtures, then we have

$$\begin{aligned} \lim_{n\to\infty} \lim_{p\to 0} \left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{a'X_i - \mu(a)}{c(\theta_0, p)}\right|^p\right)^{1/p} &\overset{*}{=} \lim_{n\to\infty}\prod_{i=1}^{n}|a'X_i - \mu(a)|^{1/n} \\ &= (a'\Sigma a)^{1/2}. \end{aligned}$$

The last equation is true because of (ii) of the following theorem. The proof of the equality (*) can be found in Stoyanov (2005).

Theorem 9. *Let X, $c : \Theta \times (0, \alpha) \to \mathbf{R}$ be as in lemma 2 and let $X_1, ..., X_n$ be i.i.d. samples with the same distribution as X. The estimator*

$$\hat{\sigma}_n(a) = \frac{1}{c(\theta_0, 1/n)} \prod_{i=1}^{n} |a'(X_i - \mu)|^{1/n}$$

(i) is unbiased, i.e.,

$$E(\hat{\sigma}_n(a)) = (a'\Sigma a)^{1/2} \textit{ for all } a \in \mathbf{R}$$

(ii) is consistent, i.e.,

$$P(|\hat{\sigma}_n(a) - (a'\Sigma a)^{1/2}| > \epsilon) \to 0 \; (n \to \infty).$$

Proof. (i) follows directly from lemma 2. For statement (ii), we notice that we can show

$$P(|\hat{\sigma}_n(a) - (a'\Sigma a)^{1/2}| > \epsilon) \quad \leq \quad \frac{1}{\epsilon^2}\left(\frac{c(\theta_0, 2/n)}{c(\theta_0, 1/n)} - 1\right)(a'\Sigma a),$$

due to the Markov inequation. Then (ii) follows from the equation (11) in lemma 2. □

Note, that $\sigma_n^2(a)$, $a \in \mathbf{R}^d$, is a biased but consistent estimator for $(a'\Sigma a)$.

For the rest of this section we concentrate on α-stable sub-Gaussian random vectors. In this case the dispersion matrix is unique since the scalar-valued random variable $W \sim S_{\alpha/2}((\cos\frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)$ has a well-defined scale parameter which determines the scale parameter of the dispersion matrix. Since an α-stable sub-Gaussian random vector is symmetric, all linear combinations $a'X$ are also symmetric due to theorem 1. We obtain $\theta = (\alpha, \beta)$ with $\beta = 0$ and one can show

$$\begin{aligned} c(\alpha, p)^p \quad &= \quad c(\alpha, \beta = 0, p)^p = 2^p \frac{\Gamma(\frac{p+1}{2})\Gamma(1 - p/\alpha)}{\Gamma(1 - p/2)\sqrt{\pi}} \\ &= \quad \frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\Gamma\left(1 - \frac{p}{\alpha}\right). \end{aligned}$$

(For the proof, see Hardin (1984) and Stoyanov (2005).) With theorems 8 and 9, we derive two estimators for the scale parameter $\sigma(a)$ of projections of α-stable sub-Gaussian random vectors. The first one is

$$\hat{\sigma}_n(p, a) \quad = \quad \frac{1}{n}\left(\frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\Gamma\left(1 - \frac{p}{\alpha}\right)\right)^{-1}\sum_{i=1}^{n}|a'X_i - \mu(a)|^p$$

based on theorem 8. The second one is

$$\begin{aligned} \hat{\sigma}_n(a) \quad &= \quad \frac{1}{c(\alpha, 1/n)}\prod_{i=1}^{n}(|a'X_i - \mu(a)|)^{1/n} \\ &= \quad \left(\frac{2}{\pi}\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1 - \frac{1}{n\alpha}\right)\right)^{-n} \cdot \\ &\quad\quad \prod_{i=1}^{n}(|a'X_i - \mu(a)|)^{1/n}. \end{aligned}$$

based on theorem 9. We reconstruct the stable dispersion matrix from the projections as shown in the equations (13) and (14).

4.3 Estimation of the parameter α

We assume that the data $X_1, ..., X_n \in \mathbf{R}^d$ follow a sub-Gaussian α-stable distribution. We propose the following algorithm to obtain the underlying parameter α of the distribution.

(i) Generate i.i.d. samples $u_1, u_2, ..., u_n$ according to the uniform distribution on the unit hypersphere $\mathcal{S}^{d-1}$.

(ii) For all i from 1 to n estimate the index of stability α_i with respect to the data $u_i'X_1, u_i'X_2, ..., u_i'X_n$, using an unbiased and fast estimator $\hat{\alpha}$ for the index.

(iii) Calculate the index of stability of the distribution by

$$\hat{\alpha} = \frac{1}{n}\sum_{k=1}^{n} \hat{\alpha}_k.$$

The algorithm converges to the index of stability α of the distribution. (For further information we refer to Rachev and Mittnik (2000).)

4.4 Simulation of α-stable sub-Gaussian distributions

Efficient and fast multivariate random number generators are indispensable for modern portfolio investigations. They are important for Monte-Carlo simulations for VaR, which have to be sampled in a reasonable timeframe. For the class of elliptical distributions we present a fast and efficient algorithm which will be used for the simulation of α-stable sub-Gaussian distributions in the next section. We assume the dispersion matrix Σ to be positive semi-definite. Hence we obtain for the Cholesky decomposition $\Sigma = AA'$ a unique full-rank lower-triangular matrix $A \in \mathbf{R}^{r\times r}$. We present a generic algorithm for generating multivariate elliptically-distributed random vectors. The algorithm is based on the stochastic representation of corollary 2. For the generation of our samples, we use the following algorithm:

Algorithm for $EC_r(\mu, R; \psi_{sub})$ simulation

(i) Set $\Sigma = AA'$, via Cholesky decomposition.

(ii) Sample a random number from W.

(iii) Sample d independent random numbers $Z_1, ..., Z_d$ from a $N_1(0,1)$ law.

(iv) Set $U = Z/||Z||$ with $Z = (Z_1, ..., Z_d)$.

(v) Return $X = \mu + \sqrt{W}AU$

If we want to generate random number with a $E_d(\mu, \Sigma, \psi_{sub})$ law with the algorithm, we choose $W \stackrel{d}{=} S_{\alpha/2}(\cos(\frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)||Z||^2$, where Z is $N_d(0, Id)$ distributed. It can be shown that $||Z||^2$ is independent of both W as well as $Z/||Z||$.

5 Emprical analysis of the estimators

In this section, we evaluate two different estimators for the dispersion matrix of an α-stable sub-Gaussian distribution using boxplots. We are primarily interested in estimating the off-diagonal entries, since the diagonal entries σ_{ii} are essentially only the square of the scale parameter σ. Estimators for the scale parameter σ have been analyzed in numerous studies. Due to corollary 1 and theorem 9, the estimator

$$\hat{\sigma}_{ij}^{(1)}(n) = \frac{(\hat{\sigma}_n(e_i + e_j))^2 - (\hat{\sigma}_n(e_i - e_j))^2}{2} \tag{15}$$

is a consistent estimator for σ_{ij} and the second estimator

$$\hat{\sigma}_{ij}^{(2)}(n,p) = \frac{2}{c_{\alpha,0}(p)^p}\hat{\sigma}_n(e_j)^{2-p}\frac{1}{n}\sum_{k=1}^{n} X_{ki}X_{kj}^{<p-1>}. \tag{16}$$

is consistent because of proposition 4 for $i \neq j$. We analyze the estimators empirically.

For an empirical evaluation of the estimators described above, it is sufficient to exploit the two-dimensional sub-Gaussian law since for estimating σ_{ij} we only need the ith and jth component of the data $X_1, X_2, ..., X_n \in \mathbf{R}^d$. For a better understanding of the speed of convergence of the estimators, we choose different sample sizes ($n = 100, 300, 500, 1000$). Due to the fact that asset returns exhibit an index of stability in the range between 1.5 and 2, we only consider the values $\alpha = 1.5, 1.6, ..., 1.9$. For the empirical analysis of the estimators, we choose the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

The corresponding dispersion matrix is

$$\Sigma = AA' = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}.$$

5.1 Empirical analysis of $\hat{\sigma}_{ij}^{(1)}(n)$

For the empirical analysis of $\hat{\sigma}_{ij}^{(1)}(n)$, we generate samples as described in the previous paragraph and use the algorithm described in section 4.4. The generated samples follow an α-stable sub-Gaussian distribution, i.e., $X_i \sim E_2(0, \Sigma, \psi_{sub}(., \alpha))$, $i = 1, ..., n$, where A is defined above. Hence, the value of the off-diagonal entry of the dispersion matrix σ_{12} is 11.

In Figures 4 through 7, we illustrate the behavior of the estimator $\hat{\sigma}_{ij}^{(1)}(n)$ for several sample sizes and various values for the tail index, i.e., $\alpha = 1.5, 1.6, ..., 1.9$. We demonstrate the behavior of the estimator using boxplots based on 1,000 sample runs for each setting of sample length and parameter value.

In general, one can see that for all values of α the estimators are median-unbiased. By analyzing the figures, we can additionally conclude that all estimators are slightly skewed to the right. Turning our attention to the rate of convergence of the estimates towards the median value of 11, we examine the boxplots. Figure 4 reveals that for a sample size of $n = 100$ the interquartile range is roughly equal to four for all values of α. The range diminishes gradually for increasing sample sizes until which can be seen in Figures 4 to 7. Finally in Figure 7, the interquartile range is equal to about 1.45 for all values of α. The rate of decay is roughly $n^{-1/2}$. Extreme outliers can be observed for small sample sizes larger than twice the median, regardless of the value of α. For $n = 1,000$, we have a maximal error around about 1.5 times the median. Due to right-skewness, extreme values are observed mostly to the right of the median.

5.2 Empirical analysis of $\hat{\sigma}_{ij}^{(2)}(n, p)$

We examine the consistency behavior of the second estimator as defined in (16) again using boxplots. In Figures 5 through 12 we depict the statistical behavior of the estimator. For generating independent samples of various lengths for $\alpha = 1.5, 1.6, 1.7, 1.8$, and 1.9, and two different values of p we use the algorithm described in Section 4.4.[4] For the values of p, we select 1.0001 and 1.3, respectively. A value for p closer to one leads to improved properties of the estimator as will be seen.

In general, we can observe that the estimates are strongly skewed. This is more pronounced for lower values of α while skewness vanishes slightly for increasing α. All figures display a noticeable bias in the median towards low values. Finally, as will be seen, $\hat{\sigma}_{ij}^{(1)}(n)$ seems more appealing than $\hat{\sigma}_{ij}^{(2)}(n, p)$.

For a sample length of $n = 100$, Figures 8 and 9 show that the bodies of the boxplots which are represented by the innerquartile ranges are as high as 4.5 for a lower value of p and α. As α increases, this effect vanishes slightly. However, results are worse for $p = 1.3$ as already indicated. For sample lenghts of $n = 300$, Figures 10 and 11 show interquartile ranges between 1.9 and 2.4 for lower values of p. Again, results are worse for $p = 1.3$. For $n = 500$, Figures 12 and 13 reveal ranges between 1.3 and 2.3 as α increases. Again, this worsens when p increases. And finally for samples of length $n = 1,000$, Figures 14 and 15 indicate that for $p = 1.00001$ the interquartile ranges extend between 1 for $\alpha = 1.9$ and 1.5 for $\alpha = 1.5$. Depending on α, the same pattern but on a worse level is displayed for $p = 1.3$.

It is clear from the statistical analysis that concerning skewness and median bias, the estimator $\hat{\sigma}_{ij}^{(1)}(n)$ has properties superior to estimator $\hat{\sigma}_{ij}^{(2)}(n, p)$ for both values of p. Hence, we use estimator $\hat{\sigma}_{ij}^{(1)}(n)$.

6 Application to the DAX 30

For the empirical analysis of the DAX30 index, we use the data from the Karlsruher Kapitaldatenbank. We analyze data from May 6, 2002 to March 31, 2006. For each company listed in the DAX30, we consider $1,000$ daily log-returns in the study period.[5]

6.1 Model check and estimation of the parameter α

Before fitting an α-stable sub-Gaussian distribution, we assessed if the data are appropriate for a sub-Gaussian model. This can be done with at least two different methods.

[4] In most of these plots, extreme estimates had to be removed to provide for a clear display of the boxplots.

[5] During our period of analysis Hypo Real Estate Holding AG was in the DAX for only 630 days. Therefore we exclude this company from further treatment leaving us with 29 stocks.

In the first method, we analyze the data by pursuing the following steps (also Nolan (2005)):

(i) For every stock X_i, we estimate $\hat{\theta} = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\sigma}_i, \hat{\mu}_i)$, $i = 1, ..., d$.

(ii) The estimated $\hat{\alpha}_i$'s should not differ much from each other.

(iii) The estimated $\hat{\beta}_i$'s should be close to zero.

(iv) Bivariate scatterplots of the components should be elliptically contoured.

(v) If the data fulfill criteria (ii)-(iv), a sub-Gaussian model can be justified. If there is a strong discrepancy to one of these criteria we have to reject a sub-Gaussian model.

In Table 1, we depict the maximum likelihood estimates for the DAX30 components. The estimated $\hat{\alpha}_i$, $i = 1, ..., 29$, are significantly below 2, indicating leptokurtosis. We calculate the average to be $\bar{\alpha} = 1.6$. These estimates agree with earlier results from Höchstötter, Fabozzi and Rachev (2005). In that work, stocks of the DAX30 are analyzed during the period 1988 through 2002. Although using different estimation procedures, the results coincide in most cases. The estimated $\hat{\beta}_i$, $i = 1, ..., 29$, are between -0.1756 and 0.1963 and the average, $\bar{\beta}$, equals -0.0129. Observe the substantial variability in the α's and that not all β's are close to zero. These results agree with Nolan (2005) who analyzed the Dow Jones Industrial Average. Concerning item (iv), it is certainly not feasible to look at each bivariate scatterplot of the data. Figure 16 depicts randomly chosen bivariate plots. Both scatterplots are roughly elliptical contoured.

The second method to analyze if a dataset allows for a sub-Gaussian model is quite similar to the first one. Instead of considering the components of the DAX30 directly, we examine randomly chosen linear combinations of the components. We only demand that the Euclidean norm of the weights of the linear combination is 1. Due to the theory of α-stable sub-Gaussian distributions, the index of stability is invariant under linear combinations. Furthermore, the estimated $\hat{\beta}$ of linear combination should be close to zero under the sub-Gaussian assumption. These considerations lead us to the following model check procedure:

(i) Generate i.i.d. samples $u_1, ..., u_n \in \mathbf{R}^d$ according to the uniform distribution on the hypersphere $\mathcal{S}_{d-1}$.

(ii) For each linear combination $u_i'X$, $i = 1, ..., n$, estimate $\theta_i = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\sigma}_i, \hat{\mu}_i)$.

(iii) The estimated $\hat{\alpha}_i$'s should not differ much from each other.

(iv) The estimated $\hat{\beta}_i$'s should be close to zero.

(v) Bivariate scatterplots of the components should be elliptically contoured.

(vi) If the data fulfill criteria (ii)-(v) a sub-Gaussian model can be justified.

If we conclude after the model check that our data are sub-Gaussian distributed, we estimate the α of the distribution by taking the mean $\bar{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\hat{\alpha}_i$. This approach has the advantage compared to the former one that we incorporate more information from the dataset and we can generate more sample estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$. In the former approach, we analyze only the marginal distributions.

Figure 17 depicts the maximum likelihood estimates for 100 linear combinations due to (ii). We observe that the estimated $\hat{\alpha}_i$, $i = 1, ..., n$, range from 1.5 to 1.84. The average, $\bar{\alpha}$, equals 1.69. Compared to the first approach, the tail indices increase, meaning less leptokurtosis, but the range of the estimates decreases. The estimated $\hat{\beta}_i$'s, $i = 1, ..., n$, lie in a range of -0.4 and 0.4 and the average, $\bar{\beta}$, is -0.0129. In contrast to the first approach, the variability in the β's increases. It is certainly not to be expected that the DAX30 log-returns follow a pure i.i.d. α stable sub-Gaussian model, since we do not account for time dependencies of the returns. The variability of the estimated $\hat{\alpha}$'s might be explained with GARCH-effects such as clustering of volatility. The observed skewness in the data[6] cannot be captured by a sub-Gaussian or any elliptical model. Nevertheless, we observe that the mean of the β's is close to zero.

6.2 Estimation of the stable DAX30 dispersion matrix

In this section, we focus on estimating the sample dispersion matrix of an α-stable sub-Gaussian distribution based on the DAX30 data. For the estimation procedure, we use the estimator $\hat{\sigma}_{ij}^{(1)}(n)$, $i \neq j$ presented in Section 5. Before applying this estimator, we center each time series by subtracting its sample mean. Estimator $\hat{\sigma}_{ij}^{(1)}(n)$ has the disadvantage that it cannot handle zeros. But after centering the data, there are no zero log-returns in the time series. In general, this is a point which has to be considered carefully.

For the sake of clearity, we display the sample dispersion matrix and covariance matrix as heat maps, respectively. Figure 18 is a heat map of the sample dispersion matrix of the α-stable sub-Gaussian distribution. The sample dispersion matrix is positive definite and has a very similar shape and structure as the sample covariance matrix which is depicted in Figure 19. Dark blue colors correspond to low values, whereas dark red colors depict high values.

Figure 20 (a) and (b) illustrate the eigenvalues λ_i, $i = 1, ...29$, of the sample dispersion matrix and covariance matrix, respectively. In both Figures, the first eigenvalue is significantly larger than the others. The amounts of the eigenvectors decline in similar fashion.

Figures 21 (a) and (b) depict the cumulative proportion of the total variability explained by the first k principal components corresponding to the k largest eigenvalues. In both figures, more than 50% is explained by the first principal component. We observe that the first principal component in the stable case explains slightly more variability than in the ordinary case, e.g., 70% of the total amount of dispersion is captured by the first six stable components whereas in the normal case, only 65% is explained. In contrast to the normal PCA the stable components are not independent but quasi-uncorrelated. Furthermore, in the case of $\alpha = 1.69$, the coefficient of tail

[6]The estimated $\hat{\beta}$'s differ sometimes significantly from zero.

dependence for two principal components, say Y_i and Y_j, is

$$\lambda(Y_i, Y_j, 0, 1.69) = \frac{\int_0^{\sqrt{1/2}} \frac{s^{1.69}}{\sqrt{1-s^2}} ds}{\int_0^1 \frac{s^{1.69}}{\sqrt{1-s^2}} ds} \approx 0.21$$

due to theorem 6 for all $i \neq j$, $i, j = 1, ..., 29$.

In Figure 22 (a), (b),(c), and (d) we show the first four eigenvectors of the sample dispersion matrix, the so-called vectors of loadings. The first vector is positively weighted for all stocks and can be thought of as describing a kind of index portfolio. The weights of this vector do not sum to one but they can be scaled to be so. The second vector has positive weights for technology titles such as Deutsche Telekom, Infineon, SAP, Siemens and also to the non-technology companies Allianz, Commerzbank, and Tui. The second principal component can be regarded as a trading strategy of buying technology titles and selling the other DAX30 stocks except for Allianz, Commerzbank, and Tui. The first two principal components explain around 56% of the total variability. The vectors of loadings in (c) and (d) correspond to the third and fourth principal component, respectively. It is slightly difficult to interpret this with respect to any economic meaning, hence, we consider them as pure statistical quantities. In conclusion, the estimator $\hat{\sigma}_{ij}(n)$, $i \neq j$, offers a simple way to estimate the dispersion matrix in an i.i.d. α-stable sub-Gaussian model. The results delivered by the estimator are reasonable and consistent with economic theory. Finally, we stress that a stable PCA is feasible.

7 Conclusion

In this paper we present different estimators which allow one to estimate the dispersion matrix of any normal variance mixture distribution. We analyze the estimators theoretically and show their consistency. We find empirically that the estimator $\hat{\sigma}_{ij}^{(1)}(n)$ has better statistical properties than the estimator $\hat{\sigma}_{ij}^{(2)}(n, p)$ for $i \neq j$. We fit an α-stable sub-Gaussian distribution to the DAX30 components for the first time. The sub-Gaussian model is certainly more realistic than a normal model, since it captures tail dependencies. But it has still the drawback that it cannot incorporate time dependencies.

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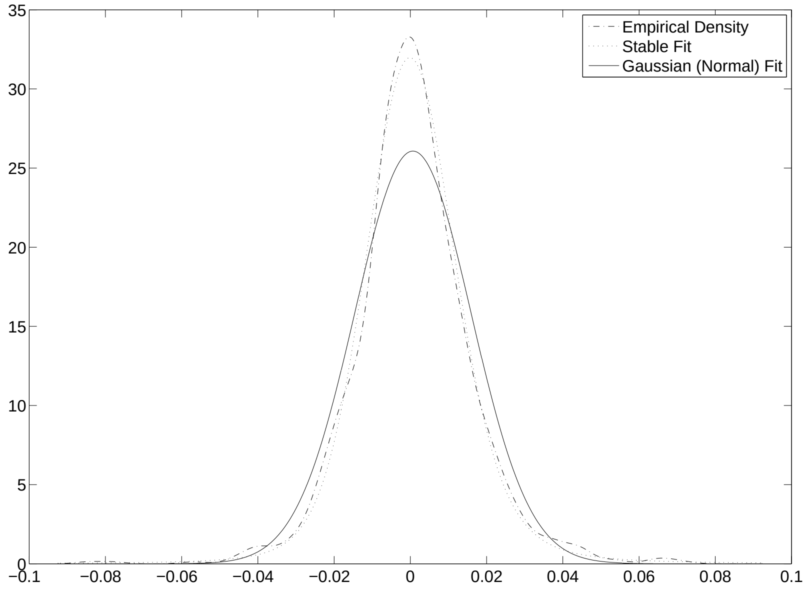


Figure 1: Kernel denisty plots of Adidas AG: empirical, normal, and stable fits.

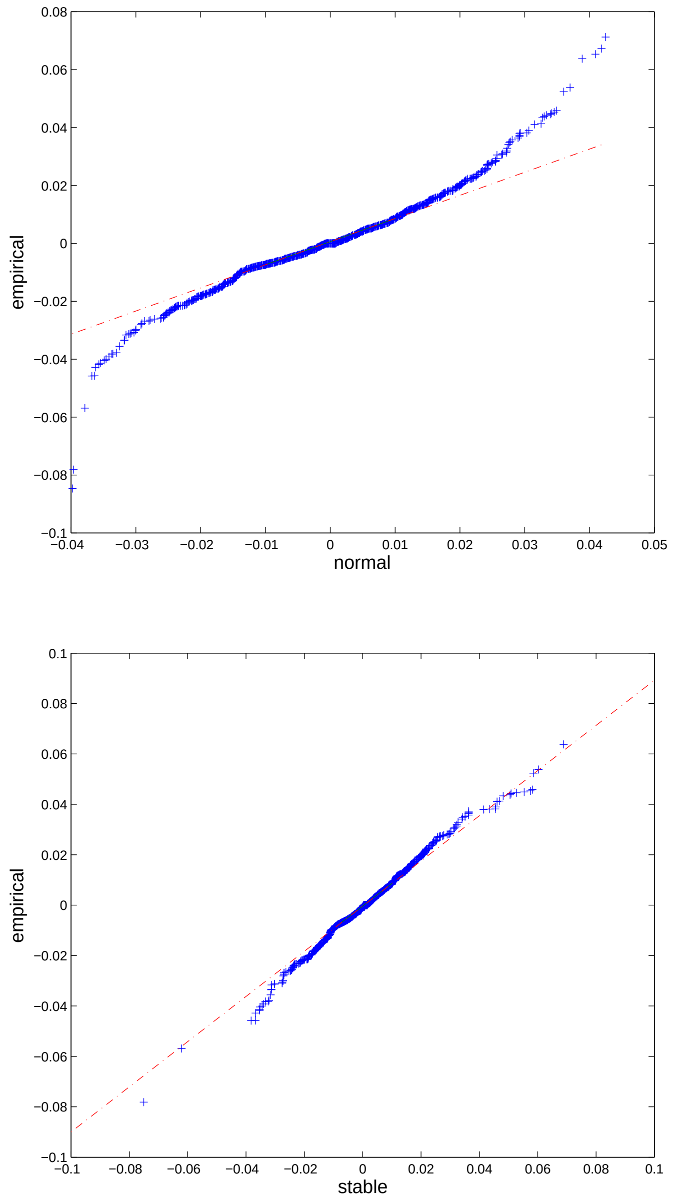
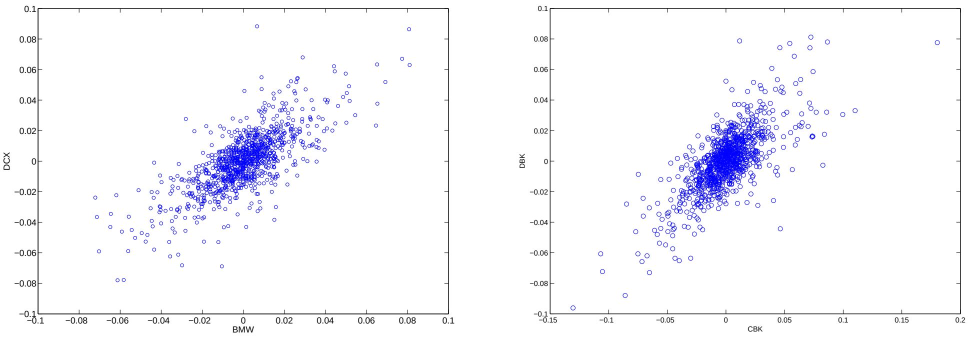


Figure 2: Adidas AG quantile plots of empirical return percentiles vs normal (top) and stable (bottom) fits.



(a) (b)

Figure 3: Bivariate scatterplot of BMW versus DaimlerChrysler and Commerzbank versus Deutsche Bank. Depicted are daily log-returns from May 6, 2002 through March 31, 2006.

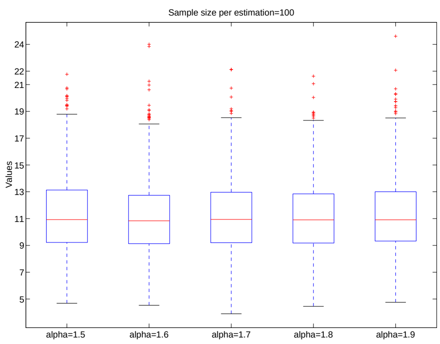


Figure 4: Sample size 100

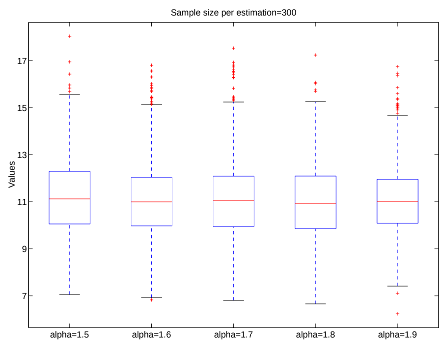


Figure 5: Sample size 300

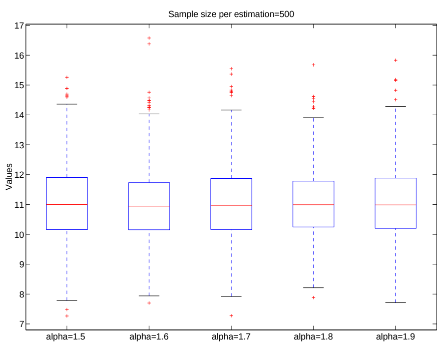


Figure 6: Sample size 500

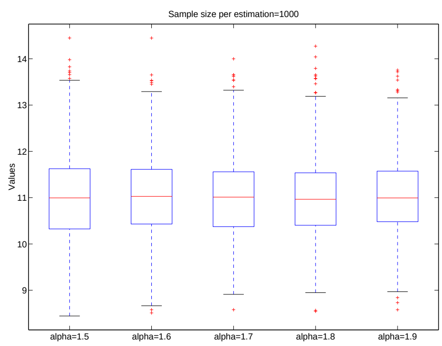


Figure 7: Sample size 1000

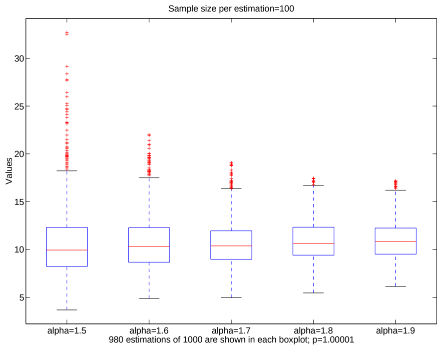


Figure 8: Sample size 100, p=1.00001

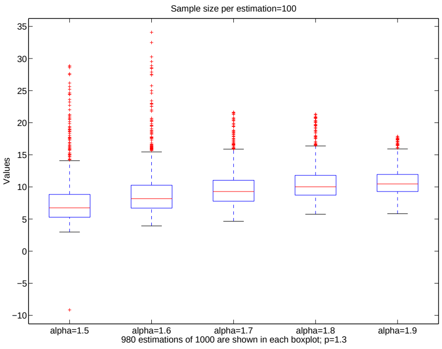


Figure 9: Sample size 100, p=1.3

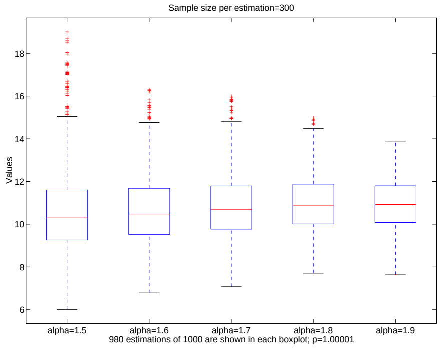


Figure 10: Sample size 300, p=1.00001

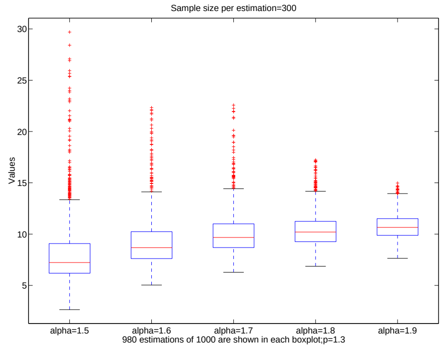


Figure 11: Sample size 300,p=1.3

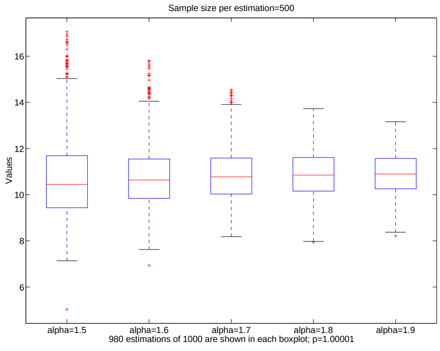


Figure 12: Sample size 500

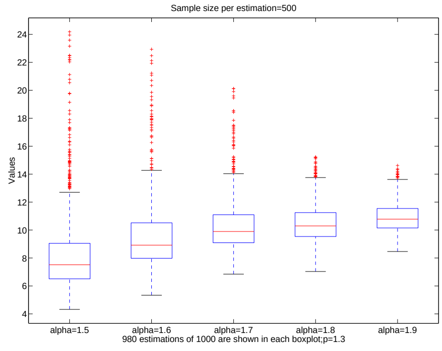


Figure 13: Sample size 500

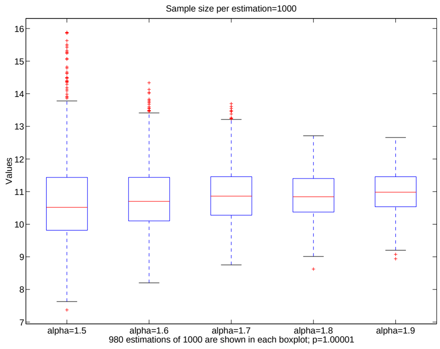


Figure 14: Sample size 1000

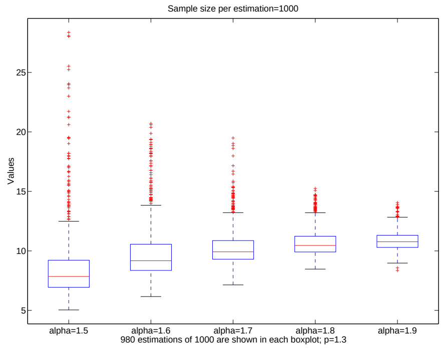
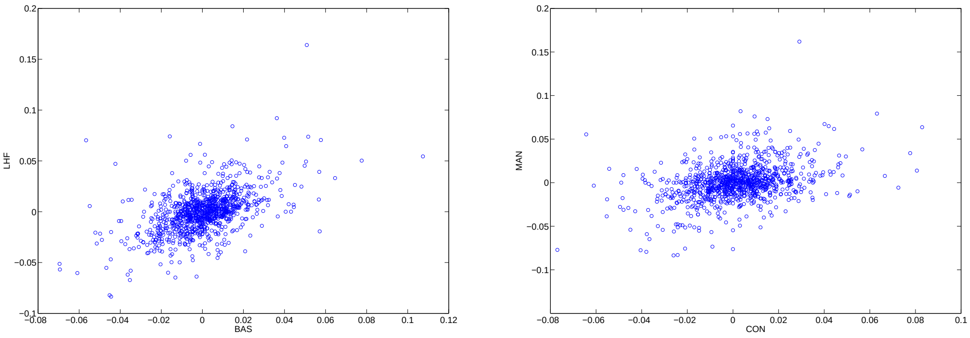


Figure 15: Sample size 1000

Name	Ticker Symbol	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\mu}$
Addidas	ADS	1.716	0.196	0.009	0.001
Allianz	ALV	1.515	-0.176	0.013	-0.001
Atlanta	ALT	1.419	0.012	0.009	0.000
BASF	BAS	1.674	-0.070	0.009	0.000
BMW	BMW	1.595	-0.108	0.010	0.000
Bayer	BAY	1.576	-0.077	0.011	0.000
Commerzbank	CBK	1.534	0.054	0.012	0.001
Continental	CON	1.766	0.012	0.011	0.002
Daimler-Chryser	DCX	1.675	-0.013	0.011	0.000
Deutsch Bank	DBK	1.634	-0.084	0.011	0.000
Deutsche Brse	DB1	1.741	0.049	0.010	0.001
Deutsche Post	DPW	1.778	-0.071	0.011	0.000
Telekom	DTE	1.350	0.030	0.009	0.000
Eon	EOA	1.594	-0.069	0.009	0.000
FresenMed	FME	1.487	0.029	0.010	0.001
Henkel	HEN3	1.634	0.103	0.008	0.000
Infineon	IFX	1.618	0.019	0.017	-0.001
Linde	LIN	1.534	0.063	0.009	0.000
Lufthansa	LHA	1.670	0.030	0.012	-0.001
Man	MAN	1.684	-0.074	0.013	0.001
Metro	MEO	1.526	0.125	0.011	0.001
MncherRck	MUV2	1.376	-0.070	0.011	-0.001
RWE	RWE	1.744	-0.004	0.010	0.000
SAP	SAP	1.415	-0.093	0.011	-0.001
Schering	SCH	1.494	-0.045	0.009	0.000
Siemens	SIE	1.574	-0.125	0.011	0.000
Thyssen	TKA	1.650	-0.027	0.011	0.000
Tui	TUI	1.538	0.035	0.012	-0.001
Volkswagen	VOW	1.690	-0.024	0.012	0.000
Average values		$\bar{\alpha} = 1,6$	$\bar{\beta} = -0,0129$		

Table 1: Stable parameter estimates using the maximum likelihood estimator



(a) (b)

Figure 16: Bivariate Scatterplots of BASF and Lufthansa in (a); and of Continental and MAN in (b).

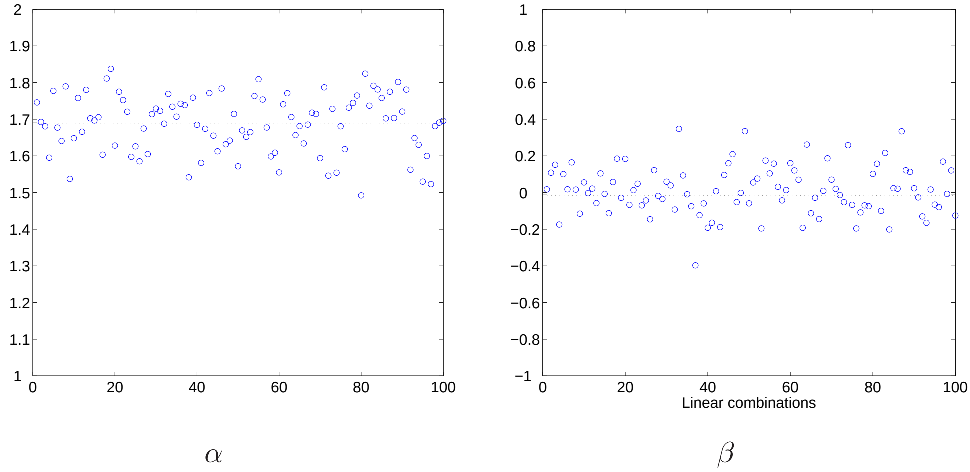


Figure 17: Scatterplot of the estimated α's and β's for 100 linear combinations.

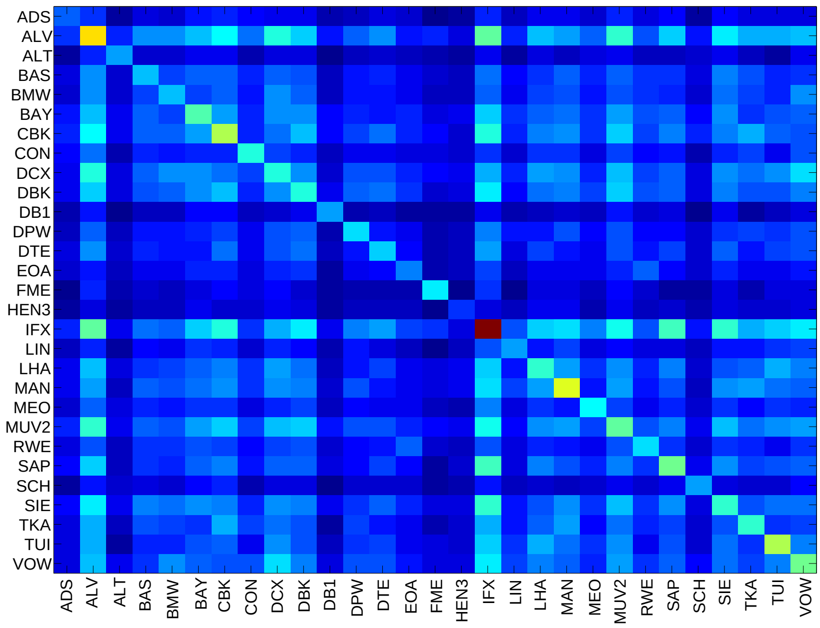


Figure 18: Heat map of the sample dispersion matrix. Dark blue colors corresponds to low values (min=0.0000278), to blue, to green, to yellow, to red for high values (max=0,00051)

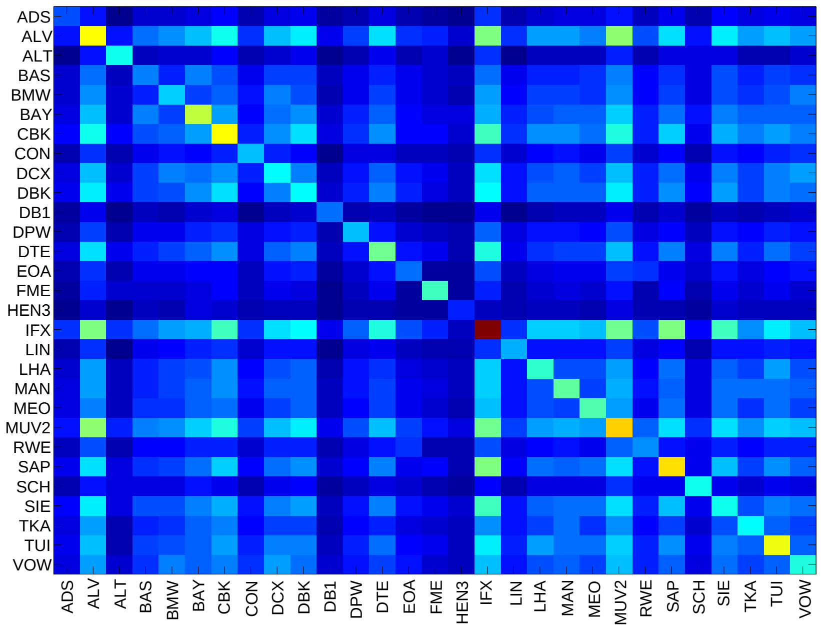
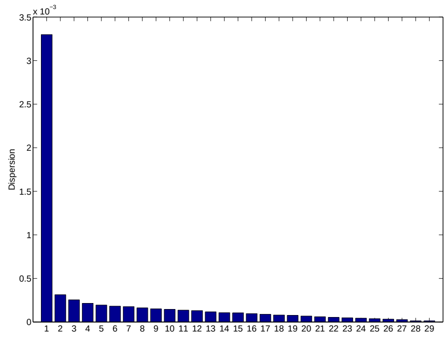
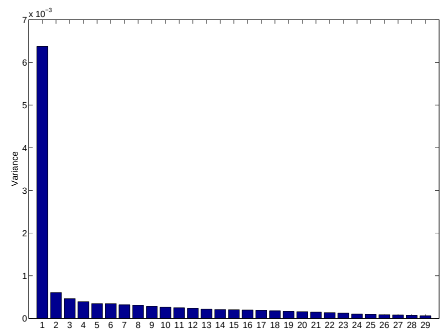


Figure 19: Heat map of the sample covariance matrix. Dark blue colors corresponds to low values (min=0.000053), to blue, to green, to yellow, to red for high values (max=0,00097)

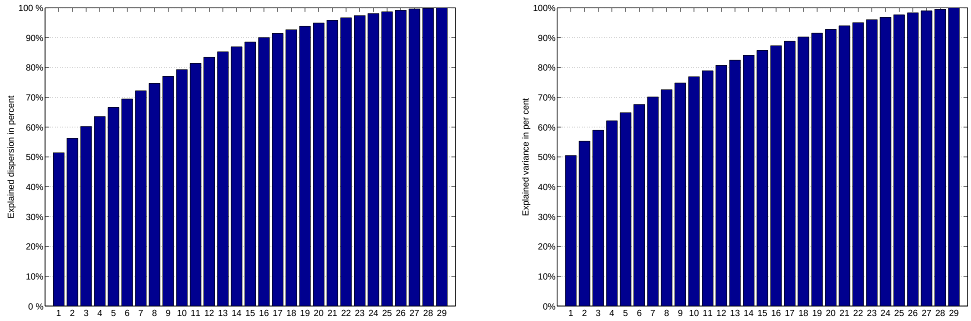


(a)



(b)

Figure 20: Barplots (a) and (b) depict the eigenvalues of the sample dispersion matrix and the sample covariance matrix.



(a) (b)

Figure 21: Barplot (a) and (b) show the cumulative proportion of the total dispersion and variance explained by the components, i.e., $\sum_{i=1}^{k} \lambda_i / \sum_{i=1}^{29} \lambda_i$.

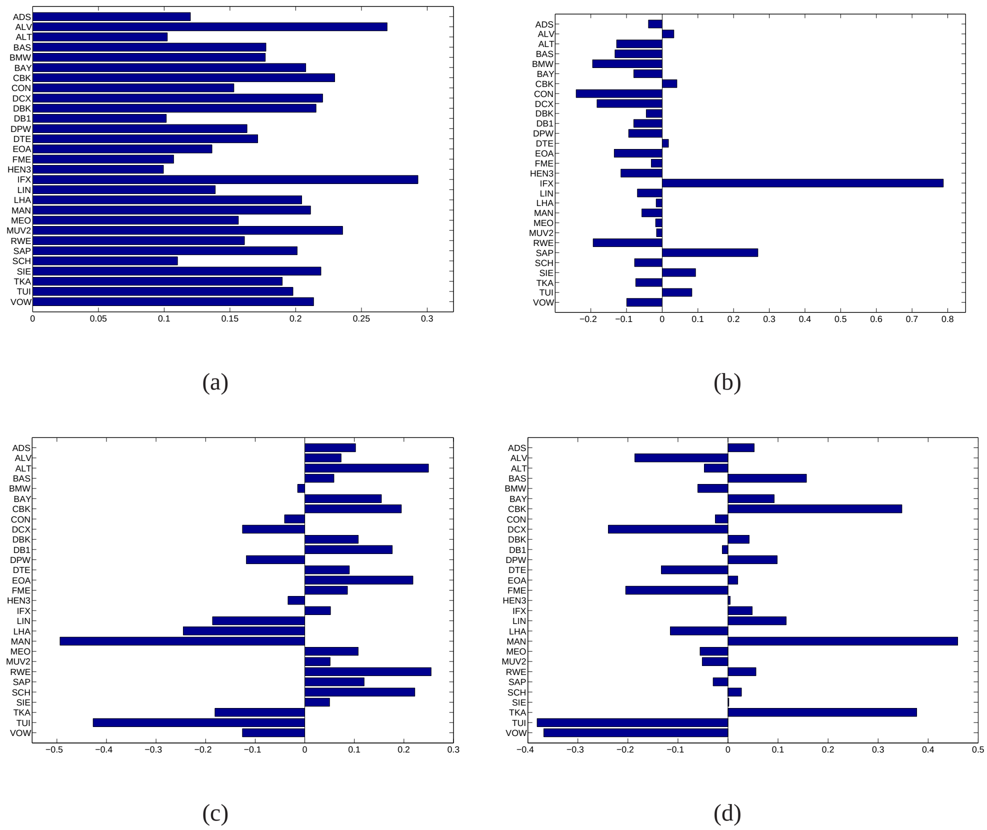


Figure 22: Barplot summarizing the loadings vectors g_1, g_2, g_3 and g_4 defining the first four principal components:(a) factor 1 loadings; (b) factor 2 loadings; (c) factor 3 loadings; and (d) factor 4 loadings