Tempered infinitely divisible distributions and processes

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Abstract. In this paper, we construct the new class of tempered infinitely divisible (TID) distributions. Taking into account the tempered stable distribution class, as introduced by in the seminal work of Rosińsky [10], a modification of the tempering function allows one to obtain suitable properties. In particular, TID distributions may have exponential moments of any order and conserve all proper properties of the Rosiński setting. Furthermore, we prove that the modified tempered stable distribution is TID and give some further parametric example.

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1 Introduction

The formal and elegant definition of tempered stable distributions and processes has been proposed in the work of Rosiński [10] where a completely monotone function is chosen to transform the Lévy measure of a stable distribution. Tempered stable distributions may have all moments finite and exponential moments of some order. The idea of selecting a different tempering function has been already considered in the literature, see [4]. In this paper, by following the approach of Rosiński [10] and considering a particular family of tempering functions, a new class of distributions is introduced with the same suitable properties of the tempered stable class, but with the advantage that it may admit exponential moments of any order. By multiplying the Lévy measure of a stable distribution with a positive definite radial function, see [13], instead of with a completely monotone function as in [10], we obtain the class of tempered infinitely divisible (TID) distributions. In some cases, the characteristic function of a TID random variable is extendible to an entire function on $\mathbb{C}$, that is, it admits any exponential moment.

Some practical problems in the field of mathematical finance have motivated our studies. Furthermore, we want to fill a gap in the literature. The modified tempered stable (MTS) distribution is not a tempered stable distribution of the Rosiński type [8] even though its properties are very close to that class. We will prove that the MTS distribution is in the TID class.

Although this distributional family is constructed by tempering the Lévy measure of a stable distribution, any stability property is lost. We will proceed as following. In Section 2, basic definitions and distributional properties are given. Working with the Lévy measure may be a difficult task, therefore a spectral measure R is needed to figure out all characteristics of this class. This measure describes all distributional properties and allows one to obtain a close formula for the characteristic function. Since TID distributions are by construction infinitely divisible, a TID Lévy process can be considered. In Section 3, TID processes are analyzed. If the time scale increases, the TID process looks like a Gaussian process; conversely, if the time scale decreases, it looks like a stable process. Furthermore, under some condition on the tempering function, the change of measure problem between stable and TID processes can be solved. In Section 4, a view toward simulation is given. Taking into consideration [9, 10], a series representation is derived in terms of a measure Q, as already proved for the tempered stable class.

Similar to the tempered stable framework, this class of distribution has an infinite dimensional parametrization by a family of measures [14], making it difficult to use. For this reason, in Section 5 some parametric examples in one dimension are proposed, and characteristic functions are derived.

2 Tempered infinitely divisible distribution

It is well known that the Lévy measure M_0 of an α-stable distribution on $\mathbb{R}^d$ can be written in polar coordinates in the form

$$M_0(dr, du) = r^{-\alpha-1} dr \, \sigma(du), \tag{2.1}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on the unit sphere S^{d-1}.

Theorem 2.1. *If μ_0 is an α-stable distribution, then its characteristic function has the form*

$$\hat{\mu}_0(y) = \begin{cases} \exp\left\{-c_\alpha \int_{S^{d-1}} |\langle y, u\rangle|^\alpha (1 - i \tan \frac{\pi\alpha}{2} sgn\langle y, u\rangle)\sigma(du) + i\langle y, a\rangle\right\}, & \alpha \neq 1, \\ \exp\left\{-c_1 \int_{S^{d-1}} (|\langle y, u\rangle| + i\frac{2}{\pi}\langle y, u\rangle \log |\langle y, u\rangle|)\sigma(du) + i\langle y, a\rangle\right\}, & \alpha = 1, \end{cases} \tag{2.2}$$

where $a \in \mathbb{R}^d$ and

$$c_\alpha = \begin{cases} |\Gamma(-\alpha) \cos(\frac{\pi\alpha}{2})|, & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1. \end{cases}$$

Proof. See [12, Theorem 14.10]. □

Definition 2.2. *If Y is an α-stable random vector with characteristic function (2.2), we will write $Y \sim S_\alpha(\sigma, a)$.*

Taking into account the approach of [10], we want to modify the radial component of M_0 and obtain a probability distribution with lighter tails than stable ones. A TID distribution is defined by tempering the radial term of M_0 as follows.

Definition 2.3. *Let μ be a infinitely divisible probability measure on $\mathbb{R}^d$ without gaussian part. We call μ tempered infinitely divisible (TID) if its Lévy measure M can be written in polar coordinates as*

$$M(dr, du) = r^{-\alpha-1} q(r, u) dr\, \sigma(du) \tag{2.3}$$

where α is a real number $\alpha \in [0, 2)$, σ a finite measure on the unit sphere S^{d-1} and $q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$ is a Borel function defined by

$$q(r, u) := \int_0^\infty e^{-r^2 s^2/2} Q(ds|u), \tag{2.4}$$

with $\{Q(\cdot|u)\}_{u \in S^{d-1}}$ a measurable family of Borel measure on $(0, \infty)$. If $q(0+, u) = 1$ for each $u \in S^{d-1}$, μ is referred to as a proper TID. The function q is called a tempering function.

In the case where $\{Q(\cdot|u)\}_{u \in S^{d-1}}$ are finite non-negative Borel measures on $(0, \infty)$, $q(\cdot, u)$ are positive definite radial functions on $\mathbb{R}^d$. By [13], the following results holds.

Theorem 2.4. *A continuous function $\varphi : (0, \infty) \to \mathbb{R}$ is positive definite and radial on $\mathbb{R}^d$ for all d if and only if it is of the form*

$$\varphi(r) = \int_0^\infty e^{-r^2 s^2} \mu(ds),$$

where μ is a finite non-negative Borel measure on $(0, \infty)$.

Define a measure Q on $\mathbb{R}^d$ by

$$Q(A) := \int_{S^{d-1}} \int_0^\infty I_A(ru) Q(dr|u)\sigma(du), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

It is easy to check that $Q(\{0\}) = 0$. We also define a measure R on $\mathbb{R}^d$ by

$$R(A) := \int_{\mathbb{R}^d} I_A(\frac{x}{\|x\|^2})\|x\|^\alpha Q(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.6}$$

The measure R is equivalent to the measure Q and clearly $R(\{0\}) = 0$. By definition of R, for each Borel function F, the following equality is satisfied

$$\int_{\mathbb{R}^d} F(x)R(dx) = \int_{\mathbb{R}^d} F(\frac{x}{\|x\|^2})\|x\|^\alpha Q(dx), \tag{2.7}$$

in the sense that when one sides exists then the other exists and are equal. By choosing

$$F(x) = I_A(\frac{x}{\|x\|^2})\|x\|^\alpha,$$

then Q can be written as

$$Q(A) := \int_{\mathbb{R}^d} I_A(\frac{x}{\|x\|^2})\|x\|^\alpha R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.8}$$

Sometimes the only knowledge of the Lévy measure cannot be enough to obtain analytical properties of tempered infinitely divisible distributions. Therefore, the definitions of measures Q and R allow one to overcome this problem and to obtain explicit analytic formulas and more explicit calculations. The following result allows one to figure out relations between the Lévy measure M and the measure R above defined.

Proposition 2.5. *Let μ be a TID distribution, the corresponding Lévy measure M be defined as in (2.3) and R as in (2.6). Then M can be written in the form*

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(tx)t^{-\alpha-1}e^{-t^2/2}dtR(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{2.9}$$

if and only if the measure R on $\mathbb{R}^d$ satisfies the following conditions, $R(\{0\}) = 0$ and

$$\begin{cases} \int_{\mathbb{R}^d}(\|x\|^2 \wedge \|x\|^\alpha)R(dx) < \infty, & 0 < \alpha < 2, \\ \int_{\mathbb{R}^d}(\log(1 + \|x\|) + 1)R(dx) < \infty, & \alpha = 0. \end{cases} \tag{2.10}$$

Proof. Let μ be a TID distribution with Lévy measure M. First we will prove that there exists at least one measure R, defined in (2.6), such that M, defined in (2.3), can be written in the form (2.9). To show this result, we take the measure R as in (2.6) and for each $A \in \mathcal{B}(\mathbb{R}^d)$, by considering (2.4), (2.5), (2.6), (2.7) and Fubini theorem, we have

$$
\begin{aligned}
M(A) &= \int_{S^{d-1}} \int_0^\infty I_A(ru) r^{-\alpha-1} q(r,u) dr \sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} Q(ds|u) dr \sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(ru) r^{-\alpha-1} e^{-r^2 s^2/2} dr \right) Q(ds|u) \sigma(du) \\
&= \int_{S^{d-1}} \int_0^\infty \left(\int_0^\infty I_A(\frac{t}{s}u) t^{-\alpha-1} e^{-t^2/2} dt \right) s^\alpha Q(ds|u) \sigma(du) \\
&= \int_0^\infty \left(\int_{S^{d-1}} \int_0^\infty I_A(t\frac{u}{s}) s^\alpha Q(ds|u) \sigma(du) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} I_A(t\frac{x}{\|x\|^2}) \|x\|^\alpha Q(dx) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} I_A(ty) R(dy) \right) t^{-\alpha-1} e^{-t^2/2} dt \\
&= \int_{\mathbb{R}^d} \int_0^\infty I_A(ty) t^{-\alpha-1} e^{-t^2/2} dt R(dy)
\end{aligned}
\tag{2.11}
$$

Conversely, given a measure R, let Q be the measure defined by (2.8) and let us consider the decomposition $Q(dr, du) = Q(dr|u)\sigma(du)$, where σ is a finite measure on S^{d-1}. Thus, we can define $q(r,u)$ by (2.4) and the computation (2.11) proves that M can be written in the form (2.3).

Now we want to prove that M is a Lévy measure if and only if (2.10) holds. Suppose M a Lévy measure, then we have

$$
\int_{\|x\|\leq 1} \|x\|^2 M(dx) < \infty
$$

and by considering (2.9) and $\alpha \neq 0$, we obtain

$$
\begin{aligned}
\infty > \int_{\|x\|\leq 1} \|x\|^2 M(dx) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq 1} \|x\|^2 \int_0^1 t^{1-\alpha} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \|x\|^2 \int_0^{1/\|x\|} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq e^{-1/2}(2-\alpha)^{-1} \int_{\|x\|\leq 1} \|x\|^2 R(dx) + e^{-1/2}(2-\alpha)^{-1} \int_{\|x\|>1} \|x\|^\alpha R(dx)
\end{aligned}
$$

thus, we obtain the desired inequality

$$
\int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(dx) < \infty.
$$

Now, let us consider the case $\alpha = 0$. By definition of the Lévy measure we have

$$
\int_{\|x\|\geq 1} M(dx) < \infty
$$

and by considering (2.9) with $\alpha = 0$, the following inequalities are satisfied

$$\begin{aligned}
\infty > \int_{\|x\|>1} M(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&= \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq 1} K R(dx) + e^{-1/2} \int_{\|x\|>1} \log(\|x\|) R(dx)
\end{aligned}$$

where K is a finite constant. Then, also when $\alpha = 0$, condition (2.10) is a necessary condition. Conversely, now we prove that (2.10) is also sufficient. Suppose that there is a measure R satisfying (2.10). Then the measure M can be written in the form (2.9). If $\alpha \neq 0$, we can write

$$\begin{aligned}
&\int_{\|x\|\leq 1} \|x\|^2 M(dx) = \\
&\quad = \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\frac{1}{\|x\|}} t^{1-\alpha} e^{-t^2/2} dt R(dx) \\
&\quad \leq \int_{\|x\|\leq 1} \|x\|^2 \int_0^{\infty} t^{1-\alpha} e^{-t^2/2} dt R(dx) + \frac{1}{2-\alpha} \int_{\|x\|>1} \|x\|^{\alpha} R(dx) \\
&\quad = 2^{-\frac{\alpha}{2}} \Gamma(1 - \frac{\alpha}{2}) \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \frac{1}{2-\alpha} \int_{\|x\|>1} \|x\|^{\alpha} R(dx) < \infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{\|x\|>1} M(dx) &= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1-\alpha} e^{-t^2/2} dt R(dx) \\
&\leq C \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-3} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-\alpha-1} dt R(dx) \\
&= \frac{C}{2} \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \frac{1}{\alpha} \int_{\|x\|>1} \|x\|^{\alpha} R(dx),
\end{aligned}$$

where $C := \sup_{t\geq 1} t^{2-\alpha} e^{-t^2/2}$. Thus M is a Lévy measure.

Considering the case where $\alpha = 0$, we obtain

$$\begin{aligned}
\int_{\|x\|\leq 1} \|x\|^2 M(dx) &= \int_{\mathbb{R}^d} \|x\|^2 \int_0^{\frac{1}{\|x\|}} t e^{-t^2/2} dt R(dx) \\
&\leq \int_{\mathbb{R}^d} \|x\|^2 (1 - e^{-\frac{1}{2\|x\|^2}}) R(dx) \\
&\leq \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \int_{\|x\|>1} R(dx)
\end{aligned}$$

and

$$\int_{\|x\|>1} M(dx) =$$
$$= \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx)$$
$$\leq \int_{\|x\|\leq 1} \int_{\frac{1}{\|x\|}}^{\infty} \frac{1}{t(1+t^2/2)} dt R(dx) + \int_{\|x\|>1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-1} e^{-t^2/2} dt R(dx)$$
$$\leq \int_{\|x\|\leq 1} \|x\|^2 R(dx) + \int_{\|x\|>1} (\log(\|x\|) + e^{-1/2}) R(dx).$$

Thus, M is a Lévy measure.

Now, in order to show that (2.9) is well defined, we want to show that R is uniquely determined. We will prove it by contradiction. Let R_1 and R_2 be two measures on $\mathbb{R}^d$ satisfying (2.9). Then, by previous argument, (2.10) has to be satisified also. By contradiction, we suppose that there exists a Borel set A such that $R_1(A) \neq R_2(A)$. By equation (2.8), we can define Q_1 and Q_2 from R_1 and R_2 and consider the polar representation

$$Q_i(dr, du) = Q_i(dr|u)\sigma(du)$$

where σ is a probability measure on S^{d-1} and $\{Q_i(\cdot|u)\}_{u\in S^{d-1}}$ are measurable families of Borel measure on $(0, \infty)$. Without any loss of generality, we assume that σ is not the null measure on S^{d-1}. If $\alpha \neq 0$, by definition (2.8) and conditions (2.10), the inequality

$$\infty > \int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^{\alpha}) R_i(dx) = \int_{\mathbb{R}^d} (\|x\|^{-2} \wedge \|x\|^{-\alpha}) \|x\|^{\alpha} Q_i(dx)$$
$$= \int_{S^{d-1}} \int_0^{\infty} (s^{\alpha-2} \wedge 1) Q_i(ds|u).$$

holds. Therefore, the tempering function

$$q_i(r, u) = \int_0^{\infty} e^{-r^2 s^2/2} Q_i(ds|u)$$

is well defined. Since $R_1(A) \neq R_2(A)$ also $Q_1(A) \neq Q_2(A)$. By assumption, R_i verifies (2.10). Then, by using the same calculus done to obtain (2.11), we can find $M(A)$ and write

$$\int_{S^{d-1}} \int_0^{\infty} I_A(ru) r^{-\alpha-1} (q_1(r, u) - q_2(r, u)) dr \sigma(du) = 0$$

and we find the contradiction. A similar argument shows the uniqueness of the measure R also in the case $\alpha = 0$. □

Remark 2.6. *The case $\alpha = 0$ is consider only for completeness and the theory will be not completely extended to this limiting case. It may be an interesting case in some applications.*

Remark 2.7. *If alpha* $\in (0,2)$ *and* R *satisfies the following additional inequality*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty, \quad 0 < \alpha < 2, \tag{2.12}$$

we will call μ *a proper TID distribution. The measure* Q *has the form*

$$Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} \|x\|^\alpha R(dx), \quad 0 < \alpha < 2, \tag{2.13}$$

In this case Q *is a finite measure and it can be represented in polar coordinates as* $Q(dr, du) = Q(dr|u)\sigma(du)$*, where* $Q(\cdot|u)$ *are finite measures and* σ *is a finite measure on* S^{d-1}.

Definition 2.8. *The unique measure* R *in (2.9) is called a spectral measure of the corresponding TID distribution. We will call* R *the Rosiński measure [14].*

We focus on the following result.

Remark 2.9. *If* μ *is a proper TID distribution, then* Q *is a finite measure and* $\{Q(\cdot|u)\}_{u\in S^{d-1}}$ *is a measurable family of finite Borel measures on* $(0,\infty)$*. Since the equation* $q(0+, u) = 1$ *holds, they are probability measures. Furthermore, for any fixed* $u \in S^{d-1}$*, function* $q(\cdot, u)$ *are positive definite radial functions.*

Taking into consideration Lemma 2.14 of [10], we want to figure out the relation between parameters of the proper TID distribution and stable ones.

Proposition 2.10. *Let* M *be a Lévy measure of a proper TID distribution, as in (2.3), with corresponding spectral measure* R*. Then, the Lévy measure* M_0 *of an* α*-stable distribution, given in (2.1), can be written in the following form*

$$M_0(A) = \int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d) \tag{2.14}$$

and additionally

$$\sigma(B) = \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right) \|x\|^\alpha R(dx), \qquad B \in \mathcal{B}(S^{d-1}). \tag{2.15}$$

Proof. By definitions (2.6) and (2.5) we obtain for $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_0^\infty I_A(tx) t^{-\alpha-1} dt R(dx) &= \int_{\mathbb{R}^d} \int_0^\infty I_A(t\frac{x}{\|x\|^2}) t^{-\alpha-1} \|x\|^\alpha dt Q(dx) \\
&= \int_{\mathbb{R}^d} \int_0^\infty I_A(s\frac{x}{\|x\|^2}) s^{-\alpha-1} \|x\|^\alpha ds Q(dx) \\
&= \int_{S^{d-1}} \int_0^\infty I_A(su) s^{-\alpha-1} ds \sigma(du) \\
&= M_0(A).
\end{aligned}$$

Since we are considering a proper TID distribution, then, by remark 2.9, $Q(\cdot|u)$ are finite measures on $(0,\infty)$. Therefore we can write

$$\begin{aligned}
\int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right) \|x\|^\alpha R(dx) &= \int_{\mathbb{R}^d} I_B\left(\frac{x}{\|x\|}\right) Q(dx) \\
&= \int_B \int_0^\infty Q(ds|u)\sigma(du) = c\sigma(B).
\end{aligned}$$

Since $Q(ds|u)$ are probability measures, then $c = 1$ and (2.15) holds. □

2.1 Distributional properties

A TID distribution may have moments and also exponential moments of any order. The behavior of the tails depends on the measure R.

Proposition 2.11. *Let μ be a TID distribution with Lévy measure M given by (2.9) and $\alpha \in (0,2)$. Then*

(a) For $p \in (0,\alpha)$

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty;$$

(b) $\int_{\mathbb{R}^d} \|x\|^\alpha \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^\alpha \log(\|x\|) R(dx) < \infty$;

(c) If $p > \alpha$, then

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty;$$

(d) For each $\theta > 0$, we have

$$\int_{\mathbb{R}^d} e^{\theta\|x\|} \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^{-(\alpha+1)} e^{\frac{\theta^2\|x\|^2}{2}} R(dx) < \infty.$$

(e) If $\alpha = 0$ and $p > 0$, then

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \iff \int_{\|x\|>1} \|x\|^p R(dx) < \infty;$$

Proof. It is well known that moments conditions for μ are related to the corresponding conditions for $M_{|\{\|x\|>1\}}$, see [12].

Let we consider $p > 0$. Then we obtain

$$\begin{aligned}\int_{\|x\|>1} \|x\|^p M(dx) = & \int_{\|x\|\le 1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt R(dx) \\ & + \int_{\|x\|>1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt R(dx) \\ = & \; I^{(1)}(x) + I^{(2)}(x).\end{aligned}$$

By (2.10), the following inequality holds

$$I^{(1)}(x) \le C \int_{\|x\|\le 1} \int_{\frac{1}{\|x\|}}^{\infty} t^{-3} dt R(dx) \le \frac{C}{2} \int_{\|x\|\le 1} \|x\|^2 R(dx) < \infty \tag{2.16}$$

where $C := \sup_{t\ge 1} t^{p+2-\alpha} e^{-t^2/2}$. The inequality (2.16) shows that the integral $I^{(1)}(x)$ is always finite.

If $p < \alpha$, then

$$I^{(2)}(x) \le \int_{\|x\|>1} \|x\|^p \int_{\frac{1}{\|x\|}}^{\infty} t^{p-\alpha-1} dt R(dx) = \frac{1}{\alpha - p} \int_{\|x\|>1} \|x\|^\alpha R(dx) < \infty,$$

by inequality (2.10), condition (a) is fulfilled. If $p = \alpha$, we have

$$I^{(2)}(x) \leq \int_{\|x\|>1} \|x\|^{\alpha} \int_{\frac{1}{\|x\|}}^{1} t^{-1} dt R(dx) + \int_{\|x\|>1} \|x\|^{\alpha} \int_{1}^{\infty} e^{-t^2/2} dt R(dx)$$
$$= \int_{\|x\|>1} \|x\|^{\alpha} (\log(\|x\|) + C_1) R(dx)$$

where C_1 is a finite constant, and from the other side,

$$I^{(2)}(x) \geq e^{-1/2} \int_{\|x\|>1} \|x\|^{\alpha} \log(\|x\|) R(dx).$$

Therefore condition (b) is satisfied. Now, we suppose $p > \alpha$. Let us define

$$\bar{C} = \int_{1}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt.$$

Then, the following inequality holds

$$I^{(2)}(x) \geq \bar{C} \int_{\|x\|>1} \|x\|^{p} R(dx)$$

and furthermore,

$$I^{(2)}(x) \leq \int_{\|x\|>1} \|x\|^{p} \int_{0}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt R(dx).$$

By changing variables in the integral, we have

$$\int_{0}^{\infty} t^{p-\alpha-1} e^{-t^2/2} dt = 2^{(p-\alpha)/2-1} \int_{0}^{\infty} z^{(p-\alpha)/2-1} e^{-z} dz = 2^{(p-\alpha)/2-1} \Gamma\Big(\frac{p-\alpha}{2}\Big),$$

thus

$$I^{(2)}(x) \leq 2^{(p-\alpha)/2-1} \Gamma\Big(\frac{p-\alpha}{2}\Big) \int_{\|x\|>1} \|x\|^{p} R(dx).$$

This proves (c).

In order to prove (d), we consider the integral

$$\int_{\|x\|>1} e^{\theta\|x\|} M(dx) = \int_{\mathbb{R}^d} \int_{\frac{1}{\|x\|}}^{\infty} e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt R(dx)$$

and we define

$$I_{\theta}(x) := \int_{\frac{1}{\|x\|}}^{\infty} e^{\theta t\|x\|} t^{-(\alpha+1)} e^{-t^2/2} dt.$$

It is easy to check that as $\|x\| \to 0$, then $I_{\theta}(x)$ goes to 0 exponentially fast. Now, let us consider the case $\|x\| \to \infty$. We have

$$I_{\theta}(x) = e^{\theta^2\|x\|^2/2} \int_{\frac{1}{\|x\|}}^{\infty} t^{-(\alpha+1)} e^{-(t-\theta\|x\|)^2/2} dt.$$

Define $K_\theta(x)$ by

$$K_\theta(x) := \int_{\frac{1}{\|x\|}}^{\infty} t^{-(\alpha+1)} e^{-(t-\theta\|x\|)^2/2} dt,$$

by changing variables in the integral we obtain

$$\begin{aligned} K_\theta(x) &= \int_{\frac{1}{\|x\|}-\theta\|x\|}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= \int_{\frac{1}{\|x\|}-\theta\|x\|}^{-\frac{\theta\|x\|}{2}} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt + \int_{-\frac{\theta\|x\|}{2}}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &= K_\theta^{(1)}(x) + K_\theta^{(2)}(x). \end{aligned}$$

Furthermore, the following inequality is satisfied

$$\begin{aligned} K_\theta^{(1)}(x) &\leq e^{-\theta^2\|x\|^2/8} \int_{\frac{1}{\|x\|}-\theta\|x\|}^{-\frac{\theta\|x\|}{2}} (t+\theta\|x\|)^{-(\alpha+1)} dt \\ &\leq C\|x\|^{\alpha+2} e^{-\theta^2\|x\|^2/8}. \end{aligned}$$

and for $\|x\| \to \infty$,

$$\begin{aligned} K_\theta^{(2)}(x) &\leq \int_{-\frac{\theta\|x\|}{2}}^{\infty} (t+\theta\|x\|)^{-(\alpha+1)} e^{-t^2/2} dt \\ &\sim (\theta\|x\|)^{-(\alpha+1)} \int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)}, \end{aligned}$$

It follows that, for $\|x\| \to \infty$,

$$K_\theta(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)},$$

and

$$I_\theta(x) \sim \sqrt{2\pi}(\theta\|x\|)^{-(\alpha+1)} e^{\theta^2\|x\|^2/2},$$

therefore condition (d) holds. Part (e) can be proved with a similar argument to (c). □

Remark 2.12. *If the measure R has a bounded support, then $E(e^{\theta\|X\|}) < \infty$ for all $\theta > 0$. We have exponential moments of any order.*

As we said before, sometimes it is more convenient to work with the measure R; in order to find some distributional property of a TID distribution. Taking into account Proposition 2.8 of [10], we will show a result about finite variation.

Proposition 2.13. *Let M be the Lévy measure of a TID distribution, and R a measure as in (2.9) and (2.6). These conditions are equivalent*

(i) $\int_{\|x\|\leq 1} \|x\| M(dx) < \infty$

(ii) $\int_{\|x\|\leq 1} \|x\| R(dx) < \infty$ *and* $\alpha \in (0, 1)$

Proof. Suppose condition (i) is fulfilled. Choose $r \geq 1$ such that $R(\{\|x\| \leq r\}) \neq 0$. We can write the following relations

$$\begin{aligned}
\int_{\|x\|\leq 1} \|x\| M(dx) &= \int_{\mathbb{R}^d} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&\geq \int_{\|x\|\leq r} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&= 2^{-\frac{1}{2}(\alpha+1)} \int_{\|x\|\leq r} \|x\| \int_0^{1/(2\|x\|^2)} z^{-\frac{1}{2}(\alpha+1)} e^{-z} dz R(dx) \\
&\geq 2^{-\frac{1}{2}(\alpha+1)} \int_{\|x\|\leq r} \|x\| R(dx) \int_0^{1/(2r^2)} z^{-\frac{1}{2}(\alpha+1)} e^{-z} dz.
\end{aligned}$$

By condition (i), we obtain $\alpha < 1$ and

$$\int_{\|x\|\leq 1} \|x\| R(dx) < \infty.$$

Conversely, if (ii) holds, then

$$\begin{aligned}
&\int_{\|x\|\leq 1} \|x\| M(dx) = \\
&= \int_{\|x\|\leq 1} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) + \int_{\|x\|>1} \|x\| \int_0^{1/\|x\|} t^{-\alpha} e^{-t^2/2} dt R(dx) \\
&\leq \int_{\|x\|\leq 1} \|x\| \int_0^{\infty} t^{-\alpha} e^{-t^2/2} dt R(dx) + \frac{1}{1-\alpha} \int_{\|x\|>1} \|x\|^\alpha R(dx) \\
&= 2^{-\frac{\alpha}{2}-\frac{1}{2}} \Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\|x\|\leq 1} \|x\| R(dx) + \frac{1}{1-\alpha} \int_{\|x\|>1} \|x\|^\alpha R(dx) < \infty,
\end{aligned}$$

which proves the converse. □

2.2 Characteristic function of a TID distribution

It is well known that given a Lévy measure of a infinitely divisible distribution, we have an explicit formula for the characteristic function, see [12]. Sometimes, working with a Lévy measure of the form (2.3) may be difficult and, as a consequence, we will provide an expression for the characteristic function of a TID distribution with respect to the measure R. The measure R allows one to obtain explicit analytic formulas and more explicit calculations. Let us now define functions

$$\psi_\alpha(s) = \int_0^\infty (e^{ist} - 1 - ist) t^{-\alpha-1} e^{-t^2/2} dt, \tag{2.17}$$

and

$$\psi_\alpha^0(s) = \int_0^\infty (e^{ist} - 1) t^{-\alpha-1} e^{-t^2/2} dt. \tag{2.18}$$

In order to find a more useful form for the characteristic function of a TID distribution, we will need the following results.

Lemma 2.14. *The following limits are verified*

$$\begin{array}{ll} \lim_{s\to 0} s^{-2}\psi_\alpha(s) = -2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2}), & \alpha\in(0,2) \\ \lim_{s\to\infty} s^{-1}\psi_0(s) = -i\sqrt{\frac{\pi}{2}}, & \alpha=0 \\ \lim_{s\to\infty} s^{-1}\psi_\alpha(s) = -2^{-\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{\alpha}{2})i, & \alpha\in(0,1) \\ \lim_{s\to\infty}(s^{-1}\psi_1(s)+i\log s) = -\frac{\pi}{2}+i, & \alpha=1 \\ \lim_{s\to\infty} s^{-\alpha}\psi_\alpha(s) = \Gamma(-\alpha)e^{-i\alpha\frac{\pi}{2}}, & \alpha\in(1,2) \end{array} \tag{2.19}$$

Furthermore, if $\alpha\in(0,1)$ we have

$$\begin{array}{l} \lim_{s\to\infty} s^{-1}\psi^0_\alpha(s) = 2^{-\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{\alpha}{2})i, \\ \lim_{s\to\infty} s^{-\alpha}\psi^0_\alpha(s) = \Gamma(-\alpha)e^{-i\alpha\frac{\pi}{2}}, \end{array} \tag{2.20}$$

Then, there exists for each α a finite positive constant C_α such that for all $s\in\mathbb{R}$ the following inequalities are fulfilled

$$\begin{array}{rcll} C_\alpha^{-1}(s^2\wedge|s|^{\alpha\vee 1}) & \le|\psi_\alpha(s)|\le & C_\alpha(s^2\wedge|s|^{\alpha\vee 0,1}), & \alpha\neq 1, \\ C_1^{-1}[s^2\wedge|s|(1+\log^+|s|)] & \le|\psi_1(s)|\le & C_1[s^2\wedge|s|(1+\log^+|s|)], & \alpha=1, \\ C_\alpha^{-1}(s^2\wedge|s|^{\alpha}) & \le|\psi^0_\alpha(s)|\le & C_\alpha(s^2\wedge|s|^{\alpha}), & \alpha\in(0,1). \\ C_0^{-1}[1+\log(1+s)] & \le|\psi_0(s)|\le & C_0[1+\log(1+s)], & \alpha=0. \end{array} \tag{2.21}$$

Proof. By solving the limit and using [12, Lemma 14.11], (2.19) and (2.20) are verified. □

Lemma 2.15. *Let us consider the confluent equation*

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ax=0. \tag{2.22}$$

Then the solution of this differential equation is

$$y=AM(a,c;z)+BU(a,c;z)$$

where A and B are constant and $M(a,c;z)$ is the Kummer's or confluent hypergeometric function of first kind [1, 13.1.2] and $U(a,c;z)$ is the confluent hypergeometric function of second kind [1, 13.1.3].

Proof. For a complete overview on confluent hypergeometric function see [15] or [1]. □

Lemma 2.16. *Let $\alpha\in(0,2)$, $\alpha\neq 1$. Then we have*

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{z^n}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right) &= \Gamma\Big(-\frac{\alpha}{2}\Big)M\left(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{z}{2}\Big)^2\right) \\ &\quad + z\Gamma\Big(\frac{1-\alpha}{2}\Big)M\left(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{z}{2}\Big)^2\right) \end{aligned} \tag{2.23}$$

Proof. Since the series converges, we can split it in two parts

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right)=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}\Gamma\left(n-\frac{\alpha}{2}\right)$$
$$+\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\Gamma\left(n+\frac{1-\alpha}{2}\right).$$

By the Legendre duplication formula [1, 6.1.18], we obtain the following equalities

$$(2n)! = n!2^{2n}\left(\frac{1}{2}\right)_n,$$
$$(2n+1)! = n!2^{2n}\left(\frac{3}{2}\right)_n,$$

and we define the Pochhammer's symbols as $(a)_n = \Gamma(n+a)/\Gamma(a)$ [1, 6.1.22]. Thus, we obtain

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\Gamma\left(\frac{n-\alpha}{2}\right)=$$
$$=\Gamma\left(-\frac{\alpha}{2}\right)\sum_{n=0}^{\infty}\frac{z^{2n}}{n!2^{2n}}\frac{\left(-\frac{\alpha}{2}\right)_n}{\left(\frac{1}{2}\right)_n}+z\Gamma\left(\frac{1-\alpha}{2}\right)\sum_{n=0}^{\infty}\frac{z^{2n}}{n!2^{2n}}\frac{\left(\frac{1-\alpha}{2}\right)_n}{\left(\frac{3}{2}\right)_n}$$
$$=\Gamma\left(-\frac{\alpha}{2}\right)M\left(-\frac{\alpha}{2},\frac{1}{2};\left(\frac{z}{2}\right)^2\right)+z\Gamma\left(\frac{1-\alpha}{2}\right)M\left(\frac{1-\alpha}{2},\frac{3}{2};\left(\frac{z}{2}\right)^2\right).$$

□

Theorem 2.17. *(Characteristic function) Let μ be a TID distribution with Lévy measure given by (2.9), $\alpha \in [0,2)$ and $\alpha \neq 1$. If the distribution has finite mean, i.e. $\int_{\mathbb{R}^d}\|x\|\mu(dx) < \infty$, then*

$$\hat{\mu}(y) = \exp\left\{\int_{\mathbb{R}^d}\psi_\alpha(\langle y,x\rangle)R(dx)+i\langle y,m\rangle\right\} \tag{2.24}$$

where

$$\begin{aligned}\psi_\alpha(s) &= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad+i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad-i\sqrt{2}s\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned} \tag{2.25}$$

and $m = \int_{\mathbb{R}^d} x\mu(dx)$. Furthermore, if $0 < \alpha < 1$, the characteristic function can be written in an alternative form

$$\hat{\mu}(y) = \exp\left\{\int_{\mathbb{R}^d}\psi^0_\alpha(\langle y,x\rangle)R(dx)+i\langle y,m_0\rangle\right\} \tag{2.26}$$

where

$$\begin{aligned}\psi_\alpha(s) &= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\quad + i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned}\tag{2.27}$$

Proof. First, integrals (2.24) and (2.26) are well defined due to conditions (2.10) and (2.21) of Lemma 2.14. It is well known that if the mean is finite, that is if the first absolute moment exists, i.e. $\int_{\mathbb{R}^d}\|x\|\mu(dx)<\infty$, then $\hat{\mu}$ can be written as

$$\hat{\mu}=\exp\Big(\int_{\mathbb{R}^d}(e^{i\langle y,x\rangle}-1-i\langle y,x\rangle)\nu(dx)+i\langle y,m\rangle\Big)$$

where $m=\int_{\mathbb{R}^d}x\mu(dx)$. By (2.9), we obtain the equality (2.24), where, if $\alpha\in[0,2)$

$$\psi_\alpha(s)=\int_0^\infty(e^{ist}-1-ist)t^{-\alpha-1}e^{-t^2/2}dt,\tag{2.28}$$

If $\alpha\in[0,1)$ and $\int_{\|x\|\le 1}\|x\|R(dx)<\infty$, by Proposition 2.13 $\int_{\|x\|\le 1}\|x\|\nu(dx)<\infty$, in which case $\hat{\mu}$ can be written as

$$\exp\Big(\int_{\mathbb{R}^d}(e^{i\langle y,x\rangle}-1)\nu(dx)+i\langle y,m_0\rangle\Big),$$

where m_0 is the drift as defined in [12]. By (2.9), we obtain the equality (2.26), where

$$\psi^0_\alpha(s)=\int_0^\infty(e^{ist}-1)t^{-\alpha-1}e^{-t^2/2}dt,\tag{2.29}$$

and, furthermore, the equality

$$\psi_\alpha(s)=\psi^0_\alpha(s)-is\int_0^\infty t^{-\alpha}e^{-t^2/2}dt\tag{2.30}$$

holds. Now we will prove (2.25) and (2.27). If $\alpha\in(0,1)$, we obtain by equality (2.23)

$$\begin{aligned}&\int_0^\infty(e^{ist}-1)t^{-\alpha-1}e^{-t^2/2}dt=\\ &\quad=\sum_{n=1}^\infty\frac{(is)^n}{n!}\int_0^\infty t^{n-\alpha-1}e^{-t^2/2}dt\\ &\quad=\sum_{n=1}^\infty\frac{(is)^n}{n!}2^{\frac{1}{2}(n-\alpha-2)}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\\ &\quad=2^{-\frac{\alpha}{2}-1}\sum_{n=1}^\infty\frac{(i\sqrt{2}s)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\\ &\quad=2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)\\ &\qquad+i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)-\Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg).\end{aligned}$$

With a similar calculus, if $\alpha \in (0,2)$ and $\alpha \neq 1$, we obtain

$$\int_0^\infty (e^{ist} - 1 - ist)t^{-\alpha-1}e^{-t^2/2}dt =$$

$$= \sum_{n=2}^{\infty} \frac{(is)^n}{n!} \int_0^\infty t^{n-\alpha-1}e^{-t^2/2}dt$$

$$= \sum_{n=2}^{\infty} \frac{(is)^n}{n!} 2^{\frac{1}{2}(n-\alpha-2)}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)$$

$$= 2^{-\frac{\alpha}{2}-1} \sum_{n=2}^{\infty} \frac{(i\sqrt{2}s)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)$$

$$= 2^{-\frac{\alpha}{2}-1}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)$$
$$+ i\sqrt{2}s\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}s}{2}\Big)^2\Big)$$
$$- \sqrt{2}is\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big) - \Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg)$$

□

Remark 2.18. *With a similar technique, the characteristic exponent also can be calculated also for both cases* $\alpha = 0$ *and* $\alpha = 1$.

Definition 2.19. *We will write* $X \sim TID_\alpha(R,m)$ *to indicate that* X *is a TID random variable with characteristic function (2.24) and* $X \sim TID^0_\alpha(R,m_0)$ *to indicate that* X *is a TID random variable with characteristic function (2.26). The constant* m *is exactly the mean* $m = E[X]$.

3 TID processes

In this section, we will introduce TID processes. By Definition 2.3, if μ is a TID distribution, it is infinitely divisible and therefore there exists a Lévy process $\{X(t) : t \geq 0\}$ such that μ is the distribution of $X(1)$.

3.1 Short and long time behavior

The following theorems will show the different behavior of a TID process for different time scale. If one decreases the time scale, a TID process looks like a stable process; otherwise, if one increases the time scale, it looks like a Brownian motion. To figure out this different time behavior, we consider the time rescaled process

$$\{X_h(t) : t \geq 0\} = \{X(ht) : t \geq 0\}, \tag{3.1}$$

where $h > 0$.

Theorem 3.1. *(Short time behavior) Let $\{X(t) : t \geq 0\}$ be a TID Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1)$ has spectral measure R.*

(a) *Let us consider a TID process with $X(1) \sim TID^0_\alpha(R, 0)$, if $\alpha \in (0, 1)$ and with $X(1) \sim TID_\alpha(R, 0)$, if $\alpha \in (1, 2)$. Assume that*

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty \tag{3.2}$$

and let σ be the finite measure on S^{d-1} defined in (2.15). Then

$$h^{-1/\alpha} X_h \xrightarrow{d} Y, \tag{3.3}$$

as $h \to 0$, where $\{Y(t) : t \geq 0\}$ is a strictly α-stable Lévy process with $Y(1) = S_\alpha(\sigma, 0)$.

(b) *Let us consider a TID process with $X(1) \sim TID_\alpha(R, 0)$, if $\alpha = 1$. Assume that*

$$\int_{\mathbb{R}^d} \|x\| \, |\log \|x\|| \, R(dx) < \infty. \tag{3.4}$$

Then

$$h^{-1/\alpha} X_h - a_h \xrightarrow{d} Y,$$

where

$$a_h(t) = t \log h \int_{\mathbb{R}^d} x R(dx),$$

and $\{Y(t) : t \geq 0\}$ is an α-stable Lévy process with $Y(1) \sim S_1(\sigma, b)$ with

$$b = \int_{\mathbb{R}^d} x(1 - \log \|x\|) R(dx).$$

Proof. Since $\{h^{-1/\alpha} X_h(t) : t \geq 0\}$ is a Lévy process, by [5, Theorem 13.17], it is enough to show the convergence in distribution of $h^{-1/\alpha} X_h(1)$ to $Y(1)$. By a Paul Lévy theorem (also called the continuity theorem) [2, Theorem 2, p.508], the convergence in distribution can be proved by considering the pointwise convergence of the respective characteristic functions.

First, we want to prove (a). If $\alpha(0, 1)$, we obtain

$$\begin{aligned} E[e^{i\langle y, h^{-1/\alpha} X_h(1)\rangle}] &= E[e^{i\langle h^{-1/\alpha} y, X(h)\rangle}] \\ &= \exp\left\{ \int_{\mathbb{R}^d} h\psi^0_\alpha(h^{-1/\alpha}\langle y, x\rangle) R(dx) \right\}, \end{aligned} \tag{3.5}$$

The upper bounds (2.21) of Lemma 2.14 and condition (3.2) allow one to apply the dominated convergence theorem to the above integral. By definitions (2.25) and (2.27) it easy to check that $\psi^0_\alpha(-s) = \overline{\psi^0_\alpha(s)}$ and $\psi_\alpha(-s) = \overline{\psi_\alpha(s)}$. Now, by (2.20) and [12, Theorem 14.10], we calculate the limit $h \to 0$ under the integral (3.5)

$$\begin{aligned} \lim_{h\to 0} h\psi^0_\alpha(h^{-1/\alpha}\langle y, x\rangle) &= \Gamma(-\alpha)|\langle y, x\rangle|^\alpha \exp\left\{ -i\frac{\alpha\pi}{2} sgn\langle y, x\rangle \right\} \\ &= \Gamma(-\alpha) \cos\frac{\alpha\pi}{2} |\langle y, x\rangle|^\alpha \left(1 - i \tan\frac{\alpha\pi}{2} sgn\langle y, x\rangle\right). \end{aligned}$$

Therefore, under the assumption $\alpha \in (0,1)$, (3.3) holds. A similar argument proves (a) also in the case $\alpha \in (1,2)$. Let us consider the case $\alpha = 1$. By definition of a_h, the equality

$$E[\exp\{i\langle y, h^{-1}X_h(1) - a_h(1)\rangle\}] = \exp\left\{\int_{\mathbb{R}^d} (h\psi_1(h^{-1}\langle y,x\rangle) - i\langle y,x\rangle \log h)R(dx)\right\}. \tag{3.6}$$

is fulfilled, then, by assumption (3.4) and [10, Theorem 3.1], (b) holds. □

Now, we will prove that if one increases the time scale, a TID process looks like a Brownian motion.

Theorem 3.2. *(Long time behavior) Let $\{X(t) : t \geq 0\}$ be a TID Lévy process in $\mathbb{R}^d$ such that the distribution of $X(1) \sim TID_\alpha(R, 0)$ and $\alpha \in (0,2)$. Assume that*

$$\int_{\mathbb{R}^d} \|x\|^2 R(dx) < \infty. \tag{3.7}$$

Then

$$h^{-\frac{1}{2}} X_h \xrightarrow{d} B,$$

as $h \to \infty$, where $\{B(t) : t \geq 0\}$ is a Brownian motion with characteristic function

$$E[e^{i\langle y, B(t)\rangle}] = \exp\left\{-t2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2})\int_{\mathbb{R}^d} \langle y,x\rangle^2 R(dx)\right\}. \tag{3.8}$$

Proof. Since $\{h^{-\frac{1}{2}}X_h(t) : t \geq 0\}$ is a Lévy process, by [5, Theorem 13.17], it is enough to show the convergence in distribution of $h^{-\frac{1}{2}}X_h(1)$ to $B(1)$. By the continuity theorem [2, Theorem 2, p.508], the convergence in distribution can be proved by considering the pointwise convergence of the respective characteristic functions. By considering equality (2.24), we can write

$$\begin{aligned} E[e^{i\langle y, h^{-\frac{1}{2}}X_h(1)\rangle}] &= E[e^{i\langle h^{-\frac{1}{2}}y, X(h)\rangle}] \\ &= \exp\left\{\int_{\mathbb{R}^d} h\psi(h^{-\frac{1}{2}}\langle y,x\rangle)R(dx)\right\}. \end{aligned}$$

The upper bounds (2.21) and condition (3.7) allow one to apply the dominated convergence theorem to the above integral and by considering (2.19) we obtain

$$\lim_{h\to\infty} h\psi(h^{-\frac{1}{2}}\langle y,x\rangle) = -2^{-\frac{\alpha}{2}-1}\Gamma(1-\frac{\alpha}{2})\langle y,x\rangle^2,$$

which verifies (3.8). □

3.2 Change of measure

In this section, a result on density transformations between stable and TID processes is considered.

Theorem 3.3. *Let P_0 and P be probability measures on $(\Omega, \mathcal{F})$ such that the canonical process $\{X(t) : t \geq 0\}$ is a Lévy α-stable process $S_\alpha(\sigma, a)$ under P_0, while it is a proper TID process $TID_\alpha(R, b)$ under P, where σ is given by equation (2.15). Then*

(i) *$P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if*

$$\int_{S^{d-1}} \int_0^1 \Big(1 - q(r,u)\Big)^2 r^{-\alpha-1} dr\sigma(du) < \infty, \tag{3.9}$$

and

$$b - a = \begin{cases} 0, & 0 < \alpha < 1 \\ \int_{\mathbb{R}^d} x(\log\|x\| + \frac{\log 2}{2} - 1 + \frac{\gamma}{2})R(dx), & \alpha = 1 \\ 2^{-\frac{1}{2}-\frac{\alpha}{2}}\Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\mathbb{R}^d} xR(dx), & 1 < \alpha < 2 \end{cases} \tag{3.10}$$

Condition (3.9) implies that the integral exist. Furthermore, if either (3.9) or (3.10) fails, then $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are singular for all $t > 0$.

(ii) *If (3.9) and (3.10) hold, then for each $t > 0$*

$$\frac{dP}{dP_{0\,|\mathcal{F}_t}} = e^{Z_t}, \tag{3.11}$$

where $\{Z_t : t \geq 0\}$ is a Lévy process on $(\Omega, \mathcal{F}, P_0)$ given by

$$Z_t = \lim_{\delta \downarrow 0} \Bigg\{ \sum_{\{s \leq t : \|\Delta X_s\| > \delta\}} \log q\Big(\|\Delta X_s\|, \frac{\Delta X_s}{\|\Delta X_s\|}\Big) + t \int_{S^{d-1}} \int_\delta^\infty (1 - q(r,u)) r^{-\alpha-1} dr\sigma(du) \Bigg\}.$$

The convergence is uniform in t on any bounded interval, P_0-a.s..

Proof. First, we will prove part (i). By equalities (2.1) and (2.3), we have

$$\frac{dM}{dM_0}(x) = q\Big(\|x\|, \frac{x}{\|x\|}\Big), \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{3.12}$$

and for each $A \in \mathcal{B}(\mathbb{R}^d)$

$$\int_A q\Big(\|x\|, \frac{x}{\|x\|}\Big) M_0(dx) = \int_{S^{d-1}} \int_0^\infty I_A(ru) q(r,u) r^{-\alpha-1} dr\sigma(du) = M(A).$$

By Theorem 33.1 in [12], define the function $\phi(x)$ by

$$\frac{dM}{dM_0}(x) = e^{\phi(x)}.$$

Thus, we have

$$\phi(x) = \log q\Big(\|x\|, \frac{x}{\|x\|}\Big),$$

and $P_{0|\mathcal{F}_t}$ and $P_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$ if and only if

$$\int_{\mathbb{R}^d} \Big(e^{\frac{\phi(x)}{2}-1} - 1\Big)^2 M_0(dx) < \infty \tag{3.13}$$

and

$$B_\alpha = 0, \tag{3.14}$$

where B_α is defined by

$$B_\alpha = \begin{cases} b + \int_{\|x\|\le 1} xM(dx) - (a + \int_{\|x\|\le 1} xM_0(dx)) - \int_{\|x\|\le 1} x(M - M_0)(dx), & 0 < \alpha < 1, \\ b - \int_{\|x\|>1} xM(dx) - (a - c\int_{S^{d-1}} u\sigma(du)) - \int_{\|x\|\le 1} x(M - M_0)(dx), & \alpha = 1, \\ b - \int_{\|x\|>1} xM(dx) - (a - \int_{\|x\|>1} xM_0(dx)) - \int_{\|x\|\le 1} x(M - M_0)(dx), & 1 < \alpha < 2. \end{cases}$$

In the case $\alpha = 1$, $c = 1 - \gamma$, where γ is the Euler constant. The inequality (3.13) can be written as

$$\int_{\mathbb{R}^d} \left(1 - q^{1/2}\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx) < \infty \tag{3.15}$$

Since the integrand is bounded and M_0 is a Lévy measure, we may focus our attention only on integration over $\{\|x\| \le 1\}$. Applying elementary inequalities

$$\frac{1}{4}(1-y)^2 \le (1-\sqrt{y})^2 \le (1-y)^2$$

for $y \in [0, 1]$, inequality (3.15) becomes

$$\int_{\{\|x\|\le 1\}} \left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx),$$

and writing the above integral in polar coordinates, we obtain (3.9). Now, we will prove the equivalence between conditions (3.10) and (3.14). By finiteness of the integral above and Hölder inequality, we have

$$\begin{aligned} \int_{\|x\|\le 1} \|x\|(M_0 - M)(dx) &= \int_{\|x\|\le 1} \|x\|\left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right)M_0(dx) \\ \le \left(\int_{\|x\|\le 1} \|x\|^2 M_0(dx)\right)^{1/2} & \left(\int_{\|x\|\le 1} \left(1 - q\left(\|x\|, \frac{x}{\|x\|}\right)\right)^2 M_0(dx)\right)^{1/2} < \infty \end{aligned} \tag{3.16}$$

If $0 < \alpha < 1$, then $B_\alpha = b - a = 0$ by (3.10). Suppose $1 < \alpha < 2$, then we have

$$\int_{\|x\|>1} \|x\|M(dx) = \int_{\|x\|>1} \|x\|q\left(\|x\|, \frac{x}{\|x\|}\right)M_0(dx).$$

and, since we are considering a proper TID process

$$q\left(\|x\|, \frac{x}{\|x\|}\right) \le 1,$$

and M_0 is the Lévy measure of an α-stable distribution with $1 < \alpha < 2$, we obtain

$$\int_{\|x\|>1} \|x\|M(dx) \le \int_{\|x\|>1} \|x\|M_0(dx) < \infty.$$

Furthermore, by (3.16)

$$\int_{\mathbb{R}^d} \|x\|(M_0 - M)(dx) < \infty.$$

By using (2.9) and (2.14) and integrating by parts, the following result is obtained

$$\begin{aligned}\int_{\mathbb{R}^d} \|x\|(M_0 - M)(dx) &= \int_{\mathbb{R}^d} \int_0^\infty \|x\| t^{-\alpha}(1 - e^{-t^2/2}) dt R(dx) \\ &= -2^{-(1+\alpha)/2}\Gamma\Big(\frac{1}{2} - \frac{\alpha}{2}\Big) \int_{\mathbb{R}^d} \|x\| R(dx) < \infty.\end{aligned}$$

With a similar calculus, we can write

$$B_\alpha = \int_{\mathbb{R}^d} x(M_0 - M)(dx) + b - a = -2^{-(1+\alpha)/2}\Gamma(\frac{1}{2} - \frac{\alpha}{2}) \int_{\mathbb{R}^d} x R(dx) + b - a = 0,$$

where the last equality follows by (3.10), proving (3.14). It remains to verify (3.14) in the case $\alpha = 1$. By taking into account (3.16),

$$\begin{aligned}\infty > \int_{\|x\|\le 1} \|x\|(M_0 - M)(dx) &= \int_{\mathbb{R}^d} \|x\| \int_0^{1/\|x\|} t^{-1}(1 - e^{-t^2/2}) dt R(dx) \\ \ge \frac{1}{4}\int_{\|x\|\le 1} \|x\| \int_1^{\frac{1}{\|x\|}} t^{-1} dt R(dx) &= \frac{1}{4}\int_{\|x\|\le 1} \|x\| |\log \|x\| R(dx).\end{aligned}$$

Since the last integral is finite, the integral in (3.10) is well defined and we can calculate

$$\begin{aligned}\int_{\|x\|\le 1} x(M_0 - M)(dx) &= \int_{\mathbb{R}^d} x \int_0^{1/\|x\|} t^{-1}(1 - e^{-t^2/2}) dt R(dx) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} x\Big(E_1\Big(\frac{1}{2\|x\|^2}\Big) - 2\log\|x\| - \log 2 + \gamma\Big) R(dx)\end{aligned}$$

by changing variable and equation 5.1.39 in [1], where the function $E_1(x)$ is the exponential integral function defines by

$$E_1(x) = \int_x^\infty t^{-1}e^{-t}.$$

By part (b) of Proposition 2.11, the first moment is finite, thus

$$\int_{\|x\|>1} x M(dx) < \infty$$

and

$$\begin{aligned}\int_{\|x\|>1} x M(dx) &= \int_{\mathbb{R}^d} x \int_{1/\|x\|}^\infty t^{-1}e^{-t^2/2} dt R(dx) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} x E_1\Big(\frac{1}{2\|x\|^2}\Big) R(dx).\end{aligned}$$

Adding together the above results, we have

$$\begin{aligned}B_1 = &\; b - \frac{1}{2}\int_{\mathbb{R}^d} xE_1\Big(\frac{1}{2\|x\|^2}\Big)R(dx) - a + (1-\gamma)\int_{S^{d-1}} u\sigma(du) \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} x\Big(E_1\Big(\frac{1}{2\|x\|^2}\Big) - 2\log\|x\| - \log 2 + \gamma\Big)R(dx) \\ = &\; b - a + (1-\gamma)\int_{\mathbb{R}^d} xR(dx) - \int_{\mathbb{R}^d} x\Big(\log\|x\| + \frac{\log 2}{2}\Big)R(dx) + \frac{\gamma}{2}\int_{\mathbb{R}^d} xR(dx) = 0\end{aligned}$$

By considering the remark in [12, Notes page 236], we can complete the proof of part (i). Indeed, since M and M_0 are mutually absolutely continuous by (3.12), $P_{0|\mathcal{F}_t}$ and $P_{|\mathcal{F}_t}$ are mutually absolutely continuous or singular for all $t > 0$.

Part (ii) is an application of Theorem 33.2 of [12], where the form of Radon-Nikodym derivative is specified for two mutually absolutely continuous Lévy processes.

□

4 Simulation of proper TID laws and processes

There are different methods to simulate Lévy processes, but most of these methods are not suitable for the simulation of TID processes due to the complicated structure of their Lévy measure. The usual method of the inverse of the Lévy measure is difficult to implement, even if the spectral measure R has a simple form, readers are referred to [10]. To overcome this problem, we will find a shot noise representation for proper TID distributions, and consequently also TID processes, without constructing any inverse. The representation, we will show, is based on results in [9] and [10].

Let M be the Lévy measure of a proper TID distribution on $\mathbb{R}^d$, given by (2.3), and Q and R corresponding measures defined in (2.5) and (2.6). Let us define $\|\sigma\|$ as

$$\|\sigma\| := \sigma(S^{d-1}), \tag{4.1}$$

and by equalities (2.13) and (2.15), we obtain

$$\|\sigma\| = Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty.$$

Let $\{v_j\}$ be an i.i.d. sequence of random vector in $\mathbb{R}^d$ with distribution $Q/\|\sigma\|$. Let $\{u_j\}$ be an i.i.d. sequence of uniform random variables on $(0, 1)$ and let $\{e_j\}$ and $\{e'_j\}$ be i.i.d. sequences of exponential random variables with parameters 1. Furthermore, we assume that $\{v_j\}$, $\{u_j\}$, $\{e_j\}$ and $\{e'_j\}$ are independent. We consider $\gamma_j = e'_1 + \ldots + e'_j$ and, by definition of $\{e'_j\}$, $\{\gamma_j\}$ is a Poisson point process on $(0, \infty)$ with Lebesgue intensity measure. Now, we will prove a useful lemma.

Lemma 4.1. *Under the above definitions, let us define the function*

$$H(\gamma_j, (v_j, e_j, u_j)) := \left(\left(\frac{\alpha\gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}. \tag{4.2}$$

Then, for every non-empty set $A \in \mathcal{B}(\mathbb{R}^d)$, the equality

$$\int_0^\infty P(H(s,(v_1,e_1,u_1)) \in A)ds = M(A)$$

is verified.

Proof. Let A be a set of the form

$$A = \left\{ x \in \mathbb{R}^d : \|x\| > a, \frac{x}{\|x\|} \in B \right\},$$

where $a > 0$ and $B \in \mathcal{B}(S^{d-1})$. Then, we can write

$$\begin{aligned}
&\int_0^\infty P(H(s,(v_1,e_1,u_1)) \in A)ds = \\
&\quad = \int_0^\infty P\left(\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2} e_1^{1/2} u_1^{1/\alpha} \|v_1\|^{-1}\right) \frac{v_1}{\|v_1\|} \in A\right) ds, \\
&\quad = E \int_0^\infty I\left(\left(\frac{\alpha s}{\|\sigma\|}\right)^{-1/\alpha} > a, \sqrt{2} e_1^{1/2} u_1^{1/\alpha} > a\|v_1\|, \frac{v_1}{\|v_1\|} \in B\right) ds \\
&\quad = \frac{\|\sigma\| a^{-\alpha}}{\alpha} EI\left(\sqrt{2} e_1^{1/2} u_1^{1/\alpha} > a\|v_1\|, \frac{v_1}{\|v_1\|} \in B\right) \\
&\quad = \frac{a^{-\alpha}}{\alpha} \int_B \int_0^\infty P\left(\sqrt{2} e_1^{1/2} u_1^{1/\alpha} > as\right) Q(ds|u)\sigma(du).
\end{aligned}$$

By conditioning, the probability in the integral can be calculated

$$\begin{aligned}
P\left(\sqrt{2} e_1^{1/2} u_1^{1/\alpha} > as\right) &= \int_0^1 \int_{\frac{a^2 s^2}{2u^{2/\alpha}}}^\infty e^{-x} dx du \\
&= \int_0^1 e^{-\frac{a^2 s^2}{2u^{2/\alpha}}} du \\
&= a^\alpha \alpha \int_a^\infty e^{-r^2 s^2/2} r^{-\alpha-1} dr,
\end{aligned}$$

therefore we obtain

$$\begin{aligned}
\int_0^\infty P(H(s,(v_1,e_1,u_1)) \in A)ds &= \int_B \int_0^\infty \int_a^\infty e^{-r^2 s^2/2} r^{-\alpha-1} dr Q(ds|u)\sigma(du) \\
&= \int_B \int_a^\infty q(r,u) r^{-\alpha-1} dr \sigma(du) = M(A).
\end{aligned}$$

□

First, we consider a simple case.

Theorem 4.2. *($\alpha \in (0,1)$ and symmetric case) Suppose that all the above assumption are fulfilled. If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ and Q is symmetric, the series*

$$S_0 = \sum_{j=1}^\infty \left(\left(\frac{\alpha \gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1}\right) \frac{v_j}{\|v_j\|}. \tag{4.3}$$

converges a.s.. Furthemore, we have that $S_0 \sim TID_\alpha^0(R,0)$ for $\alpha \in (0,1)$ and $S_0 \sim TID_\alpha(R,0)$ for $\alpha \in [1,2)$.

Proof. To prove this theorem, we are going to use [9, Theorem 4.1] and [10, Theorem 5.1]. If H is defined as in (4.2), we can apply Lemma 4.1. Let us consider the case $\alpha \in (0,1)$, then by Proposition 2.13 we can write

$$\int_0^\infty E(\|H(s,(v_1,e_1,u_1))\| I(\|H(s,(v_1,e_1,u_1))\| \leq 1))ds = \int_{\|x\|\leq 1} \|x\| M(dx) < \infty$$

and [9, Theorem 4.1(A)] proves the theorem in the case $\alpha \in (0,1)$.

If $\alpha \in [1,2)$, then by Proposition 2.11 we have

$$\int_0^\infty E(\|H(s,(v_1,e_1,u_1))\| I(\|H(s,(v_1,e_1,u_1))\| > 1))ds = \int_{\|x\|>1} \|x\| M(dx) < \infty$$

and by [9, Theorem 4.1(B)], we can consider a series

$$\bar{S}_0 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - c_j \right]$$

which converges a.s. and $\bar{S}_0 \sim TID_\alpha(R,0)$, where

$$c_j = \int_{j-1}^{j} E\left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} \right] ds. \tag{4.4}$$

If Q is symmetric, c_j is equal to zero. It follows that $S_0 = \bar{S}_0$. This completes the proof. □

Now we consider the non-symmetric case.

Theorem 4.3. *(Non-symmetric case) Under the above notation, suppose $\alpha \in [1,2)$, Q is non-symmetric and additionally that*

$$\int_{\mathbb{R}^d} \|x\| |\log \|x\|| R(dx) < \infty \tag{4.5}$$

when $\alpha = 1$ and that

$$\int_{\mathbb{R}^d} \|x\| R(dx) < \infty \tag{4.6}$$

when $\alpha \in (1,2)$. Then, the series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha \gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - \left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + b \tag{4.7}$$

where

$$\begin{aligned} x_0 &= E \frac{v_j}{\|v_j\|} = \|\sigma\|^{-1} \int_{S^{d-1}} u \sigma(du), \\ x_1 &= \int_{\mathbb{R}^d} x R(dx), \end{aligned}$$

$$b = \begin{cases} \zeta\left(\frac{1}{\alpha}\right) \alpha^{-1/\alpha} \|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right) x_1, & 1 < \alpha < 2, \\ \left(\frac{3}{2}\gamma - \frac{\log 2}{2} + \log \|\sigma\|\right) x_1 - \int_{\mathbb{R}^d} x \log \|x\| R(dx), & \alpha = 1, \end{cases} \tag{4.8}$$

ζ denotes the Riemann zeta function and γ is the Euler constant, converges a.s.. Furthermore, we have that $S_1 \sim TID_\alpha(R,0)$.

Proof. To prove this theorem, we are going to use [9, Theorem 4.1] and [10, Theorem 5.1]. If H is defined as in (4.2), we can apply Lemma 4.1. If $\alpha \in [1,2)$, then by Proposition 2.11 we have

$$\int_0^\infty E(\|H(s,(v_1,e_1,u_1))\| I(\|H(s,(v_1,e_1,u_1))\| > 1))ds = \int_{\|x\|>1} \|x\| M(dx) < \infty$$

and [9, Theorem 4.1(B)], we can consider a series

$$S_1 = \sum_{j=1}^{\infty} \left[\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} - c_j \right]$$

which converges a.s. and $S_1 \sim TID_\alpha(R,0)$, where

$$c_j = \int_{j-1}^{j} E\left[\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_j^{1/2} u_j^{1/\alpha} \|v_j\|^{-1} \right) \frac{v_j}{\|v_j\|} \right] ds.$$

We have to prove that the equality

$$\sum_{j=1}^{\infty} \left[\left(\frac{\alpha j}{\|\sigma\|} \right)^{-1/\alpha} x_0 - c_j \right] = b \tag{4.9}$$

holds, where b is given by (4.8).

First consider the case $\alpha \in (1,2)$. Define for $j \geq 1$ [10, equation (5.8)]

$$c_j' = \int_{j-1}^{j} E\left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \frac{v_1}{\|v_1\|} \right] ds = \frac{\alpha^{1-1/\alpha} \|\sigma\|^{1/\alpha}}{\alpha - 1} [j^{1-1/\alpha} - (j-1)^{1-1/\alpha}] x_0. \tag{4.10}$$

We have

$$\|c_j' - c_j\| \leq \int_{j-1}^{j} E\left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_1^{1/2} u_1^{1/\alpha} \|v_1\|^{-1} \right] \right\} ds.$$

Furthermore by [10, equation 5.9], for every $\theta > 0$ the equality

$$\int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \theta \right] \right\} ds = \frac{\|\sigma\|}{\alpha(\alpha-1)} \theta^{1-\alpha} \tag{4.11}$$

holds. Using this identity for $\theta = \sqrt{2} e_1^{1/2} u_1^{1/\alpha} \|v_1\|^{-1}$ pointwise, we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \|c_j' - c_j\| &= E \int_0^\infty \left\{ \left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} - \left[\left(\frac{\alpha s}{\|\sigma\|} \right)^{-1/\alpha} \wedge \sqrt{2} e_1^{1/2} u_1^{1/\alpha} \|v_1\|^{-1} \right] \right\} ds \\
&= \frac{\|\sigma\|}{\alpha(\alpha-1)} E\left[2^{\frac{1}{2}(1-\alpha)} e_1^{\frac{1}{2}(1-\alpha)} u_1^{-1+\frac{1}{\alpha}} \|v_1\|^{\alpha-1} \right] \\
&= 2^{\frac{1}{2}(1-\alpha)} \Gamma\left(\frac{3}{2} - \frac{\alpha}{2} \right) \frac{\|\sigma\|}{\alpha - 1} E\|v_1\|^{\alpha-1} \\
&= -2^{-(1+\alpha)/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{2} \right) \int_{\mathbb{R}^d} \|x\| R(dx) < \infty.
\end{aligned} \tag{4.12}$$

By using (4.11) we obtain

$$
\begin{aligned}
\sum_{j=1}^{\infty}(c_j'-c_j) &= E\Big\{\int_0^\infty \Big(\Big(\frac{\alpha s}{\|\sigma\|}\Big)^{-1/\alpha} - \Big[\Big(\frac{\alpha s}{\|\sigma\|}\Big)^{-1/\alpha} \wedge \sqrt{2} e_1^{1/2} u_1^{1/\alpha}\|v_1\|^{-1}\Big]\Big) ds \frac{v_1}{\|v_1\|}\Big\} \\
&= E\Big[\frac{\|\sigma\|}{\alpha(\alpha-1)} 2^{\frac{1}{2}(1-\alpha)} e_1^{\frac{1}{2}(1-\alpha)} u_1^{-1+\frac{1}{\alpha}} \|v_1\|^{\alpha-1} \frac{v_1}{\|v_1\|}\Big] \\
&= 2^{\frac{1}{2}(1-\alpha)}\Gamma\left(\frac{3}{2}-\frac{\alpha}{2}\right)\frac{1}{\alpha-1}\int_{\mathbb{R}^d} x\|x\|^{\alpha-2} Q(dx) \\
&= -2^{-(1+\alpha)/2}\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)\int_{\mathbb{R}^d} xR(dx) \\
&= -2^{-(1+\alpha)/2}\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right) x_1
\end{aligned}
$$

Then we have

$$
\sum_{j=1}^{n}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j'\right] = \left(\sum_{j=1}^{n} j^{-1/\alpha} - \frac{\alpha}{\alpha-1} n^{1-1/\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha} x_0.
$$

From a classical formula [1, 23.2.9],

$$
\sum_{j=1}^{n} j^{-z} - \frac{n^{1-z}}{1-z} = \zeta(z) + z\int_n^\infty \frac{s-[s]}{s^{z+1}} ds, \qquad \mathrm{Re}(z) > 0,\ \mathrm{Re}(z) \neq 1, \tag{4.13}
$$

we obtain

$$
\sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j'\right] = \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha} x_0
$$

and we can write

$$
\begin{aligned}
\sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j\right] &= \sum_{j=1}^{\infty}\left[\left(\frac{\alpha j}{\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j'\right] + \sum_{j=1}^{\infty}(c_j'-c_j) \\
&= \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}\|\sigma\|^{1/\alpha} x_0 - 2^{-(1+\alpha)/2}\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right) x_1 = b
\end{aligned}
$$

which proves (4.9). Now, we consider the case $\alpha = 1$. By the same computation above, define for $j \leq 2$

$$
c_j' = \int_{j-1}^{j} E\left[\left(\frac{s}{\|\sigma\|}\right)^{-1/\alpha}\frac{v_1}{\|v_1\|}\right] ds = (\log j - \log(j-1))\|\sigma\| x_0, \tag{4.14}
$$

and put $c_1' = 0$. For every $\theta > 0$ [10, equation (5.14)], we have

$$
\begin{aligned}
&\int_1^\infty \left\{\left(\frac{s}{\|\sigma\|}\right)^{-1} - \left[\left(\frac{s}{\|\sigma\|}\right)^{-1} \wedge \theta\right]\right\} ds \\
&= \{\theta - \|\sigma\|\log\theta + \|\sigma\|\log\|\sigma\| - \|\sigma\|\} I(\theta \leq \|\sigma\|) \qquad (4.15) \\
&\leq \|\sigma\|\log^+\left(\frac{\|\sigma\|}{\theta}\right).
\end{aligned}
$$

By assumption (4.18), we can write

$$\begin{aligned}\sum_{j=1}^{\infty}\|c_j'-c_j\| &= E\int_1^\infty\left\{\left(\frac{s}{\|\sigma\|}\right)^{-1}-\left[\left(\frac{s}{\|\sigma\|}\right)^{-1}\wedge\sqrt{2}e_1^{1/2}u_1\|v_1\|^{-1}\right]\right\}ds\\ &\le \|\sigma\|E\log^+\left(\frac{\|\sigma\|\|v_1\|}{\sqrt{2}e_1^{1/2}u_1}\right)\\ &\le \|\sigma\|(|\log\|\sigma\||+E|\log\|v_1\||+E|\log\sqrt{2}e_1^{1/2}u_1|)\\ &= \|\sigma\||\log\|\sigma\||+\int_{\mathbb{R}^d}|\log\|x\|\|\|x\|R(dx)+K\|\sigma\|<\infty,\end{aligned}\tag{4.16}$$

where $K=E|\log\sqrt{2}e_1^{1/2}u_1|<\infty$.

Before computing the series $\sum_{j=1}^{\infty}(c_j'-c_j)$, we recall some useful relations [10]. For every $\theta>0$

$$\int_0^1\left(\frac{s}{\|\theta\|}\right)^{-1}\wedge\theta ds=\theta I(\theta\le\|\sigma\|)+\{\|\sigma\|-\|\sigma\|\log\|\sigma\|+\|\sigma\|\log\theta\}I(\theta>\|\sigma\|)$$

and by (4.15) we get

$$-\int_0^1\left(\frac{s}{\|\theta\|}\right)^{-1}\wedge\theta\,ds+\int_1^\infty\left\{\left(\frac{s}{\|\theta\|}\right)^{-1}-\left[\left(\frac{s}{\|\theta\|}\right)^{-1}\wedge\theta\right]\right\}=\|\sigma\|(\log\|\sigma\|-\log\theta-1).$$

By using this formula for $\theta=\sqrt{2}e_1^{1/2}u_1\|v_1\|^{-1}$ we get

$$\begin{aligned}\sum_{j=1}^{\infty}(c_j'-c_j) &= E\Bigg\{\Bigg[-\int_0^1\left(\left(\frac{s}{\|\theta\|}\right)^{-1}\wedge\sqrt{2}e_1^{1/2}u_1\|v_1\|^{-1}\right)ds\\ &\qquad+\int_1^\infty\left(\left(\frac{s}{\|\theta\|}\right)^{-1}-\left[\left(\frac{s}{\|\theta\|}\right)^{-1}\wedge\sqrt{2}e_1^{1/2}u_1\|v_1\|^{-1}\right]\right)ds\Bigg]\frac{v_1}{\|v_1\|}\Bigg\}\\ &= \|\sigma\|E\left\{(\log\|\sigma\|+\log\|v_1\|-\log(\sqrt{2}e_1^{1/2}u_1)-1)\frac{v_1}{\|v_1\|}\right\}.\end{aligned}$$

The following expectation can be calculated

$$E\log(\sqrt{2}e_1^{1/2}u_1)=\frac{\log 2}{2}-1-\frac{1}{2}\gamma,$$

where $\gamma=-\int_0^\infty\log(x)e^{-x}dx$ is the Euler constant, see [1, 6.1.3]. By equation (2.7), the series above can be rewritten as

$$\begin{aligned}\sum_{j=1}^{\infty}(c_j'-c_j) &= \|\sigma\|E\left\{\frac{v_1}{\|v_1\|}(\log\|\sigma\|+\log\|v_1\|+\frac{1}{2}\gamma-\frac{\log 2}{2}\right\}\\ &= \int_{\mathbb{R}^d}\frac{x}{\|x\|}\left(\log\|\sigma\|+\log\|x\|+\frac{1}{2}\gamma-\frac{\log 2}{2}\right)Q(dx)\\ &= \int_{\mathbb{R}^d}x\left(\log\|\sigma\|-\log\|x\|+\frac{1}{2}\gamma-\frac{\log 2}{2}\right)R(dx)\\ &= \left(\frac{1}{2}\gamma-\frac{\log 2}{2}+\log\|\sigma\|\right)x_1-\int_{\mathbb{R}^d}x\log\|x\|R(dx).\end{aligned}$$

By [10, Theorem 5.1], the equality

$$\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c'_j\right] = \gamma x_1$$

holds, where γ is the Euler constant, thus we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c_j\right] &= \sum_{j=1}^{\infty}\left[\left(\frac{j}{\|\sigma\|}\right)^{-1}x_0 - c'_j\right] + \sum_{j=1}^{\infty}(c'_j - c_j)\\ &= \left(\frac{3}{2}\gamma - \frac{\log 2}{2} + \log\|\sigma\|\right)x_1 - \int_{\mathbb{R}^d} x\log\|x\| R(dx) = b,\end{aligned}$$

which completes the proof. □

A series representation for TID processes can be obtained.

Theorem 4.4. *Under the above notation and assumptions, given a fixed $T > 0$, let $\{\tau_j\}$ be a i.i.d. sequence of uniform random variables in $[0, T]$. Assume $\{\tau_j\}$ independent of the random sequences $\{v_j\}$, $\{u_j\}$, $\{e_j\}$ and $\{\gamma_j\}$.*

(i) If $\alpha \in (0, 1)$, or if $\alpha \in [1, 2)$ and Q is symmetric, set for every $t \in [0, T]$

$$X_0(t) = \sum_{j=1}^{\infty} I_{(0,t]}(\tau_j)\left(\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}, \tag{4.17}$$

then the series converges a.s. uniformly in $t \in [0, T]$ to a Lévy process such that $X_0(t) \sim TID^0_\alpha(tR, 0)$ if $\alpha \in (0, 1)$ and $X_0(t) \sim TID^0_\alpha(tR, 0)$ if $\alpha \in [1, 2)$.

(ii) If $\alpha \in [1, 2)$, Q is non-symmetric and additionally

$$\int_{\mathbb{R}^d} \|x\|| \log\|x\|| R(dx) < \infty \tag{4.18}$$

when $\alpha = 1$ and that

$$\int_{\mathbb{R}^d} \|x\| R(dx) < \infty \tag{4.19}$$

when $\alpha \in (1, 2)$, then, the series

$$\begin{aligned}X_1(t) = \sum_{j=1}^{\infty}\Bigg[I_{(0,t]}(\tau_j)&\left(\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}\\ &- \frac{t}{T}\left(\frac{\alpha j}{T\|\sigma\|}\right)^{-1/\alpha}x_0\Bigg] + tb_T.\end{aligned} \tag{4.20}$$

where

$$b_T = \begin{cases} \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}T^{-1}(T\|\sigma\|)^{1/\alpha}x_0 - 2^{-(1+\alpha)/2}\Gamma\left(\frac{1}{2} - \frac{\alpha}{2}\right)x_1, & 1 < \alpha < 2,\\ (\frac{3}{2}\gamma - \frac{\log 2}{2} + \log T\|\sigma\|)x_1 - \int_{\mathbb{R}^d} x\log\|x\| R(dx), & \alpha = 1,\end{cases} \tag{4.21}$$

the series converges a.s. uniformly in $t \in [0, T]$ to a Lévy process such that $X_1(t) \sim TID_\alpha(tR, 0)$.

Proof. It is enough to show the convergence in distribution of series (4.17) and (4.20) for a fixed t, see [9, 10]. By the same arguments of Lemma 4.1, we obtain

$$\int_0^\infty P(H(s,(v_1,e_1,u_1,\tau_1)) \in A)ds = tM(A)$$

where we define

$$H(\gamma_j,(v_j,e_j,u_j,\tau_j)) := I_{(0,t]}(\tau_j)\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}. \quad (4.22)$$

By following the proof of Theorem 4.2, (i) is verified in the case $\alpha \in (0,1)$. By Proposition 2.11 if $\alpha \in [1,2)$, then $\int_{\|x\|>1}\|x\|M(dx) < \infty$. By [9, Theorem 4.1(B)] we can consider the series

$$\bar{X}_1(t) = \sum_{j=1}^\infty \left[I_{(0,t]}(\tau_j)\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|} - a_j^T\right]$$

which converges a.s. and $\bar{X}_1(t) \sim TID_\alpha(tR,0)$, where

$$a_j^T = \int_{j-1}^j E\left[I_{(0,t]}(\tau_j)\left(\left(\frac{\alpha\gamma_j}{\|\sigma\|}\right)^{-1/\alpha} \wedge \sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|}\right]ds.$$

If Q is symmetric then $a_j^T = 0$ and (i) is proved. To complete the proof, by following [10, Theorem 5.3] and equation (4.4), c_j can be viewed as a function of the measure Q, thus we have

$$a_j^T(t) = \frac{t}{T}c_j(TQ).$$

By Theorem 4.3, where TQ and TR have to be considered instead of Q and R, we have

$$\sum_{j=1}^\infty \left[\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} x_0 - c_j(TQ)\right]$$
$$= \begin{cases} \zeta\left(\frac{1}{\alpha}\right)\alpha^{-1/\alpha}(T\|\sigma\|)^{1/\alpha}x_0 - 2^{-(1+\alpha)/2}\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)Tx_1\alpha - 1, & 1<\alpha<2, \\ \left(\frac{3}{2}\gamma - \frac{\log 2}{2} + \log T\|\sigma\|\right)Tx_1 - T\int_{\mathbb{R}^d} x\log\|x\|R(dx), & \alpha = 1. \end{cases}$$

By definition of $a_j^T(t)$, we obtain

$$\sum_{j=1}^\infty \left[\frac{t}{T}\left(\frac{\alpha\gamma_j}{T\|\sigma\|}\right)^{-1/\alpha} x_0 - a_j^T(t)\right] = tb_T,$$

which completes the proof. □

Remark 4.5. *By removing the tempering part $\sqrt{2}e_j^{1/2}u_j^{1/\alpha}\|v_j\|^{-1}$ in the shot noise representation, a well-known result for α-stable processes can be found, see [10, Theorem 5.4] or [11].*

5 Examples

A real TID law can be defined by fixing a positive definite radial function q with a measure σ on S^1 or alternatively by defining its spectral measure R. We are going to show in the following three parametric examples of TID laws in one dimension. In the first example, the measure R is the sum of two Dirac measures multiplied for opportune constants. The spectral measure R of the second example has a non-trivial bounded support and the derived TID distribution has exponential moments of any order. In the last example, the MTS distribution is considered, see [8, 6], the spectral measure is defined on an unbounded support and there exist exponential moments of some order.

5.1 Example 1: Simple TID distribution

The Lévy measure of the form

$$M(dx) = (C_+ e^{-\lambda_+^2 x^2/2} 1_{x>0} + C_- e^{-\lambda_-^2 |x|^2/2} 1_{x<0}) \frac{dx}{|x|^{\alpha+1}},$$

can be written in polar coordinates as

$$M(dr, du) = r^{-\alpha-1} q(r,u) dr\, \sigma(du)$$

where

$$q(r,1) = e^{-\lambda_+^2 r^2/2}, \quad q(r,-1) = e^{-\lambda_-^2 r^2/2},$$

and

$$\sigma(1) = C_+, \quad \sigma(-1) = C_-.$$

The positive definite radial function q, by Theorem 2.4, has the form

$$q(r,u) = \int_0^\infty e^{-r^2 s^2/2} Q(ds|u)$$

where

$$Q(ds|1) = \delta_{\lambda_+}(s)ds, \quad Q(ds|-1) = \delta_{\lambda_-}(s)ds,$$

and we have

$$Q(A) = C_+ \int_A \delta_{\lambda_+}(x)dx + C_- \int_A \delta_{-\lambda_-}(x)dx,$$

and hence the spectral measure R can be defined

$$R(A) = C_+ \int_A \lambda_+^\alpha \delta_{1/\lambda_+}(x)dx + C_- \int_A \lambda_-^\alpha \delta_{-1/\lambda_-}(x)dx.$$

Definition 5.1. *Let C_+, C_-, λ_+, λ_- strictly positive constants, $\alpha \in (0,2)$, $\alpha \neq 1$ and $\mu \in \mathbb{R}$. An infinitely divisible distribution is called the simple TID distribution with parameter (α, C_+, C_-, λ_+, λ_-) and mean μ, if its Lévy triplet is given by (0, μ, M) where*

$$M(dx) = (C_+ e^{-\lambda_+^2 x^2/2} 1_{x>0} + C_- e^{-\lambda_-^2 |x|^2/2} 1_{x<0}) \frac{dx}{|x|^{\alpha+1}}.$$

Proposition 5.2. *The characteristic function of the simple TID distribution with parameter* (α, C_+, C_-, λ_+, λ_-, μ) *becomes*

$$\phi(u) = \exp\left(iu\mu + G(iu;\alpha, C_+, \lambda_+) + G(-iu;\alpha, C_-, \lambda_-)\right) \tag{5.1}$$

where

$$\begin{aligned} G(x;\alpha,C,\lambda) &= 2^{-\alpha/2-1}C\lambda^{\alpha}\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ &\quad + \frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ &\quad - \frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big) - \Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg). \end{aligned} \tag{5.2}$$

Proof. It follows by Theorem 2.17. □

5.2 Example 2: non trivial spectral measure

In the first example, the spectral measure R has no zero mass only at two points. Now we will consider a spectral measure with power decay defined on a bounded support of $\mathbb{R}$. By taking into consideration the construction of the KR tempered stable distribution in [7], we can consider the same spectral measure R. Indeed we have

$$R(dx) = (C_+ r_+^{-p_+} I_{(0,r_+)}(x)|x|^{p_+-1} + C_- r_-^{-p_-} I_{(-r_-,0)}(x)|x|^{p_--1})\,dx. \tag{5.3}$$

By Theorem 2.17, the characteristic function of this distribution can be written in close form.

Lemma 5.3. *Let* $\alpha \in (0,2)$ *and* $\alpha \neq 1$*. Then, the following equality holds*

$$\begin{aligned} &\int\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Bigg)x^{p-1}dx \\ &\quad = \frac{x^p}{p}\Gamma\Big(-\frac{\alpha}{2}\Big)\,{}_2F_2\Big(\frac{p}{2},-\frac{\alpha}{2};1+\frac{p}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}u\frac{x^{p+1}}{p+1}\Gamma\Big(\frac{1-\alpha}{2}\Big)\,{}_2F_2\Big(\frac{1}{2}+\frac{p}{2},\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2}+\frac{p}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \end{aligned} \tag{5.4}$$

Proof. By equation (2.23), we can write

$$\begin{aligned} &\int\Bigg(\Gamma\Big(-\frac{\alpha}{2}\Big)M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\ &\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big)M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\Bigg)x^{p-1}dx \\ &\quad = \int\sum_{n=0}^{\infty}\frac{(i\sqrt{2}u)^n x^{n+p-1}}{n!}\Gamma\left(\frac{1}{2}(n-\alpha)\right)dx \end{aligned} \tag{5.5}$$

Since the series converges on each bounded interval on $\mathbb{R}$, we obtain

$$\int \sum_{n=0}^{\infty} \frac{(i\sqrt{2}u)^n x^{n+p-1}}{n!} \Gamma\left(\frac{1}{2}(n-\alpha)\right) dx = x^p \sum_{n=0}^{\infty} \frac{(i\sqrt{2}ux)^n}{n!(n+p)} \Gamma\left(\frac{1}{2}(n-\alpha)\right).$$

Furthermore, the following equalities are fulfilled

$$\begin{aligned} \frac{p}{2n+p} &= \frac{\left(\frac{p}{2}\right)_n}{\left(1+\frac{p}{2}\right)_n} \\ \frac{p+1}{2n+1+p} &= \frac{\left(\frac{1}{2}+\frac{p}{2}\right)_n}{\left(\frac{3}{2}+\frac{p}{2}\right)_n} \end{aligned}$$

and by a similar argument of Lemma 2.16, equation (5.4) is verified. □

Proposition 5.4. *The characteristic function of the TID distribution with parameter $(\alpha, C_+, C_-, \lambda_+, \lambda_-, p_+, p_-)$, mean m and with spectral measure (5.3) is*

$$\phi(u) = \exp\left(ium + C_+ B(iu; \alpha, r_+, p_+) + C_- B(-iu; \alpha, r_-, p_-)\right) \tag{5.6}$$

where

$$\begin{aligned} &B(iu; \alpha, r, p) = \\ &= 2^{-\alpha/2-1}\frac{1}{p}\Gamma\Big(-\frac{\alpha}{2}\Big)\left({}_2F_2\Big(\frac{p}{2}, -\frac{\alpha}{2}; 1+\frac{p}{2}, \frac{1}{2}; \Big(\frac{i\sqrt{2}ur}{2}\Big)^2\Big) - 1\right) \\ &\quad + 2^{-\alpha/2-1}\frac{i\sqrt{2}ur}{p+1}\Gamma\Big(\frac{1-\alpha}{2}\Big)\left({}_2F_2\Big(\frac{1}{2}+\frac{p}{2}, \frac{1}{2}-\frac{\alpha}{2}; \frac{3}{2}+\frac{p}{2}, \frac{3}{2}; \Big(\frac{i\sqrt{2}ur}{2}\Big)^2\Big) - 1\right) \end{aligned} \tag{5.7}$$

Proof. Since the support of the measure R is bounded, by Proposition 2.11 the distribution has exponential moments of any order and in particular finite mean. By Theorem 2.17, we can consider the representation (2.25) and by Lemma 5.3 the characteristic exponent can be computed. □

5.3 Example 3 : MTS distribution

A parametric example of TID distributions has been already considered in the literature, the MTS distribution, see [8, 6]. The Lévy measure of a MTS distribution is defined as

$$M(dx) = \left(C_+ \frac{\lambda_+^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ x)}{x^{\frac{\alpha+1}{2}}} 1_{x>0} + C_- \frac{\lambda_-^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- |x|)}{|x|^{\frac{\alpha+1}{2}}} 1_{x<0} \right) dx,$$

where $\lambda_+, \lambda_-, C_+, C_- > 0$, $\alpha \in (0, 2)$, and $\alpha \neq 1$. The tempering function q is of the form

$$q(r, u) = \begin{cases} (\lambda_+ r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_+ r), & u = 1 \\ (\lambda_- r)^{\frac{\alpha+1}{2}} K_{\frac{\alpha+1}{2}}(\lambda_- r), & u = -1, \end{cases} \tag{5.8}$$

and the measure σ is

$$\sigma(1) = C_+, \qquad \sigma(-1) = C_-.$$

Lemma 5.5. *Let $z > 0$ and $K_\nu(x)$ the modified Bessel function of second kind, then the equality*

$$2z^{\frac{\nu}{2}}K_\nu(2\sqrt{z}) = \int_0^\infty e^{-zt-\frac{1}{t}}t^{-\nu-1}dt \tag{5.9}$$

is satisfied.

Proof. By equality [3, 8.432(7)], we have

$$K_\nu(xp) = \frac{p^\nu}{2}\int_0^\infty e^{-\frac{xt}{2}-\frac{xp^2}{2t}}t^{-\nu-1}dt. \tag{5.10}$$

By setting $x = 2z$ and $p = 1/\sqrt{z}$, then we can write

$$K_\nu(2\sqrt{z}) = \frac{z^{-\frac{\nu}{2}}}{2}\int_0^\infty e^{-zt-\frac{1}{t}}t^{-\nu-1}dt$$

hence the equality (5.9) holds. □

Lemma 5.6. *Let μ be a MTS distribution, then*

$$Q(ds|\pm 1) = e^{-\lambda_\pm^2/2s^2}s^{-\alpha-2}\lambda_\pm^{\alpha+1}ds, \tag{5.11}$$

and

$$R(dx) = \left(C_+\lambda_+^{\alpha+1}e^{-\frac{\lambda_+^2x^2}{2}}I_{x>0} + C_-\lambda_-^{\alpha+1}e^{-\frac{\lambda_-^2x^2}{2}}I_{x<0}\right)dx. \tag{5.12}$$

Proof. By setting $\nu = \frac{\alpha+1}{2}$ and $z = (\lambda r)^2/4$ into (5.9) and changing variable $t = 2s^2/\lambda^2$, we have

$$\begin{aligned}(\lambda r)^{\frac{\alpha+1}{2}}K_{\frac{\alpha+1}{2}}(\lambda r) &= 2^{\frac{\alpha}{2}-\frac{1}{2}}\int_0^\infty e^{-\frac{r^2s^2}{2}}e^{-\frac{\lambda^2}{2s^2}}\left(\frac{2s^2}{\lambda^2}\right)^{-\frac{\alpha+3}{2}}4s\lambda^{-2}ds \\ &= 2^{\frac{\alpha}{2}-\frac{1}{2}}\int_0^\infty e^{-\frac{r^2s^2}{2}}e^{-\frac{\lambda^2}{2s^2}}s^{-\alpha-2}\lambda^{\alpha+1}2^{-\frac{\alpha}{2}+\frac{1}{2}}ds \\ &= \int_0^\infty e^{-\frac{r^2s^2}{2}}e^{-\frac{\lambda^2}{2s^2}}s^{-\alpha-2}\lambda^{\alpha+1}ds\end{aligned}$$

By applying this result into (5.8), we have

$$q(r,\pm 1) = \int_0^\infty e^{-\frac{r^2s^2}{2}}\left(e^{-\frac{\lambda^2}{2s^2}}s^{-\alpha-2}\lambda^{\alpha+1}\right)ds.$$

and obtain the equation (5.11) by the definition of $Q(ds|u)$. Moreover, for $A \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned}Q(A) &= \int_{S^0}\int_0^\infty I_A(ru)Q(dr|u)\sigma(du) \\ &= \int_0^\infty I_A(r)Q(dr|1)\sigma(1) + \int_0^\infty I_A(-r)Q(dr|-1)\sigma(-1) \\ &= C_+\lambda_+^{\alpha+1}\int_0^\infty I_A(r)e^{-\lambda_+^2/2r^2}r^{-\alpha-2}dr \\ &\quad + C_-\lambda_-^{\alpha+1}\int_0^\infty I_A(-r)e^{-\lambda_-^2/2r^2}r^{-\alpha-2}dr.\end{aligned}$$

Hence,

$$
\begin{aligned}
R(A) &= \int_{\mathbb{R}} I_A\Big(\frac{x}{|x|^2}\Big)|x|^\alpha Q(dx) \\
&= \int_{S^0}\int_0^\infty I_A\Big(\frac{ru}{r^2}\Big) r^\alpha Q(dr|u)\sigma(du) \\
&= C_+\lambda_+^{\alpha+1}\int_0^\infty I_A\Big(\frac{1}{r}\Big) r^\alpha e^{-\lambda_+^2/2r^2} r^{-\alpha-2}dr \\
&\quad + C_-\lambda_-^{\alpha+1}\int_0^\infty I_A\Big(-\frac{1}{r}\Big) r^\alpha e^{-\lambda_-^2/2r^2} r^{-\alpha-2}dr \\
&= C_+\lambda_+^{\alpha+1}\int_0^\infty I_A(x) e^{-\frac{\lambda_+^2 x^2}{2}}dx \\
&\quad + C_-\lambda_-^{\alpha+1}\int_0^\infty I_A(-x) e^{-\frac{\lambda_-^2 x^2}{2}}dx \\
&= \int_A \Big(C_+\lambda_+^{\alpha+1}e^{-\frac{\lambda_+^2 x^2}{2}} I_{x>0} + C_-\lambda_-^{\alpha+1}e^{-\frac{\lambda_-^2 x^2}{2}} I_{x<0}\Big)dx.
\end{aligned}
$$

□

Lemma 5.7. *Let $\alpha \in (0,2)$ and $\alpha \neq 1$. Then, the following equality holds*

$$
\begin{aligned}
&\int_0^\infty \bigg(\Gamma\Big(-\frac{\alpha}{2}\Big) M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\
&\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big) M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\bigg) e^{-x^2\lambda^2/2}dx \\
&\quad = \frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}} + \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big){}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big).
\end{aligned} \tag{5.13}
$$

Proof. By equation (2.23), we can write

$$
\begin{aligned}
&\int_0^\infty \bigg(\Gamma\Big(-\frac{\alpha}{2}\Big) M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big) \\
&\qquad + i\sqrt{2}ux\Gamma\Big(\frac{1-\alpha}{2}\Big) M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{i\sqrt{2}ux}{2}\Big)^2\Big)\bigg) e^{-x^2\lambda^2/2}dx \qquad (5.14) \\
&= \int_0^\infty \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big) x^n e^{-x^2\lambda^2/2}dx
\end{aligned}
$$

Since the series converges on each bounded interval on $\mathbb{R}$, by a similar argument of Lemma 2.16, we can write

$$\int_0^\infty \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)x^n e^{-x^2\lambda^2/2}dx =$$

$$= \sum_{n=0}^\infty \frac{(i\sqrt{2}u)^n}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)2^{-\frac{1}{2}+\frac{n}{2}}\lambda^{-n-1}\Gamma\Big(\frac{1}{2}(n+1)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^n \frac{1}{n!}\Gamma\Big(\frac{1}{2}(n-\alpha)\Big)\Gamma\Big(\frac{1}{2}(n+1)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^{2n} \frac{1}{2n!}\Gamma\Big(\frac{1}{2}(2n-\alpha)\Big)\Gamma\Big(\frac{1}{2}(2n+1)\Big)$$

$$+ \frac{1}{\sqrt{2}\lambda}\sum_{n=0}^\infty \Big(\frac{2iu}{\lambda}\Big)^{2n+1} \frac{1}{(2n+1)!}\Gamma\Big(\frac{1}{2}(2n+1-\alpha)\Big)\Gamma\Big(\frac{1}{2}(2n+2)\Big)$$

$$= \frac{1}{\sqrt{2}\lambda}\Gamma\Big(-\frac{\alpha}{2}\Big)\Gamma\Big(\frac{1}{2}\Big)\sum_{n=0}^\infty \Big(-\frac{u^2}{\lambda^2}\Big)^n \frac{1}{n!}\frac{\big(-\frac{\alpha}{2}\big)_n\big(\frac{1}{2}\big)_n}{\big(\frac{1}{2}\big)_n}$$

$$+ \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)\sum_{n=0}^\infty \Big(-\frac{u^2}{\lambda^2}\Big)^n \frac{1}{n!}\frac{\big(\frac{1}{2}-\frac{\alpha}{2}\big)_n(1)_n}{\big(\frac{3}{2}\big)_n}$$

$$= \frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}} + \frac{i\sqrt{2}u}{\lambda^2}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)\,{}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big).$$

□

Proposition 5.8. *The characteristic function of the MTS distribution with parameter* $(\alpha, C_+, C_-, \lambda_+, \lambda_-)$*, mean* m *and with spectral measure (5.12) is*

$$\phi(u) = \exp\left(ium + C_+H(iu;\alpha,\lambda_+) + C_-H(-iu;\alpha,\lambda_-)\right) \tag{5.15}$$

where

$$\begin{aligned} H(iu;\alpha,\lambda) \;&= \frac{\lambda^\alpha\sqrt{\pi}}{2^{\frac{\alpha}{2}+\frac{3}{2}}}\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(\Big(1+\frac{u^2}{\lambda^2}\Big)^{\frac{\alpha}{2}}-1\Big) \\ &+ \frac{i\lambda^{\alpha-1}u}{2^{\frac{\alpha}{2}+\frac{1}{2}}}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big)\Big({}_2F_1\Big(1,\frac{1}{2}-\frac{\alpha}{2};\frac{3}{2};-\frac{u^2}{\lambda^2}\Big)-1\Big). \end{aligned} \tag{5.16}$$

Proof. By definition of the measure R, by Proposition 2.11, the distribution has finite mean. By Theorem 2.17, we can consider the representation (2.25) and by Lemma 5.7 the characteristic exponent can be computed. □

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