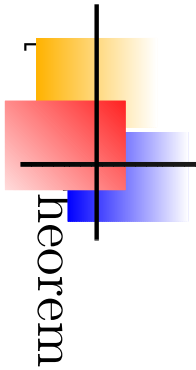

Module S-5/1 - Part 1 -
Asset Liability Management

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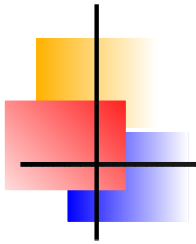


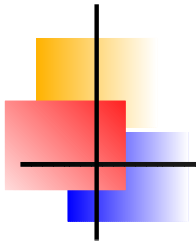
Literature Recommendations

1) Jitka Dupačová, Jan Hurt and Josef Štěpán, Stochastic Modeling in Economics and Finance, Kluwer Academic Publisher, 2002

2) Stavros A. Zenios, Lecture Notes: Mathematical modeling and its application in finance, http://www.hermes.ucy.ac.cy/zenios/teaching/399.001/index.html

3) F. Fabozzi and A. Konisbi, The Handbook of Asset/Liability Management: State-of-Art Investment Strategies, Risk Controls and Regulatory Required, Wiley.

4) Svetlozar Rachev and Stefan Mittnik, Stable Paretian Models in Finance, John Wiley & Sons Ltd., 2000



Overview

1) Introduction

2) Optimization and Risk

3) Optimization Problems, Stochastic Programing and Scenario Analysis

4) **Modeling of the Risk Factors**

5) ALM Implementation - a Pension Fund Example

3) Modeling the Risk Factors

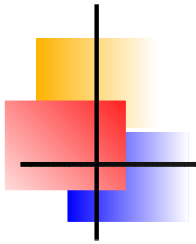
3.1) Definiton and Parameters of a Stable Distribution

3.2) Special Cases and Properties

3.3) Stable Modeling of Risk Factors

3.4) Multivariate Distributions

3.5) Dependence M odeling and Copulas



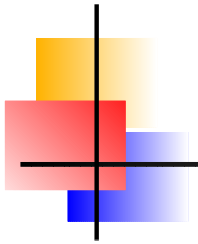
Modeling Risk Factors with Stable Distributions

Risk Factors and Financial Returns

An important task in ALM is the identification and adequate modeling of the underlying risk factors.
The dynamic of financial risk factors is well known to often exhibit some of the following phenomena:

- heavy tails
- skewness
- high-kurtotic residuals

The recognition and description of the latter phenomena goes back to the seminal papers of Mandelbrot (1963) and Fama (1965).



Stable Distributions

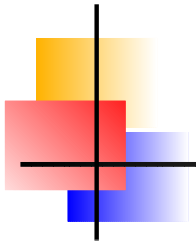
Definitions and Parameters

A stable distribution can be defined in four equivalent ways, given in the following definitions:

Definition 1. *A random variable X follows a stable distribution, if for any positive numbers A and B there exists a positive number C and a real number D such that*

$$AX_1 + BX_2 = CX + D \tag{1}$$

where X_1 and X_2 are independent copies of X and "=" denotes equality in distribution.



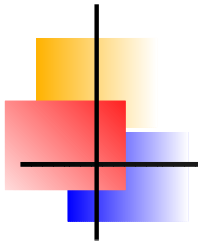
Stable Distributions

Definitions and Parameters

α is called the index of stability or characteristic exponent and for any stable random variable X, there is a number $\alpha \in (0, 2]$ such that the number C in 1 satisfies the following equation:

$$C^\alpha = A^\alpha + B^\alpha \tag{2}$$

A random variable X with index α is called α–variable. Obviously the Gaussian distribution has an index of stability of 2.



Stable Distributions

Definitions and Parameters

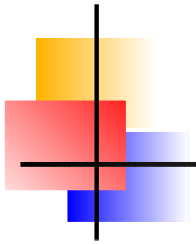
The next definition is equivalent to 1 and considers the sum of n independent copies of a stable random variable.

Definition 2. *A random variable X has a stable distribution if for any $n \geq 2$, there is a positive real number C_n and a real number D_n such that*

$$X_1 + X_2 + ... + X_n \stackrel{d}{=} C_n X + D_n \tag{3}$$

where $X_1, X_2, ..., X_n$ are independent copies of X.

Again, the number C_n and the stability index of the distribution are closely linked and we get $C_n = n^{1/\alpha}$ where the $\alpha \in (0, 2]$ is the same as in equation 2.



Stable Distributions

Definitions and Parameters

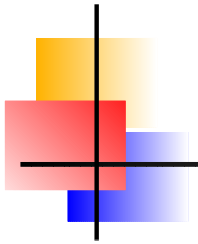
The third definition of a stable distribution is a generalisation of the central limit theorem:

Definition 3. *A random variable X is said to be stable if it has a domain of attraction, i.e., if there is a sequence of random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{d_n\}$ and real numbers $\{c_n\}$, such that*

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} \Rightarrow^d X. \tag{4}$$

The notation $\Rightarrow^d$ denotes convergence in distribution.

Definition 3 is obviously equivalent to definition 2, as one can take the $Y_i s$ to be independent and distributed like X. When $d_n = n^{1/\alpha}$, the $Y_i s$ are said to belong to the *normal* domain of attraction of X.



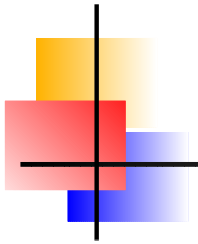
Stable Distributions

Definitions and Parameters

Finally, the last equivalent way to define a stable random variable provides information about its characteristic function.

Definition 4. *A random variable X has a stable distribution if there are parameters $0 < \alpha \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1$, and μ real such that its characteristic function has the following form:*

$$E(e^{iXt}) = \begin{cases} \exp(-\sigma^{\alpha}|t|^{\alpha}[1 - i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2}] + i\mu t), & \text{if } \alpha \neq 1, \\ \exp(-\sigma|t|[1 + i\beta \frac{2}{\pi} \text{sign}(t) \ln |t|] + i\mu t), & \text{if } \alpha = 1, \end{cases} \tag{5}$$



Stable Distributions

Definitions and Parameters

A stable distribution is defined by four parameters. The dependence of a stable random variable X from its parameters we will indicate by writing:

$$X \sim S_\alpha(\beta, \sigma, \mu)$$

where α is the the so-called index of stability $(0 < \alpha \leq 2)$:

- The lower the value of α the more leptocurtic is the distribution.
- The value of α for asset returns is often between 1 and 2.
- For $\alpha > 1$, the location parameter μ is the mean of the distribution.

Stable Distributions

Definitions and Parameters

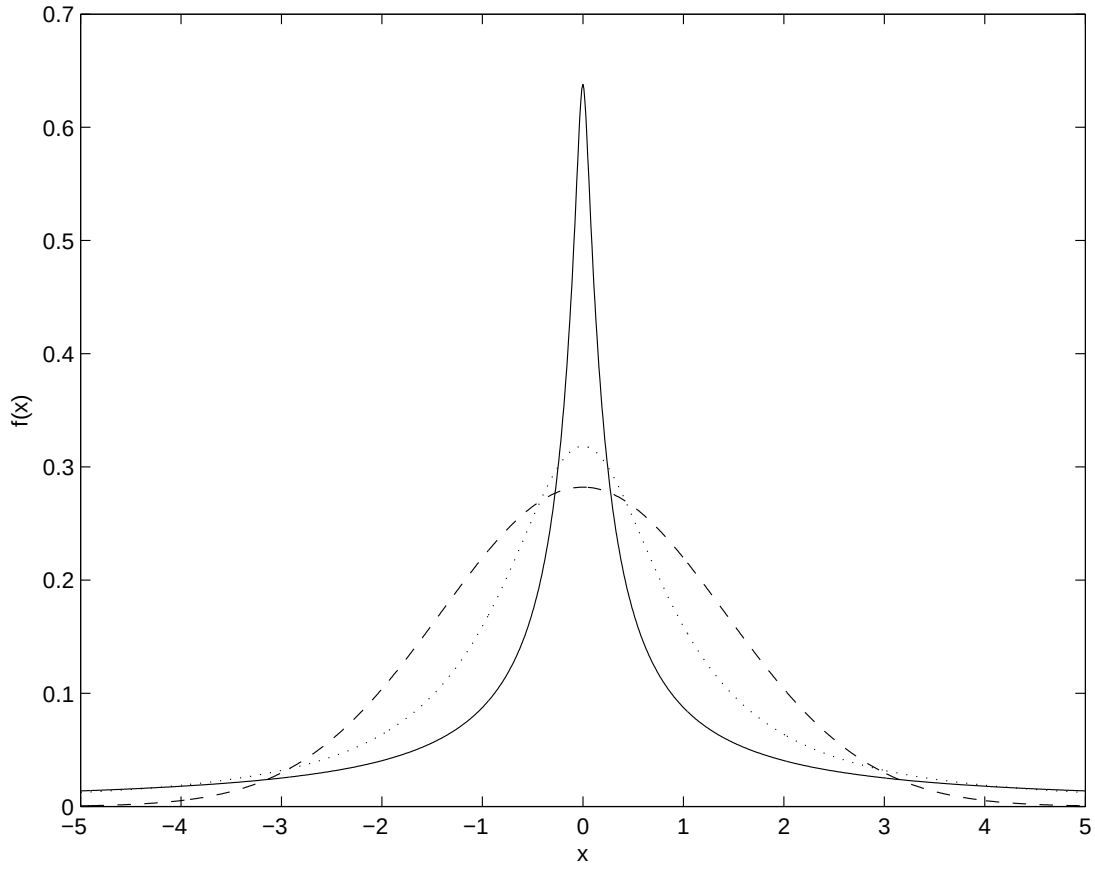
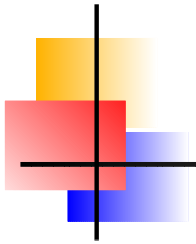


Figure 1: Probability density functions for standard symmetric α-stable random variables, $\alpha = 2$, $\alpha = 1$ (dotted) and $\alpha = 0.5$ (dashed).



Stable Distributions

Definitions and Parameters

The second parameter β is the skewness parameter $(-1 \leq \beta \leq 1)$.

- A stable distribution with $\beta = \mu = 0$ is called a symmetric α-stable distribution ($S\alpha S$).
- If $\beta < 0$, the distribution is skewed to the left.
- If $\beta > 0$, the distribution is skewed to the right.
- Obviously, the stable distribution can also capture asymmetric asset returns.

Stable Distributions

Definitions and Parameters

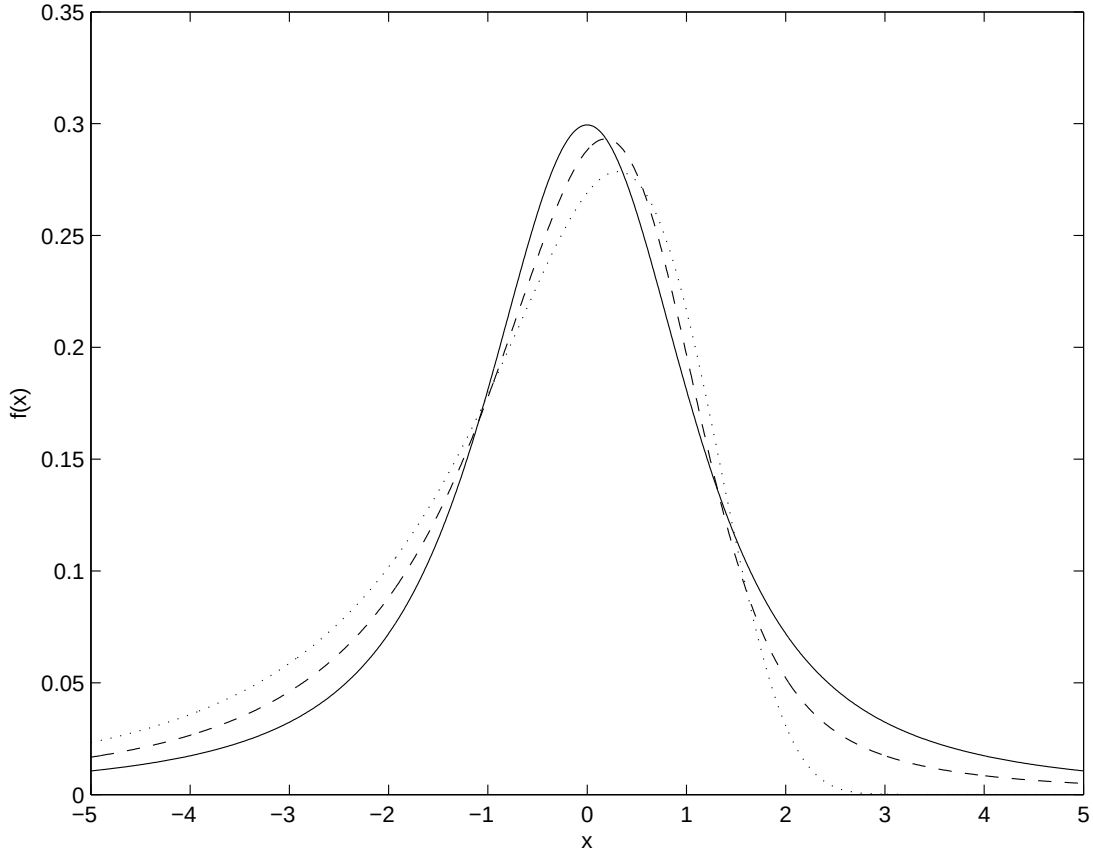
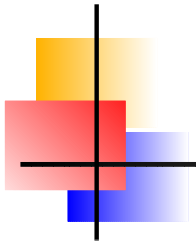


Figure 2: Probability density functions for stable random variables with $\alpha = 1.2$, β varying, $\beta = 0$, $\beta = -0.5$ (dashed) and $\beta = -1$ (dotted).

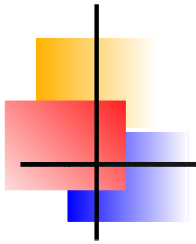


Stable Distributions

Special Cases

Generally the probability density function of a stable distribution cannot be specified in explicit form. However, there are three special cases:

- The *Gaussian distribution*
  If the index of stability $\alpha = 2$, then the stable distribution reduces to the Normal distribution, and it is $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$.
  - When $\alpha = 2$, the characteristic function becomes $Ee^{itX} = e^{-\sigma^2 t^2 + i\mu t}$.
  - This is the characteristic function of a Gaussian random variable with mean μ and variance $2\sigma^2$.



Stable Distributions

Special Cases

- The *Cauchy distribution*
  $S_1(\sigma, 0, \mu)$, whose density $f_1(x)$ is

$$f_1(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)} \tag{6}$$

  If $X \sim S_1(\sigma, 0, 0)$, then for $x > 0$,

$$P(X \leq x) = 0.5 + \frac{1}{\pi} \arctan\left(\frac{x}{\sigma}\right). \tag{7}$$

- The *Lévy distribution*
  $S_{1/2}(\sigma, 1, \mu)$, whose density

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\} \tag{8}$$

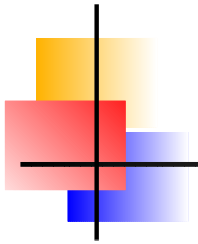
  is concentrated on (μ, ∞)

Properties

The first property mentioned is the so-called **summation stability**.

Proposition 1. *Let X_1, X_2 be independent random variables with $X_i \sim (\sigma_i, \beta_i, \mu_i)$, $i = 1, 2$. Then $X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu)$, with*

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \ \beta = \frac{\beta_1 \sigma_1^\alpha + \beta_2 \sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \ \mu = \mu_1 + \mu_2. \tag{9}$$



Stable Distributions

Properties

The second proposition concerns the parameter σ. The Gaussian distribution can be scaled by multiplication with a constant. This property extends to $0 < \alpha \leq 2$.

Proposition 2. *Let* $X \sim S_\alpha(\sigma, \beta, \mu)$ *and let* $a \in \mathbb{R}\backslash\{0\}$. *Then*

$$aX \sim S_\alpha(|a|\sigma, \arg(a)\beta, a\mu) \qquad if\ \alpha \neq 1$$

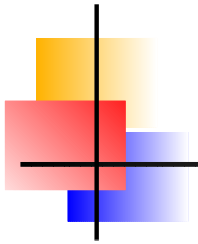
$$aX \sim S_\alpha(|a|\sigma, \arg(a)\beta, a\mu - \frac{2}{\pi}a(\ln|a|\sigma\beta) \qquad if\ \alpha = 1$$

Properties

The third proposition concerns the shift parameter μ. It was already discussed that in the case of $\alpha = 2$ the parameter μ is a shift parameter for the Gaussian distribution. The same can be inferred about μ for any admissible α:

Proposition 3. *Let* $X \sim S_\alpha(\sigma, \beta, \mu)$ *and let* a *be real constant. Then* $X + a \sim S_\alpha(\sigma, \beta, \mu + a)$.

For $1 < \alpha \leq 2$, the shift parameter μ equals the mean.



Stable Distributions

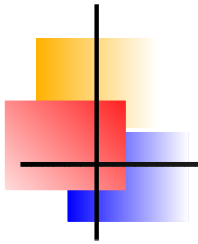
Properties

Finally, we can also interpret the last parameter β. It can be identified as a *skewness parameter*.

Proposition 4. *$X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric if and only if $\beta = 0$ and $\mu = 0$. It is symmetric about μ if and only if $\beta = 0$.*

In order to indicate that X is symmetric, i.e. $\beta = 0$ and $\mu = 0$, we write

$$X \sim S\alpha S$$



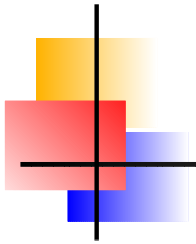
Stable Distributions

Truncated Stable Distributions

Major reasons for the so far limited use of stable distributions in applied work are

- In general there are no closed-form expressions for its probability density function.
- Numerical approximations are nontrivial and computationally demanding.
- All moments of order $\geq \alpha$ are infinite.

Following Menn and Rachev (2004) we will give a brief introduction to a new class of probability distributions that combines the modeling flexibility of stable distributions with the existence of arbitrary moments.



Stable Distributions

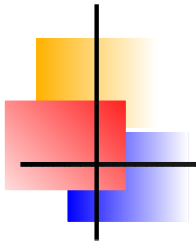
Truncated Stable Distributions

A possibility to guarantee the existence of moments of order $\geq \alpha$ is to truncate the stable distribution at certain limits and add two normally distributed tails to the distribution.

Dependent on where the truncation is conducted the distribution can still be clearly more heavy-tailed than a normal distribution but may provide finite variance.

This idea leads to the definition of a so-called smoothly truncated stable distribution.

See the lecture notes and examples in the lecture for further discussion.



Modeling of Risk Factors

Normal and Stable Distributions

We will provide some examples on the superior fit of stable distributions to financial returns compared to the Gaussian distribution that is used in most standard models of financial theory.

Various applications of stable models in finance can be found in Rachev and Mittnik (1999).

Note that due to the summation stability the sum of stable distributed random variables with identical parameter α are again alpha-stable distributed with α.

Another advantage is the number of parameters: with four parameters the distribution provides more flexibility and is capable to explain issues of financial data like skewness, excess kurtosis or heavy tails.

Modeling of Risk Factors

Normal and Stable Distributions

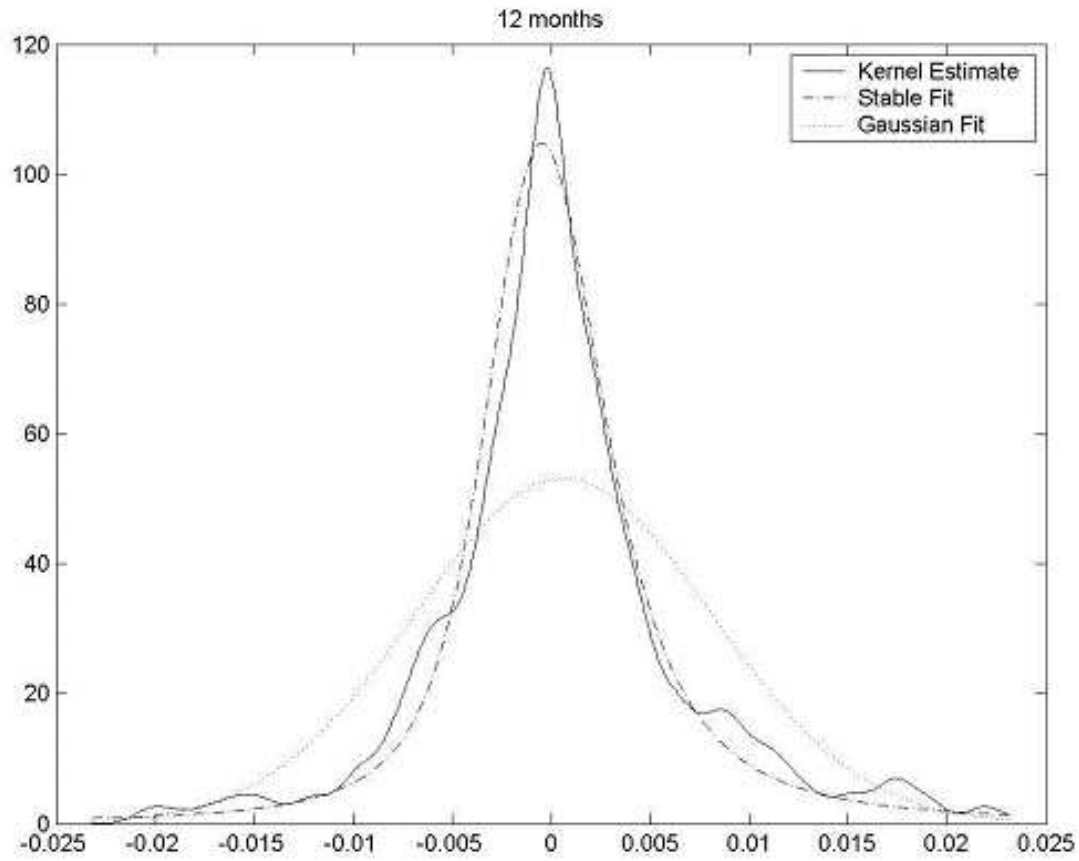


Figure 3: Fit of Gaussian and Stable distribution to 1 year Euribor rate

Normal and Stable Distributions

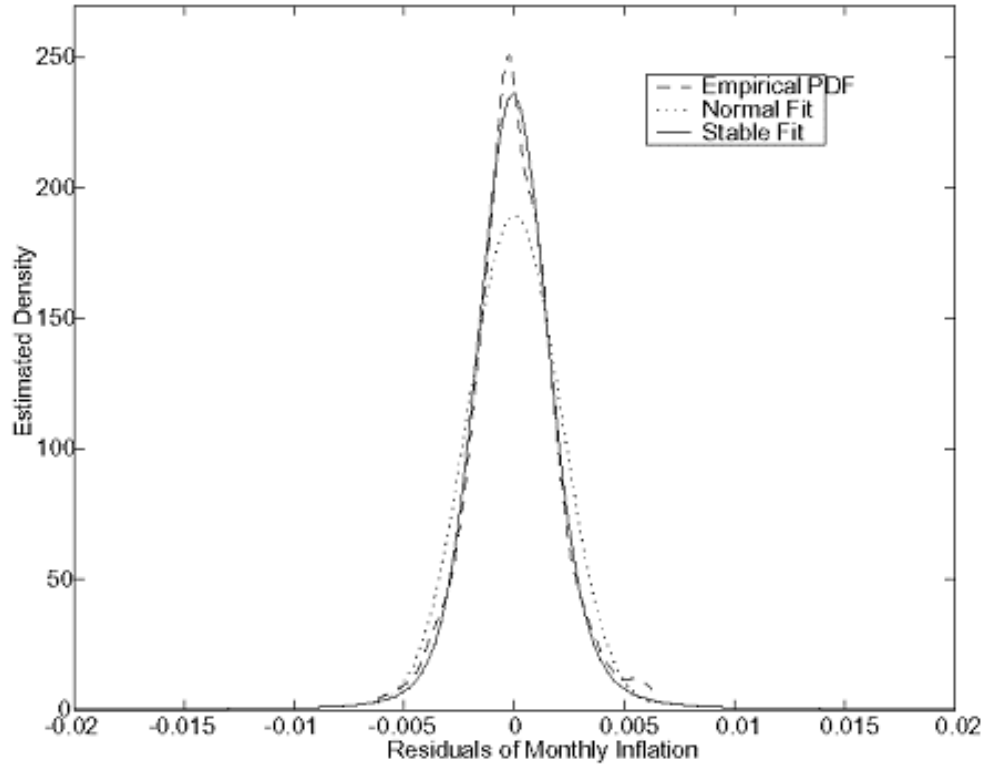
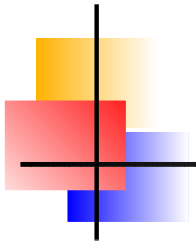


Figure 4: Fit of Gaussian and Stable distribution to residuals of monthly inflation



Modeling of Risk Factors

Normal and Stable Distributions

Distribution	Stable				Gaussian	
Parameters	alpha	beta	sigma	mu	μ	σ
Unemployment Rate	1.6691	-1.0000	0.0124	0.0316	0.0337	0.0207
Working Output	1.4474	1.0000	0.0723	0.0234	-0.0074	0.1454
Gross Domestic Product	1.6325	-1.0000	0.0830	-0.0493	-0.0495	0.1986
Consumer Price Index	1.2061	0.0880	0.3130	0.0385	0.0194	0.8058
Annual Saving	1.2849	1.0000	0.0123	0.0563	0.0433	0.0283
Personal Income	2.0000	0.1427	0.0147	0.0744	0.0744	0.0210

Table 1: Parameters of α-stable and Gaussian fit to log-returns of several US macroeconomic time series 1960-2000.

Modeling of Risk Factors

Normal and Stable Distributions

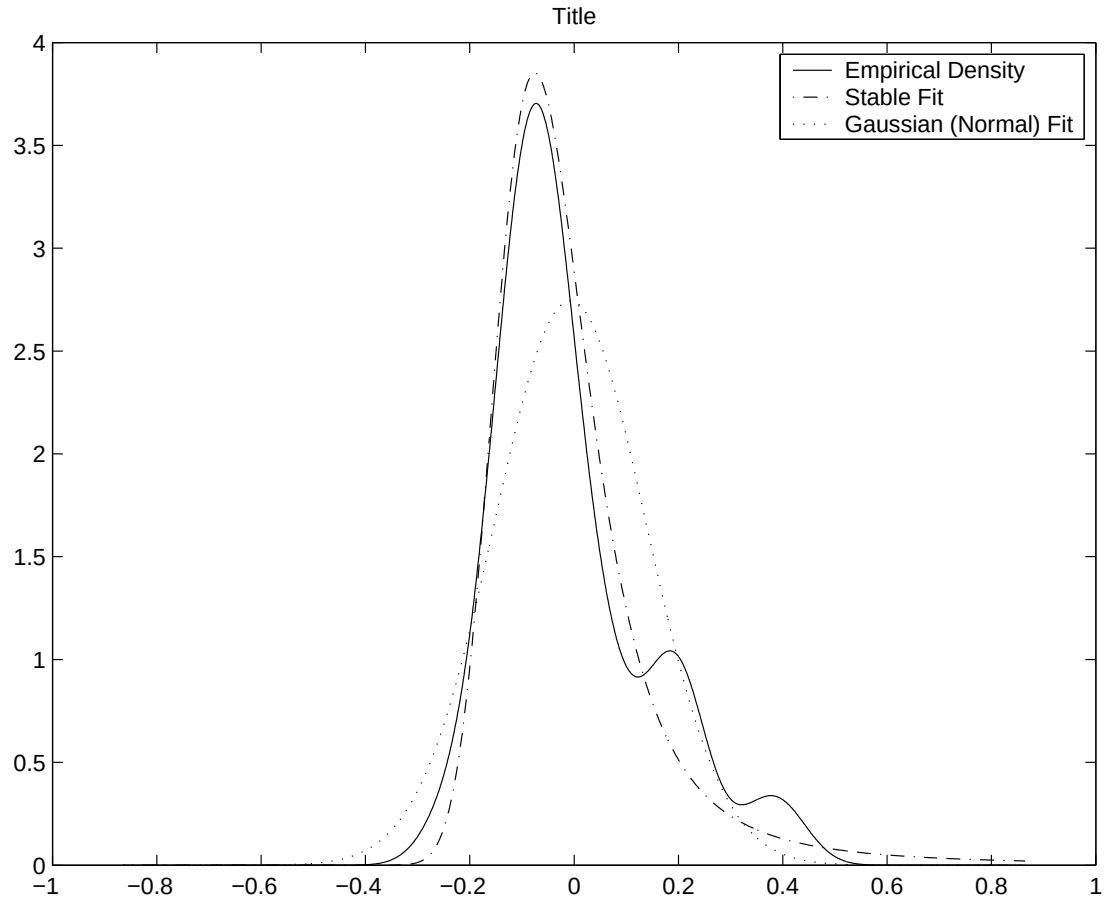


Figure 5: Normal and Stable fit to log return of Working Output per hour.

Modeling of Risk Factors

Normal and Stable Distributions

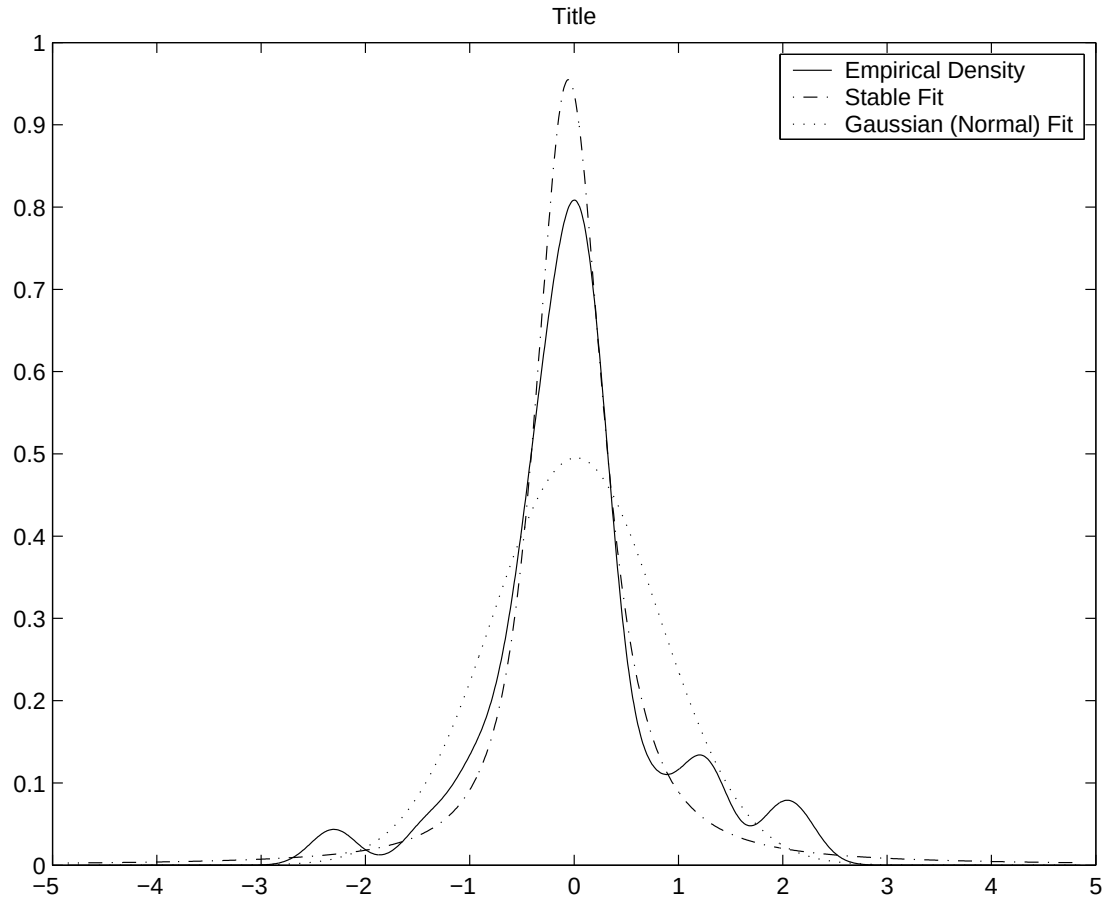


Figure 6: Normal and Stable fit to log return of GDP.

Evaluating the Fit

We also provide the goodness-of-fit measure Kolmogorov distance (KS) that measures the distance between the empirical cumulative distribution function $F_n(x)$ and the fitted CDF $F(x)$

$$KS = \max_{x \in \mathbb{R}} |F_n(x) - F(x)|. \tag{10}$$

We also considered the Anderson-Darling statistic (AD)

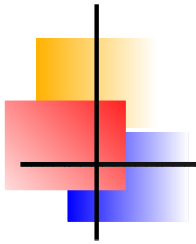
$$AD = \max_{x \in \mathbb{R}} \frac{|F_n(x) - F(x)|}{\sqrt{F(x)(1 - F(x))}} \tag{11}$$

that puts more weight to the tails of the distribution.

Evaluating the Fit

Distribution	Stable		Gaussian	
Parameters	KS	AD	KS	AD
Unemployment Rate	**0.0843**	0.5065	0.1333	**0.4077**
Working Output	**0.0965**	**0.2869**	0.1791	0.3389
Gross Domestic Product	**0.0804**	**0.5448**	0.1804	6.1008
Consumer Price Index	**0.0723**	**0.2646**	0.1833	0.5044
Annual Saving	**0.0777**	**0.1738**	0.1635	0.3521
Personal Income	0.1073	**0.1864**	**0.0783**	0.1920

Table 2: Goodness-of-Fit criteria Kolmogorov distance (KS) and Anderson-Darling statistic (AD) for Stable and Normal Distribution.



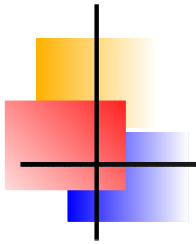
Dependence between Risk Factors

Dealing with Multivariate Distributions

Often scenarios are generated by calibrating and simulating a time-series model to a **multivariate data set**.

There are two major approaches modeling multivariate data:

- Fit a multivariate distribution.
- Fit each individual time-series with a univariate distribution and use a copula to describe the dependence structure.



Dependence between Risk Factors

Dealing with Multivariate Data Sets

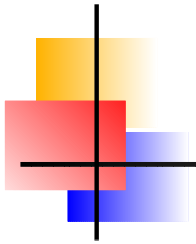
- the second approach is more flexible in the sense that it allows any type of distribution to be fit to the individual series.
- but a major problem may be the choice of the right copula.
- the first approach may be easier to implement or estimate.

Fitting Multivariate Distributions

Use a liability index l_τ and asset indices s_τ^i, $i = 1, ..., k$ to construct a multivariate scenario tree.

This is achieved by fitting a multivariate time-series model to the return vector:

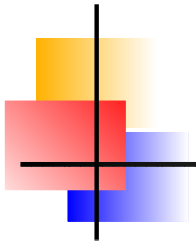
$$R_\tau = \begin{bmatrix} r_\tau^1 \\ r_\tau^2 \\ \vdots \\ r_\tau^{k-1} \end{bmatrix} = \begin{bmatrix} l_\tau / l_{\tau-1} - 1 \\ s_\tau^1 / s_{\tau-1}^1 - 1 \\ \vdots \\ s_\tau^k / s_{\tau-1}^k - 1 \end{bmatrix}. \qquad (12)$$



Multivariate Distributions

Fitting Multivariate Distributions

- Note that τ is interpreted as time, and in the previous sections, t was interpreted as the stage in a stochastic program.
- It is possible that they will coincide; however, there will usually be many smaller time periods between stages.
- For example, a time-series model may be fitted to daily or monthly data, while a stage covers a longer period like 6-months etc.



Multivariate Distributions

Fitting Multivariate Distributions

Once a time-series model is found, it is simple to

- generate sample paths for the returns and
- then convert the returns back to index values.

Vector Autoregressive Model

In terms of the multivariate approach one might calibrate a vector autoregressive (VAR) model to the data.

The general VAR(p) model for financial return data $\tilde{R}_\tau$ is

$$\tilde{R}_\tau = \Pi_1 \tilde{R}_{\tau-1} + ... + \Pi_p \tilde{R}_{\tau-p} + E_\tau, \tag{13}$$

where the innovations process $E_\tau = (e_\tau^1, ..., e_\tau^6)'$ is assumed to be white noise with covariance matrix Σ.

Vector Autoregressive Model

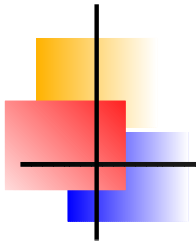
One of the advantages of VAR models is that they are easy to calibrate and it is easy to simulate scenarios from such models.

For simulation, distributional assumptions for the innovations are needed.

After estimation of the VAR(1) model, the residuals are computed by

$$\hat{E}_\tau = \tilde{R}_\tau - \hat{\Pi}_1 \tilde{R}_{\tau-1}, \tag{14}$$

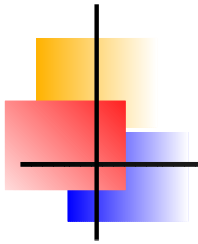
and the standardized residuals $\hat{\Sigma}^{-1/2}\hat{E}_\tau$ are plotted.



Multivariate Distributions

Vector Autoregressive Model - Residuals

- The usual assumption is that the innovations are Gaussian.
- In this case the standardized residuals should be i.i.d. multivariate Normal(0,I_n).
- However, based on the results on financial return data of the previous sections, it might also be promising to use a more flexible or heavy-tailed distribution like the α-stable or the truncated stable distribution.



Multivariate Distributions

The Multivariate Stable Model

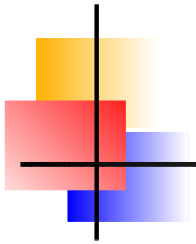
A n-dimensional random vector Z has a multivariate stable distribution if for any $a > 0$ and $b > 0$ there exists $c > 0$ and $d \in \mathbb{R}^n$ such that

$$aZ_1 + bZ_2 \stackrel{d}{=} cZ + d,$$

where Z_1 and Z_2 are independent copies of Z and $a^\alpha + b^\alpha = c^\alpha$. The characteristic function of R is given by

$$\Phi_Z(\theta) = \begin{cases} \exp\left\{-\int_{S_n} |\theta' s| \left(1 - i\text{sign}(\theta' s) \tan\frac{\pi\alpha}{2}\right) \Gamma_Z(ds) + i\theta'\mu\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\int_{S_n} |\theta' s| \left(1 + i\frac{2}{\pi}\text{sign}(\theta' s) \ln|\theta' s|\right) \Gamma_Z(ds) + i\theta'\mu\right\}, & \text{if } \alpha = 1, \end{cases}$$

where θ and μ are n-dimensional vectors.



Multivariate Distributions

The Multivariate Stable Model

- The spectral measure Γ_Z is a finite measure on the sphere in $\mathbb{R}^n$ that replaces the roles of β and σ in stable random variables.
- Again, α and μ are the index of stability and location parameter, respectively.
- A symmetric stable random vector with $\mu = 0$ is called symmetric alpha-stable ($S\alpha S$).

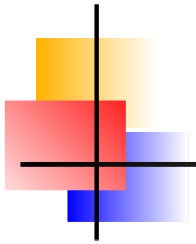
The Multivariate Stable Model

For a symmetric stable random vector ($S\alpha S$) the stable equivalent of covariance is the covariation:

$$\left[\tilde{z}^1, \tilde{z}^2\right]_\alpha = \int_{S_2} s_1 s_2^{\langle \alpha-1 \rangle} \Gamma_{(\tilde{z}^1, \tilde{z}^2)}(ds),$$

where $(\tilde{z}^1, \tilde{z}^2)$ is a $S\alpha S$ vector with spectral measure $\Gamma_{(\tilde{z}^1, \tilde{z}^2)}$ and $y^{\langle k \rangle} = |y|^k \text{sign}(x)$. Additionally, the covariation norm is given by

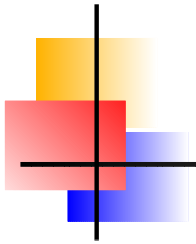
$$\|\tilde{z}^i\|_\alpha = \left(\left[\tilde{z}^i, \tilde{z}^i\right]_\alpha\right)^{1/\alpha}.$$



Correlation and general multivariate distributions

Under general multivariate distributions, correlation generally gives no indication about the degree or structure of dependence:

1. Correlation is simply a scalar measure of dependency; it cannot tell us everything we would like to know about the dependence structure of risks.
2. Possible values of correlation depend on the marginal distribution of the risks. All values between -1 and 1 are not necessarily attainable.
3. Perfectly positively dependent risks do not necessarily have a correlation of 1; perfectly negatively dependent risks do not necessarily have a correlation of -1.



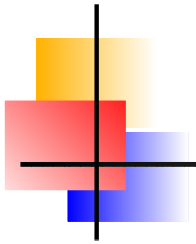
Dependence Modeling and Copulas

Correlation and general multivariate distributions

4) A correlation of zero does not indicate independence of risks.

5) Correlation is not invariant under transformations of the risks. For example, $\log(X)$ and $\log(Y)$ generally do not have the same correlation as X and Y .

6) Correlation is only defined when the variances of the risks are finite. It is not an appropriate dependence measure for very heavy-tailed risks where variances appear infinite.



Dependence Modeling and Copulas

Correlation and general multivariate distributions

For an illustration, consider the following example:

- Assume two rv's. X and Y that are lognormally distributed with $\mu_X = \mu_Y = 0$, $\sigma_X = 1$ and $\sigma_Y = 2$.
- It is possible to show that by an arbitrary specification of the joint distribution with the given marginals, it is not possible to attain any correlation in $[-1, 1]$.
- The boundaries for a maximal and a minimal attainable correlation $[\rho_{min}, \rho_{max}]$ in the given case are $[-0.090, 0.666]$.
- Allowing σ_Y to increase, the interval becomes arbitrarily small.

Correlation and general multivariate distributions

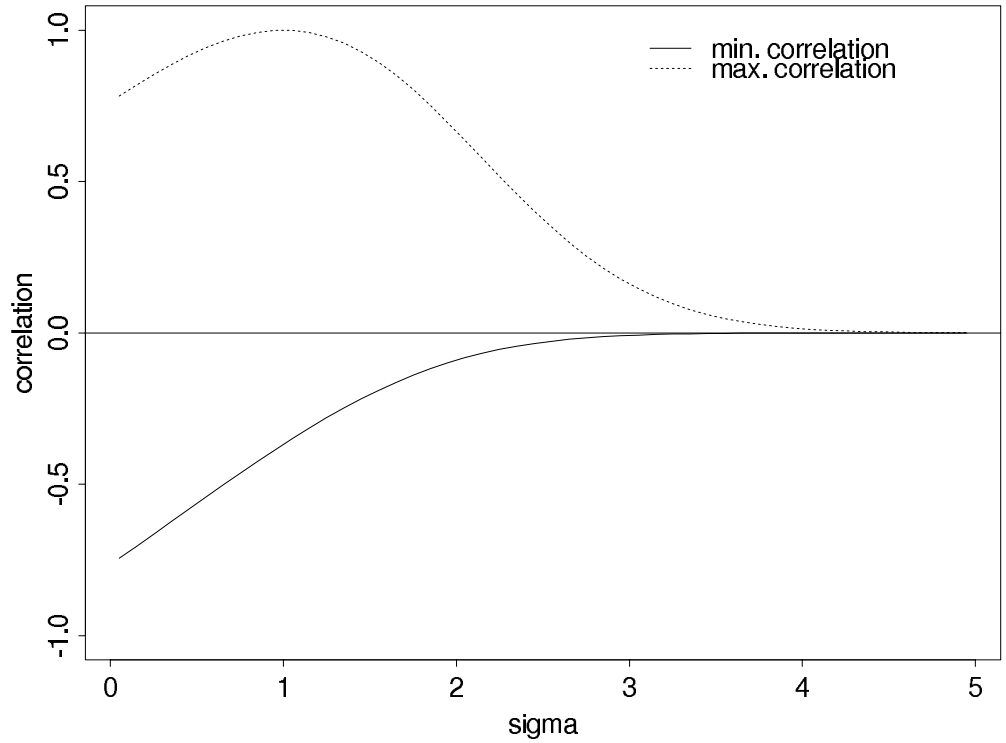
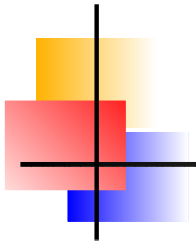


Figure 7: Maximum and minimum attainable correlation for $X \sim \text{Lognormal}(0, 1)$ and $Y \sim \text{Lognormal}(0, \text{sigma})$.



Correlation and general multivariate distributions

We conclude:

- The two boundaries represent the case where the two rv's are perfectly positive dependent (the max. correlation line) or perfectly negative dependent (the min. correlation line) respectively.
- Although the attainable interval for ρ as $\sigma_Y > 1$ converges to zero from both sides, the dependence between X and Y is by no means weak.
- This indicates that it is wrong to interpret small correlation as weak dependence.
- We have to consider alternative measures of dependence like copulas

Definition

Let $X = (X_1, \ldots, X_n)'$ be a random vector of real-valued rv's whose dependence structure is completely described by the joint distribution function

$$F(x_1, \ldots, x_n) = P(X_1 < x_1, \ldots, X_n < x_n). \tag{15}$$

Each rv X_i has a marginal distribution of F_i that is assumed to be continuous for simplicity.

Recall that the transformation of a continuous rv X with its own distribution function F results in a rv $F(X)$ which is standardly uniformly distributed.

Definition

Transforming equation (15) component-wise yields

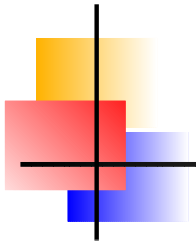
$$\begin{aligned} F(x_1,\ldots,x_n) &= P(X_1 < x_1,\ldots,X_n < x_n) \\ &= P[F_1(X_1) < F_1(x_1),\ldots,F_n(X_n) < F_n(x_n)] \\ &= C(F_1(x_1),\ldots,F_n(x_n)), \end{aligned} \tag{16}$$

where the function C can be identified as a joint distribution function with standard uniform marginals — the copula of the random vector X.
In equation (16), it can be clearly seen, how the copula combines the magrinals to the joint distribution.

Sklar's Theorem

Sklar's theorem provides a theoretic foundation for the copula concept:

Theorem 1. *Let F be a joint distribution function with continuous margins $F_1, \ldots, F_n$. Then there exists a unique copula $C : [0,1]^n \to [0,1]$ such that for all $x_1, \ldots, x_n$ in $\mathbb{R} = [-\infty, \infty]$ (16) holds. Conversely, if C is a copula and $F_1, \ldots, F_n$ are distribution functions, then the function F given by (16) is a joint distribution function with margins $F_1, \ldots, F_n$.*



Dependence Modeling and Copulas

Examples: The case of independent random variables

If the rv's X_i are independent, then the copula is just the product over the F_i:

$$C^{ind}(x_1, \ldots, x_n) = x_1 \cdot \cdots \cdot x_n.$$

Example: The Gaussian Copula

The *Gaussian* copula is defined by:

$$C_\rho^{Ga}(x,y) = \int_{-\infty}^{\Phi^{-1}(x)} \int_{-\infty}^{\Phi^{-1}(y)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp \frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\, dsdt,$$

where $\rho \in (-1,1)$ and $\Phi^{-1}(\alpha) = \inf\{ x \mid \Phi(x) \geq \alpha\}$ is the univariate inverse standard normal distribution function.

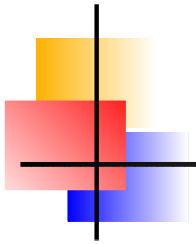
Applying C_ρ^{Ga} to two univariate standard normally distributed rv's results in a standard bivariate normal distribution with correlation coefficient ρ.

Example: The Gumbel Copula

The *Gumbel* or *logistic* copula is defined by:

$$C_{\beta}^{Gu}(x,y) = \exp\left[-\left\{(-\log x)^{\frac{1}{\beta}} + (-\log y)^{\frac{1}{\beta}}\right\}^{\beta}\right],$$

where $\beta \in (0,1]$ indicates the dependence between X and Y. $\beta = 1$ gives independence and $\beta \to 0^{+}$ leads to perfect dependence.



Dependence Modeling and Copulas

Properies: Dependence Structure and Uniqueness

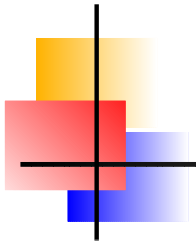
- According to theorem (1), a multivariate distribution is fully determined by its marginal distributions and a copula.
- The copula contains all information about the dependence structure between the associated random variables.
- In the case that all marginal distributions are continuous, the copula is unique and therefore often referred to as *the dependence structure* for the given combination of multivariate and marginal distribution.
- If the copula is not unique because at least one of the marginal distributions is not continous, it can still be called a *possible representation of the dependence structure*.

Properties: Invariance

A very useful feature of a copula is the fact that it is invariant under increasing and continuous transformation of the marginals:

Lemma 2. *If* $(X_1, \ldots, X_n)^t$ *has copula* C *and* $T_1, \ldots, T_n$ *are increasing continuous functions, then* $(T_1(X_1), \ldots, T_n(X_n))^t$ *also has copula* C*.*

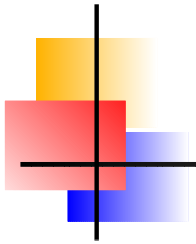
One application of lemma (2) would be that the transition from the representation of a random variable to its logarithmic representation does not change the copula. Note that the linear correlation coefficient does not have this property. It is only invariant under *linear* transformation.



Multivariate Distributions: Marginals and Copulas

Construct multivariate distribution by

- assuming some marginal distributions
- apply copula for dependence structure



Dependence Modeling and Copulas

Multivariate Distributions: Marginals and Copulas

Construct multivariate distribution by

- assuming some marginal distributions
- apply copula for dependence structure

Note: in practical approach, problem will be set up the other way round:

The multivariate distribution has to be estimated by fitting a copula to data.

Dependence Modeling and Copulas

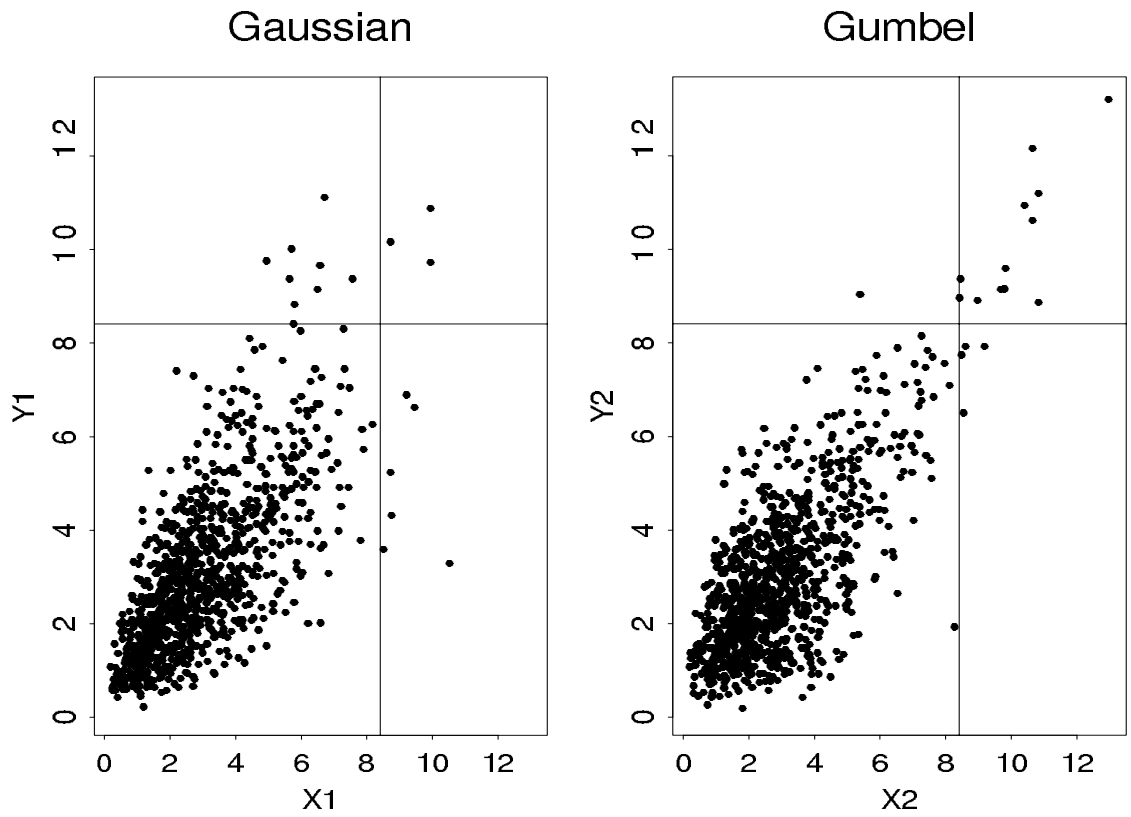


Figure 8: 1000 draws from two distributions that were constructed using Gamma(3,1) marginals and two different copulas with linear correlation of $\rho = 0.7$.

Remarks

- Despite the fact that both distributions have the same linear correlation coefficient, the dependence between X and Y is obviously quite different.
- Using the Gumbel copula, extreme events have a tendency to occur together.
- Comparing the number of draws where x and y exceed $q_{0.99}$ simultaneously: 12 for Gumbel and 3 for Gaussian.
- The probability of Y exceeding $q_{0.99}$ given that X has exceeded $q_{0.99}$ can be roughly estimated from the figure:

$$\hat{P}_{Gaussian}(X > q_{0.99}|Y > q_{0.99}) = \frac{3}{9} = 0.\bar{3}$$

$$\hat{P}_{Gumbel}(X > q_{0.99}|Y > q_{0.99}) = \frac{12}{16} = 0.75$$