Construction of probability metrics on classes of investors

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Abstract

We introduce a probability semimetric defined on a class of investors. Our approach is consistent with Cumulative Prospect Theory. We provide a sufficient condition for the set of investors' value functions which guarantees that the semimetric turns into a metric. Furthermore, we consider a quasi-semimetric which is consistent with first-order stochastic dominance and we introduce the class of investors with balanced views.

Key words: probability metrics, cumulative prospect theory

JEL Classification: C44, D81, G11

Introduction

Expected utility theory (EUT) is an accepted model describing choice under uncertainty. However, a number of alternative behavioral models for human choice have been proposed. One of them is Cumulative Prospect Theory (CPT) as proposed by Tversky and Kahneman (1992). CPT is built upon the following main observations. First, investors usually think about possible outcomes relative to a certain reference point rather than the final outcome. This is referred to as the framing effect by behavioral finance theorists. Second, investors have different attitude towards gains (outcomes which are larger than the reference point) and losses (outcomes which are less than the reference point), referred to by behavioral finance theorists as loss aversion. Finally, investors tend to overweight extreme events and underweight events with higher probability.

CPT has been applied to a diverse range of problems such as the asset allocation puzzle, the status quo bias, and various gambling and betting puzzles which appear inconsistent with standard economic rationality, see for example Benartzi and Thaler (1995) and Kahneman et al. (1991). However, the applications are in a simple discrete setting and only few attempts have been made to apply the theory in a more complicated setting, see for example Baucells and Heukamp (2006) and Hwang and Satchell (2003).

CPT arises as an alternative theory to EUT on the basis of the observations outlined above in which the utility function is replaced by a value function and the cumulative probabilities are replaced by weighted cumulative probabilities. The value function, $v(x)$, assigns values to the possible outcomes. It is non-decreasing and $v(0) = 0$ since the outcome equal to the reference point brings no value to the individual. Different functional forms for $v(x)$ have been suggested. It is often assumed that the $v(x)$ has an s-shaped form, i.e. $v(x)$ is convex for $x < 0$ and it is concave for $x > 0$, see Daniel and Tversky (1979). An illustration is provided in Figure 1. From a financial viewpoint, the framing effect means that the value function can be constructed for returns rather than wealth as in the classical EUT.

The weighted cumulative probabilities are usually modeled as a transformations of the cumulative distribution function (c.d.f.) of the prospect $F_X(x) = P(X \leq x)$ and the tail $1 - F_X(x) = P(X > x)$ depending on whether $x < 0$ or $x > 0$, respectively. The transformation for the losses is denoted by $w^-(p)$ and the one for the profits is denoted by $w^+(p)$. Both weighting functions are non-decreasing and satisfy the following conditions

$$w^-(0) = w^+(0) = 0$$
$$w^-(1) = w^+(1) = 1.$$

Empirical studies suggest that the general shape of the weighting functions is inverse s-shaped, see for example Tversky and Kahneman (1992).

According to CPT, individuals make a choice between two risky prospects X and Y by computing the subjective expected values according to the functional

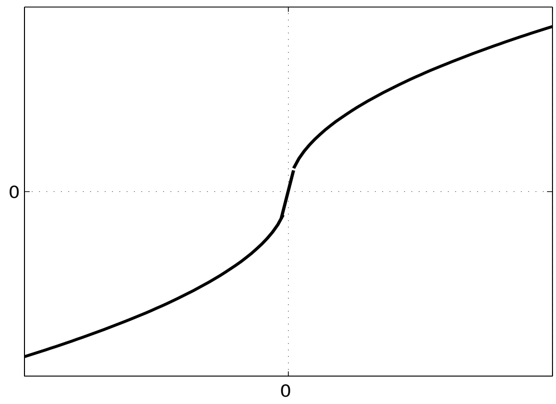

Figure 1: The graph of an s-shaped function.

$$V(X) = \int_{-\infty}^{0} v(x)d[w^{-}(F_X(x))] + \int_{0}^{\infty} v(x)d[-w^{+}(1 - F_X(x))] \qquad (1)$$

and then compare $V(X)$ and $V(Y)$, see for example Baucells and Heukamp (2006). If $V(X) \geq V(Y)$, then Y is not preferred to X. If $V(X) = V(Y)$, then the individual is indifferent. Note that if the individuals do not weight the cumulative probabilities, i.e. $w^{-}(p) = w^{-}(p) = p$, then the definition of $V(X)$ reduces to

$$V(X) = Ev(X) = \int_{-\infty}^{\infty} v(x)dF_X(x).$$

The expression (1) implies that we can map all individuals to pairs (v, w) where v is the corresponding value function and w is a shorthand for both w^{-} and w^{+}. Consider a set of individuals represented by $(v_j, w_j),\ j \in J$ and two prospects X and Y.

The range of questions that we discuss in this paper is whether it is possible draw a conclusion about the dissimilarity between the two random variables X and Y by looking at how the individuals value the two prospects. Suppose that all individuals are indifferent between the two prospects, i.e. $V_j(X) = V_j(Y),\ \forall j \in \mathcal{J}$. This would be the case if $F_X(x) = F_Y(x),\ \forall x \in \mathbb{R}$. The converse does not necessarily follow because it depends on how many and diverse the individuals are. As an extreme example, if there is only one investor then $V_1(X) = V_1(Y)$ implies equality of certain characteristics of X and Y but not the entire c.d.f.s. In this paper, we provide sufficient conditions for v_j that, if satisfied, guarantee coincidence of the corresponding c.d.f.s on condition that all individuals are indifferent between X and Y. We study this problem by defining a probability semimetric on the set of investors. We also extend the discussion to a probability quasi-semimetric which is consistent with first-order stochastic dominance order and we introduce the class of investors with balanced views.

1 Metrics construction

We begin introducing some notation and then we proceed to the semimetrics construction. The class of bounded s-shaped value functions we denote by $\mathcal{S}$. The elements of $\mathcal{S}$ are bounded real-valued functions $v(x) : \mathbb{R} \to \mathbb{R}$ with the following property,

$$v(x) = \begin{cases} v^-(x), & x < 0 \\ 0, & x = 0 \\ v^+(x), & x > 0 \end{cases}$$

where $v^-(x) < 0$ is a monotonically increasing convex function and $v^+(x) > 0$ is a monotonically increasing concave function.

Suppose that all investors which we consider are indifferent between X and Y, $V_j(X) = V_j(Y)$, for all $j \in \mathcal{J}$. Note that $\mathcal{J}$ is a general set, not necessarily a countable. In order to study the implications of this assumption on the distribution functions of X and Y, we consider the functional,

$$\zeta_{\mathcal{J}}(X,Y) = \sup_{j \in \mathcal{J}} |V_j(X) - V_j(Y)| \tag{2}$$

where

$$\begin{aligned} V_j(X) - V_j(Y) = & \int_{-\infty}^{0} v_j(x) d[w_j^-(F_X(x)) - w_j^-(F_Y(x))] \\ & + \int_0^{\infty} v_j(x) d[w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))]. \end{aligned}$$

Note that in the case of no subjective weighting, the expression above reduces to

$$V_j(X) - V_j(Y) = \int_{-\infty}^{\infty} v_j(x) d(F_X(x) - F_Y(x)).$$

The functional $\zeta_{\mathcal{J}}(X,Y)$ can be straightforwardly interpreted as the largest difference between the values assigned by the investors to X and Y running through all investors. If the functional in (2) equals zero, then this means that all investors that we consider are indifferent between X and Y. In fact, $\zeta_{\mathcal{J}}(X, Y)$ has metric properties, see Rachev (1991). More precisely, it is a semimetric: if $X \stackrel{d}{=} Y$, then $\zeta_{\mathcal{J}}(X, Y) = 0$ but the converse may not hold. Therefore, the fact that $\zeta_{\mathcal{J}}(X, Y) = 0$ does not necessarily imply equality in distribution between X and Y. It depends on how rich the set of value functions $\mathcal{J}$ is. The next theorem establishes a sufficient condition on the class of investors in order for $\zeta_{\mathcal{J}}(X, Y)$ to be a simple metric.

Theorem 1. *Suppose that the class of value functions* $\mathcal{V}_{\mathcal{J}} = \{v_j, j \in \mathcal{J}\} \subseteq \mathcal{S}$ *contains the functions*

$$v^-_{x_0,n}(x) = \begin{cases} -1/n, & x < x_0 \\ x_0 - x - 1/n, & x \in [x_0, x_0 + 1/n) \\ 0, & x \geq x_0 + 1/n \end{cases} \tag{3}$$

where $n = 1, 2, ...$ *and* $x_0 + 1/n \leq 0$ *and*

$$v^+_{x_0,n}(x) = \begin{cases} 0, & x < x_0 - 1/n \\ x - x_0 + 1/n, & x \in [x_0 - 1/n, x_0) \\ 1/n, & x \geq x_0 \end{cases} \tag{4}$$

where $n = 1, 2, ...$ *and* $x_0 - 1/n \geq 0$. *Suppose also that the weighting functions* w^- *and* w^+ *are continuous. Then,* $\zeta_{\mathcal{J}}(X, Y)$ *is a simple metric.*

Proof. First, since $\mathcal{V}_{\mathcal{J}} \subseteq \mathcal{S}$ then $\zeta_{\mathcal{J}}(X, Y) < \infty$ since the class $\mathcal{S}$ contains by construction bounded functions. In order to prove the claim, it suffices to demonstrate that $\zeta_{\mathcal{F}}(X, Y) = 0$ implies $F_X(y) = F_Y(y)$. We consider two cases and take advantage of the inequalities

$$V(X) \leq \zeta_{\mathcal{J}}(X, Y) + V(Y) \tag{5}$$

and

$$\begin{aligned} -w^-(F_X(x_0 + 1/n)) &\leq n \int_{-\infty}^{0} v^-_{x_0,n}(x) dw^-(F_X(x)) < -w^-(F_X(x_0)) \\ w^+(1 - F_X(x_0)) &< n \int_{-\infty}^{0} v^+_{x_0,n}(x) d[-w^+(1 - F_X(x))] \\ &\leq w^+(1 - F_X(x_0 - 1/n)) \end{aligned} \tag{6}$$

Case I, $y < 0$. Assume $\zeta_{\mathcal{J}}(X, Y) = 0$, apply (5) for $v(x) = n.v^-_{x_0,n}(x)$, and use the first chain of inequalities in (6),

$$\begin{aligned} -w^-(F_X(y + 1/n)) &\leq n \int_{-\infty}^{0} v^-_{y,n}(x) dw^-(F_X(x)) \\ &\leq n \int_{-\infty}^{0} v^-_{y,n}(x) dw^-(F_Y(x)) < -w^-(F_Y(y)), \end{aligned}$$

and because of the symmetry of $\zeta_{\mathcal{J}}(X, Y)$,

$$-w^-(F_Y(x_0 + 1/n)) < -w^-(F_X(y)).$$

At the limit, as $n \to \infty$, we obtain $w^-(F_X(y)) = w^-((F_Y(y))$ and because of the monotonicity and continuity of w^-, it follows that $F_X(y) = F_Y(y)$.

Case II, $y > 0$. By virtue of the same reasoning as in Case I, but using the second chain of inequalities in (6), we obtain $F_X(y) = F_Y(y)$.

Combining Case I and Case II, we conclude that $P(X \in A) = P(Y \in A)$ on all events A such that $0 \notin A$. Since the distribution functions are continuous from the right, computing the limit $\lim_{y\to 0^+} F_X(y) = \lim_{y\to 0^+} F_Y(y)$ we get $F_X(y) = F_Y(y),\ y \in \mathbb{R}$. $\square$

The result in the theorem implies that if the set of investors is so large that it contains the value functions defined in (3) and (4), then $\zeta_{\mathcal{J}}(X,Y) = 0$ indicates that the distribution functions of X and Y coincide. As we noted, if the class of investors is relatively poor, then $\zeta_{\mathcal{J}}(X,Y) = 0$ implies that only certain characteristics of X and Y agree and not the entire distribution functions. Note that the particular form of the weighting functions is immaterial. The only properties needed are that they are non-decreasing and continuous.

The reasoning outlined above can be used to construct a functional consistent with the first-degree stochastic dominance order. Consider the functional,

$$\zeta^+_{\mathcal{J}}(X,Y) = \sup_{j\in\mathcal{J}} \left(V_j(X) - V_j(Y)\right)_+ \tag{7}$$

where $(x)_+ = \max(x,0)$ which is constructed in the same way as $\zeta_{\mathcal{J}}(X,Y)$. The interpretation of (7) is the following. The distance between X and Y equals the largest difference $V_j(X) - V_j(Y)$ running through all investors who do not prefer Y to X. In this case, the condition $\zeta^+_{\mathcal{F}}(X,Y) = 0$ implies that all investors prefer Y to X because in this case $V_j(X) \leq V_j(Y),\ \forall j \in \mathcal{J}$.

Theorem 2. *The functional* $\zeta^+_{\mathcal{J}}(X,Y)$ *is a quasi-semimetric. If* $F_Y(x) \leq$

$F_X(x), \forall x \in \mathbb{R}$, *then* $\zeta^+_{\mathcal{J}}(X,Y) = 0$. *If the value functions contains the ones defined in* (3) *and* (4), *and the weighting functions are continuous, then* $\zeta^+_{\mathcal{J}}(X,Y) = 0$ *implies that* $F_Y(x) \leq F_X(x), \forall x \in \mathbb{R}$.

Proof. First we demonstrate that $\zeta^+_{\mathcal{J}}(X,Y)$ is a semimetric. If $X \stackrel{d}{=} Y$, then $\zeta^+_{\mathcal{J}}(X,Y) = 0$ due to the monotonicity and continuity of the weighting functions. The triangle inequality follows due to the properties of the $(x)_+$ function. It is also obvious that $\zeta^+_{\mathcal{J}}(X,Y) \neq \zeta^+_{\mathcal{J}}(Y,X)$.

Next, consider the expression,

$$\begin{aligned} V_j(X) - V_j(Y) = &\int_{-\infty}^{0} v_j(x) d[w_j^-(F_X(x)) - w_j^-(F_Y(x))] \\ &+ \int_0^{\infty} v_j(x) d[w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))]. \end{aligned}$$

Due to the assumed boundedness of the integrand, the properties of the weighting functions, and the fact that $v(0) = 0$, integration by parts leads to

$$\begin{aligned} V_j(X) - V_j(Y) = &-\int_{-\infty}^{0} [w_j^-(F_X(x)) - w_j^-(F_Y(x))] dv_j(x) \\ &- \int_0^{\infty} [w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))] dv_j(x). \end{aligned}$$

By assumption $F_X(x) \geq F_Y(x)$ and $v_j(x)$, $w_j^-(x)$, and $w_j^+(x)$ are non-decreasing, therefore the integrand is non-positive on the entire real line and the increments of $v(x)$ are non-negative. As a result, $V_j(X) - V_j(Y) \leq 0$ for all $j \in \mathcal{J}$ and thus $\zeta^+_{\mathcal{J}}(X,Y) = 0$.

The proof of the last claim repeats the arguments in Case I and Case II of Theorem 1. The inequality between the distribution functions appears as a result of the lack of symmetry of $\zeta^{+}_{\mathcal{J}}(X,Y)$.

□

Similarly to $\zeta_{\mathcal{J}}(X,Y)$, if the class $\mathcal{J}$ is not rich enough, then the condition $\zeta^{+}_{\mathcal{J}}(X,Y) = 0$ does not imply inequality between the distribution functions but only between certain characteristics of X and Y.

2 Upper bounds

Under certain additional assumptions on the value functions, we can derive upper bounds on $\zeta_{\mathcal{J}}(X,Y)$ and $\zeta^{+}_{\mathcal{J}}(X,Y)$. The additional assumptions concern the rate of change of $v_j(x)$ and the weighting functions they and they can be regarded as smoothness assumptions. The main result is provided in the next theorem.

Theorem 3. *Consider the set $\mathcal{V}_{\mathcal{J}}$ of value functions $v_j \in \mathcal{S}$ satisfying the Lipschitz condition $|v_j(x) - v_j(y)| \leq K_{v_j}|x-y|$ and the weighting functions satisfy the Lipschitz conditions $|w^{-}_j(x) - w^{-}_j(y)| \leq K_{w_j}|x-y|$ and $|w^{+}_j(x) - w^{+}_j(y)| \leq K_{w_j}|x-y|$ where $0 < K_{v_j}K_{w_j} \leq 1$. The following inequalities hold*

$$\zeta_{\mathcal{J}}(X,Y) \leq \int_{\mathbb{R}} |F_X(x) - F_Y(x)|dx \tag{8}$$

and

$$\zeta^{+}_{\mathcal{J}}(X,Y) \leq \int_{\mathbb{R}} (F_Y(x) - F_X(x))_{+}dx \tag{9}$$

Proof. We demonstrate directly (8). The other inequality follows by the same reasoning. Consider the expression which we used in the proof of Theorem Theorem 2,

$$
\begin{aligned}
|V_j(X) - V_j(Y)| = \Bigg| \int_{-\infty}^{0} & [w_j^-(F_X(x)) - w_j^-(F_Y(x))] dv_j(x) \\
& + \int_{0}^{\infty} [w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))] dv_j(x) \Bigg|
\end{aligned}
$$

Then

$$
\begin{aligned}
|V_j(X) - V_j(Y)| &\leq \int_{-\infty}^{0} \left| w_j^-(F_X(x)) - w_j^-(F_Y(x)) \right| dv_j(x) \\
&+ \int_{0}^{\infty} \left| w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x)) \right| dv_j(x) \\
&\leq K_{v_j} K_{w_j} \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)| \, dx \\
&\leq \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)| \, dx
\end{aligned}
$$

The inequality in (8) follows from the same arguments and taking advantage of the inequality $(w_j^{-/+}(x) - w_j^{-/+}(y))_+ \leq K_{w_j}(x - y)_+$ which holds because of the monotonic properties of the weighting functions. □

The Lipschitz conditions imply that the value function and the weighting functions do not change too quickly. For example, if we compare two outcomes x and $x + h, \ h > 0$, then the Lipschitz condition suggests that $v_j(x + h) - v(x) \leq K_{v_j} h$ which means that the difference between the assigned values by v_j of the j-th investor is bounded by $K_{v_j} h$. Likewise, we can interpret the Lipschitz condition for the weighting function.

The condition in the theorem, $0 < K_{w_j} K_{v_j} \leq 1$, means that if the value function of a given investor is changing too quickly (K_{v_j} is high), then the weighting functions of the corresponding investor should have a constant bounded from above $0 < K_{w_j} \leq 1/K_{v_j}$. In effect, the combined condition in the theorem means that the individuals that we consider are balanced in their views. A steeper value function should be compensated by a more flat weighting function and vice versa. If the value function and the weighting functions are differentiable, then the Lipschitz conditions translate into bounds on their first derivatives, $|dv_j(x)/dx| \leq K_{v_j}$ and $|dw^{-/+}(x)/dx| \leq K_{w_j}$.

The class of Lipschitz value functions includes (4) and (3) with a constant $K_v = 1/n \leq 1$. Thus, the investors with balanced views are a sufficiently large class for the semimetric $\zeta_{\mathcal{J}}(X, Y)$ to become a metric and also for the quasi-semimetric $\zeta^+_{\mathcal{J}}(X, Y)$ to be consistent with first-order stochastic dominance.

3 Conclusion

We discussed the possibility to define a probability semimetric directly on classes of investors. We provided sufficient conditions on the set of value functions transforming the probability semimetric into a metric. In particular, we consider the class of investors with balanced views. We also discuss a quasi-semimetric consistent with first-order stochastic dominance. We regard the considerations in this paper as a step towards defining measures of dispersion and, eventually, risk measures ideal for a particular class of investors, see the related discussions in Stoyanov et al. (2007) and Rachev et al. (2008).

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