# A General Framework for Term Structure Models Driven by Lévy Processes*

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**Abstract**

We describe a framework in which to generalize the Heath, Jarrow and Morton model for the term structure of interest rates. We represent the model in terms of the triplet of characteristics of the underlying semimartingales. We state and prove the necessary and sufficient conditions for absence of arbitrage in terms of the characteristics of the price process. The methodology is then extended to find sufficient conditions for absence of arbitrage in the defaultable case.

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# 1 Introduction

Our goal is to present a general framework in which to construct bond market models. We introduce the class of semimartingales and a stochastic calculus associated with it. We borrow this material from Shiryaev and Jacod [12] and we adopt their notation. We outline the bond market setting presented in Björk, Kabanov and Runggaldier [2] and give an alternative proof of the main result regarding the absence of arbitrage, which extends to the case where the Levy measure of the driving process is infinite.

Shirakawa [11] introduced an extension of the HJM approach for term structure models that allows for jumps in the forward rate dynamics. The model is driven by a Poisson process with constant intensities, in addition to a standard Brownian Motion. Jarrow, Madan [7] define a multivariate point process in terms of a sequence of stopping times at which jumps take place. The associated counting process is used as one of the driving terms. Björk, et al. [2] extend the HJM model to infinite jump space by introducing a random measure with finite compensator.

Babbs and Webber [1] and El-Jahel, Lindberg and Perraudin [3] use an alternative approach that takes into account monetary policy. The short rate is assumed to be established periodically by the authorities and hence is modelled by a pure jump process. In the latter, the intensities follow a squared Ornstein-Uhlenbeck process. In Babbs and Webber [1] the intensities depend on the short rate itself and in a Markov process which represents the state of the economy.

The class of semimartingales is particularly useful for our purpose since it includes a large variety of processes. In connection with the model in Björk, et al. [2] we will work with a special case, namely diffusion processes. Further, the class of semimartingales is invariant with respect many transformations. Ito's formula, for example, implies invariance with respect to composition with $C^2$ functions. The results involving arbitrage depend heavily on invariance after an absolutely continuous change of measure.

We begin the exposition by introducing some definitions and the notation that will be used throughout.

A *stochastic basis* is a probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ equipped with a *filtration* $\mathbf{F}= (\mathscr{F}_t)_{t\in\mathbb{R}_+}$; here *filtration* means an increasing and right-continuous family of sub-$\sigma$-fields of $\mathscr{F}$. By convention, we set $\mathscr{F}_\infty = \mathscr{F}$ and $\mathscr{F}_{\infty-} = \bigvee_{s\in\mathbb{R}_+} \mathscr{F}_s$.

The stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ is called *complete*, or equivalently is said to *satisfy the usual conditions* if the $\sigma$-field $\mathscr{F}$ is $P$-complete and if every $\mathscr{F}_t$ contains all the $P$-null sets of $\mathscr{F}$.

A *random set* is a subset of $\Omega \times \mathbb{R}_+$.

A *process*(or, an *E-valued process* is a family $X = (X_t)_{t\in\mathbb{R}_+}$ of mappings from $\Omega$ into some set $E$. Unless otherwise stated, $E$ will be $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$.

A process $X$ is called *càd* (resp.*càg*, resp.*càdlàg*) if all its paths are right-continuous (resp. are left- continuous, resp. are right-continuous and admit left-hand limits). When $X$ is càdlàg we define two other processes $X_- = (X_{t-})_{t\in\mathbb{R}_+}$ and $\Delta X = (\Delta X_t)_{t\in\mathbb{R}_+}$ by

$$\begin{aligned} X_{0-} &= X_0, \quad X_{t-} = \lim_{s\to t, s<t} X_s \ \text{ for } t > 0 \\ \Delta X_t &= X_t - X_{t-}. \end{aligned}$$

If $X$ is a process and if $T$ is a mapping: $\Omega \to \overline{\mathbb{R}}_+$, we define the *process stopped at time* $T$, denoted by $X^T$, by $X^T_t = X_{T\wedge t}$.

A random set $A$ is called *evanescent* if the the set $\{\omega : \exists t \in \mathbb{R}_+ \text{ with } (\omega, t) \in A\}$ is $P$-null. Two processes $X$ and $Y$ are called *indistinguishable* if the random set $\{X \neq Y\} = \{(\omega, t) : X_t(\omega) \neq Y_t(\omega)\}$ is evanescent.

In what follows, let $(\Omega, \mathscr{F}, \mathbf{F}, P)$ be a stochastic basis.

## 1.1 $\sigma$-Fields, Random Times

A process $X$ is *adapted* to the filtration $\mathbf{F}$ if $X_t$ is $\mathscr{F}_t$-measurable for every $t \in \mathbb{R}_+$.

A *stopping time* is a mapping $T : \Omega \to \overline{\mathbb{R}}_+$ such that $\{T \leq t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$.

If $T$ is a stopping time, we denote by $\mathscr{F}_T$ the collection of all sets $A \in \mathscr{F}$ such that $A \cap \{T \leq t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$.

If $T$ is a stopping time, we denote by $\mathscr{F}_{T-}$ the $\sigma$-field generated by $\mathscr{F}_0$ and all the sets of the form $A \cap \{t < T\}$, where $t \in \mathbb{R}_+$ and $A \in \mathscr{F}_t$.

The *optional $\sigma$-field* is the $\sigma$-field $\mathscr{O}$ on $\Omega \times \mathbb{R}_+$ that is generated by all *càdlàg* adapted processes (considered as mappings on $\Omega \times \mathbb{R}_+$). A process or random set that is $\mathscr{O}$-measurable is called *optional*.

**Proposition 1.1** *Let $X$ be an optional process. When considered as a mapping on $\Omega \times \mathbb{R}_+$, it is $\mathscr{F} \otimes \mathscr{R}_+$- measurable. Moreover, if $T$ is a stopping time, then*

*a) $X_T 1_{\{T<\infty\}}$ is $\mathscr{F}_T$-adapted (hence, $X$ is adapted).*

*b) the stopped process $X^T$ is also optional.*

Let $S, T$ be two stopping times. We define the *stochastic intervals* to be the following random sets:

$$\begin{aligned}
[\![S, T]\!] &= \{(\omega, t) \colon t \in \mathbb{R}_+, S(\omega) \leq t \leq T(\omega)\} \\
[\![S, T[\![ &= \{(\omega, t) \colon t \in \mathbb{R}_+, S(\omega) \leq t < T(\omega)\} \\
]\!]S, T]\!] &= \{(\omega, t) \colon t \in \mathbb{R}_+, S(\omega) < t \leq T(\omega)\} \\
]\!]S, T[\![ &= \{(\omega, t) \colon t \in \mathbb{R}_+, S(\omega) < t < T(\omega)\}
\end{aligned}$$

We will denote the stochastic interval $[\![T, T]\!]$ by $[\![T]\!]$ and will call it the *graph of the stopping time $T$*.

**Proposition 1.2** *If $S, T$ are two stopping times and if $Y$ is an $\mathscr{F}_S$-measurable random variable, the processes $Y1_{[\![S,T]\!]}$, $Y1_{[\![S,T[\![}$, $Y1_{]\!]S,T]\!]}$ and $Y1_{]\!]S,T[\![}$ are optional.*

Let $X$ be an adapted process. For each $n \in \mathbb{N}^*$ define a new process $X^n$ by

$$X^n = \sum_{k \in \mathbb{N}} X_{k/2^n} 1_{[\![k/2^n,(k+1)/2^n[\![}.$$

Note that if $X$ is càg then the sequence $(X^n)$ converges pointwise to $X$. Hence Proposition 1.2 yields the following result:

**Proposition 1.3** *Every process $X$ that is càg and adapted is optional.*

**Corollary 1.1** *If $X$ is a càdlàg adapted process, the processes $X_-$ and $\Delta X$ are optional.*

A random set $A$ is called *thin* if it is of the form $A = \bigcup_n [\![T_n]\!]$, where $(T_n)$ is a sequence of stopping times. If moreover the sequence $(T_n)$ satisfies $[\![T_n]\!] \cap [\![T_m]\!] = \emptyset$ for all $n \neq m$, it is called an *exhausting sequence* for $A$.

**Lemma 1.1** *Any thin random set admits an exhausting sequence of stopping times.*

**Proposition 1.4** *If $X$ is a càdlàg adapted process, the random set $\{\Delta X \neq 0\}$ is thin. An exhausting sequence for this set is called a sequence that* exhausts the jumps of $X$.

Observe that if $X$ is càdlàg then for each $\omega \in \Omega$ the set $\{t \in \mathbb{R}_+ : \Delta X_t(\omega) \neq 0\}$ is countable. Proposition 1.4 and Lemma 1.1 will be applied as follows: Suppose $(T_n)_n$ is a sequence of stopping times that exhausts the thin set $\{\Delta X \neq 0\}$. Then if $(\omega, t) \in \{\Delta X \neq 0\}$ there is a unique $n$ such that $T_n(\omega) = t$. This will be useful in analyzing the "jumps" of a process describing the behavior of asset prices.

The *predictable $\sigma$-field* is the $\sigma$-field $\mathscr{P}$ on $\Omega \times \mathbb{R}_+$ that is generated by all *càg* adapted processes (considered as mappings on $\Omega \times \mathbb{R}_+$). A process or random set that is $\mathscr{P}$- measurable is called *predictable*. Note that Proposition 1.3 implies that $\mathscr{P} \subset \mathscr{O}$.

**Theorem 1.1** *The predictable $\sigma$-field is also generated by any one of the following collections of sets:*

*(i)* $A \times \{0\}$ *where* $A \in \mathscr{F}_0$, *and* $[\![0, T]\!]$ *where $T$ is any stopping time;*

*(ii)* $A \times \{0\}$ *where* $A \in \mathscr{F}_0$, *and* $A \times (s, t]$ *where* $s < t, A \in \mathscr{F}_s$.

**Proposition 1.5** *If $X$ is a predictable process and if $T$ is a stopping time,*

*a)* $X_T 1_{\{T<\infty\}}$ *is* $\mathscr{F}_{T-}$*-measurable,*

*b) the stopped process $X^T$ is also predictable.*

**Proposition 1.6** *If $S, T$ are two stopping times and if $Y$ is an $\mathscr{F}_S$-measurable random variable, then the process $Y 1_{]\!]S,T]\!]}$ is predictable.*
*process is adapted and càg)*

**Proposition 1.7** *If $X$ is a càdlàg adapted process, then $X_-$ is a predictable process. If moreover $X$ is predictable, then $\Delta X$ is predictable.*
*immediate from the definition, since $X_-$ is adapted and càg)*

A *predictable time* is a mapping $T \colon \Omega \to \overline{\mathbb{R}}_+$ such that the stochastic interval $[\![0, T[\![$ is predictable.

**Theorem 1.2** *a) Let $X$ be an $\overline{\mathbb{R}}$-valued and $\mathscr{F} \otimes \mathscr{R}_+$-measurable process. There exists a $(-\infty, \infty]$-valued process, called the* predictable projection *of $X$ and denoted by ${}^pX$, that is determined uniquely up to an evanescent set by the following two conditions:*

*(i) it is predictable,*

*(ii)* $({}^pX)_T = E(X_T|\mathscr{F}_{T-})$ *on* $\{T < \infty\}$ *for all predictable times* $T$.

*b) Moreover, if* $T$ *is any stopping time, then*

$$ {}^p(X^T) = ({}^pX)1_{[\![0,T]\!]} + X_T 1_{]\!]T,\infty[\![}. $$

*c) Moreover, if* ${}^pX$ *is finite-valued and if* $X'$ *is a* $(-\infty, \infty]$*- valued predictable process, then* ${}^p(XX') = X'{}^p(X)$.

Note that if $M$ is a local martingale then ${}^pM = M_-$ and ${}^p(\Delta M) = 0$. Further, if $X = X_0 + M + A$ is a special semimartingale so that $A$ is predictable, then ${}^pX - X_- = \Delta A$. In particular, ${}^pX = X_-$ if and only if $\Delta A = 0$.

If $\mathscr{C}$ is a class of processes, we denote by $\mathscr{C}_{\text{loc}}$ its *localized class*, defined as such: a process $X$ belongs to $\mathscr{C}_{\text{loc}}$ if and only if there exists an increasing sequence $(T_n)$ of stopping times (depending on $X$) such that $\lim_n T_n = \infty$ a.s. and that each stopped process $X^{T_n}$ belongs to $\mathscr{C}$. The sequence $(T_n)$ is called a *localizing sequence* for $X$ (relative to $\mathscr{C}$).

## 1.2 Martingales

In this subsection we state some standard results concerning martingales.

A *martingale*(resp. *submartingale*; resp. *supermartingale*) is an adapted process $X$ on the basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$, whose $P$-almost all paths are *càdlàg*, such that every $X_t$ is integrable, and that for $s \leq t$:

$$ X_s = E(X_t|\mathscr{F}_s) \quad (\text{resp. } X_s \leq E(X_t|\mathscr{F}_s); \text{ resp. } X_s \geq E(X_t|\mathscr{F}_s)). $$

We say that a process $X$ admits a *terminal variable* $X_\infty$ if $X_t$ converges a.s. to a limit $X_\infty$ as $t \uparrow \infty$; in which case the variable $X_T$ is (a.s.) well defined for any stopping time $T$, with $X_T = X_\infty$ on $\{T < \infty\}$.

**Theorem 1.3** *Let* $X$ *be a supermartingale such that there exists an integrable random variable* $Y$ *with* $X_t = E(Y|\mathscr{F}_t)$ *for all* $t \in \mathbb{R}_+$. *Then*

*a) (Doob's limit Theorem)* $X_t$ *converges a.s. to a finite limit* $X_\infty$.

*b) (Doob's stopping Theorem) If* $S, T$ *are stopping times, the random variables* $X_S$ *and* $X_T$ *are integrable, and* $X_S \geq E(X_T|\mathscr{F}_S)$ *on the set* $\{S \leq T\}$. *In particular,* $X^T$ *is again a supermartingale.*

We denote by $\mathscr{M}$ the class of all *uniformly integrable martingales*; and denote by $\mathscr{H}^2$ the class of all *square-integrable martingales*, that is, of all martingales $X$ such that $\sup_{t\in\mathbb{R}_+} E(X_t^2) < \infty$.

**Theorem 1.4** *a) If* $X$ *is a uniformly integrable martingale, then* $X_t$ *converges a.s. and in* $L^1$ *to a terminal variable* $X_\infty$, *and* $X_T = E(X_\infty|\mathscr{F}_T)$ *for all stopping times* $T$. *Moreover,* $X$ *is square-integrable if and only if* $X_\infty$ *is square-integrable, in which case the convergence* $X_t \to X_\infty$ *also takes place in* $L^2$.

*b) If* $Y$ *is an integrable random variable, there exists a uniformly integrable martingale* $X$, *and only one up to an evanescent set, such that*
$X_t = E(Y|\mathscr{F}_t)$ *for all* $t \in \mathbb{R}_+$*; moreover,* $X_\infty = E(Y|\mathscr{F}_{\infty-})$.

**Theorem 1.5 (Doob's inequality)** *If $X$ is a square-integrable martingale,*

$$E\left(\sup_{t\in\mathbb{R}_+} X_t^2\right) \leq 4 \sup_{t\in\mathbb{R}_+} E(X_t^2) = 4E(X_\infty^2).$$

The following result gives a useful characterization of uniformly integrable martingales.

**Lemma 1.2** *Let $X$ be an adapted càdlàg process, with a terminal random variable $X_\infty$. Then $X$ is a uniformly integrable martingale if and only if for each stopping time $T$, the variable $X_T$ is integrable and satisfies $E(X_T) = E(X_0)$.*

A *local martingale*(resp. a *locally square-integrable martingale*) is a process that belongs to the localized class $\mathscr{M}_{\text{loc}}$ (resp. $\mathscr{H}^2_{\text{loc}}$).

## 1.3 Increasing Processes, Decomposition of Local Martingales

We now present some further concepts that will be needed in order to introduce the class of semimartingales. First, the definition of the integral of an optional process with respect to an adapted process of finite variation is presented. The following subsection will describe the construction of a more general "stochastic integral" with respect to a semimartingale. Then we state some important results concerning the decomposition of a local martingale. Subsequent sections will state results which characterize the structure of the individual components.

We denote by $\mathscr{V}^+$ (resp.$\mathscr{V}$) the set of all real-valued processes $A$ that are *càdlàg*, adapted, $A_0 = 0$, and whose paths are non-decreasing (resp. have finite variation over each finite interval $[0, t]$).

Let $A \in \mathscr{V}$. We denote by $\text{Var}(A)$ the *variation process of* $A$, defined by

$$\text{Var}(A)_t(\omega) = \lim_n \sum_{1\leq k\leq n} |A_{tk/n}(\omega) - A_{t(k-1)/n}(\omega)|.$$

Suppose $A \in \mathscr{V}$. For each $\omega \in \Omega$, the path $t \to A_t(\omega)$ is the distribution function of a signed measure on $\mathbb{R}^+$ (denoted $dA_t(\omega)$) that is finite on each finite interval $[0, t]$, and is finite on $\mathbb{R}^+$ if and only if $\text{Var}(A)_\infty(\omega) < \infty$.

If $A \in \mathscr{V}$ and $H$ is an optional process, we can define the *integral process* $H \cdot A$ by

$$H \cdot A_t(\omega) = \int_0^t H_s(\omega) dA_s(\omega) \tag{1}$$

if $\int_0^t |H_s(\omega)|\, d[\text{Var}(A)]_s(\omega) < \infty$.

We denote by $\mathscr{A}^+$ the set of all $A \in \mathscr{V}^+$ that are integrable: $\text{E}(A_\infty) < \infty$; and denote by $\mathscr{A}$ the set of all $A \in \mathscr{V}$ that have *integrable variation*: $\text{E}(\text{Var}(A)_\infty) < \infty$.

**Theorem 1.6** *Let $A \in \mathscr{A}_{loc}$. There exists a process, called the* compensator *of $A$ and denoted by $A^p$, which is unique up to an evanescent set, and which is characterized by being a predictable process of $\mathscr{A}_{loc}$ such that $A - A^p$ is a local martingale. Moreover, for each predictable process $H$ such that $H \cdot A \in \mathscr{A}_{loc}$ then $H \cdot A^p \in \mathscr{A}_{loc}$ and $H \cdot A^p = (H \cdot A)^p$. In particular, $H \cdot A - H \cdot A^p$ is a local martingale.*

The following are some easy properties of the compensator.

1. If $A \in \mathscr{A}_{\text{loc}}$ is predictable, then $A^p = A$.
2. If $A \in \mathscr{A}_{\text{loc}}$, then ${}^p(\Delta A) = \Delta(A^p)$.
3. If $A \in \mathscr{A}_{\text{loc}}$, then $A$ is a local martingale if and only if $A = 0$.

**Theorem 1.7** *To each pair $(M, N)$ of locally square-integrable martingales one associates a predictable process $\langle M, N\rangle \in \mathscr{V}$, unique up to an evanescent set, such that $MN - \langle M, N\rangle$ is a local martingale. Moreover,*

$$\langle M, N\rangle = \frac{1}{4}\left(\langle M + N, M + N\rangle - \langle M - N, M - N\rangle\right),$$

*and if $M, N \in \mathscr{H}^2$ then $\langle M, N\rangle \in \mathscr{A}$ and $MN - \langle M, N\rangle \in \mathscr{M}$. Furthermore, $\langle M, M\rangle$ is nondecreasing.*
*only if $M$ is quasi-left-continuous.*

The process $\langle M, N\rangle$ is called the *predictable quadratic covariation* (also the *angle bracket*) of the pair $(M, N)$. Note that $\langle M, N\rangle = \langle M - M_0, N - N_0\rangle$.

There is a bijective correspondence between the elements $M$ of $\mathscr{H}^2$ and their terminal variables $M_\infty$. If we define the inner product and the norm on $\mathscr{H}^2$ by

$$(M, N)_{H^2} = \mathrm{E} = (M_\infty N_\infty), \quad \|M\|_{H^2} = \|M_\infty\|_{L^2} \text{ where } M, N \in \mathscr{H}^2,$$

then $\mathscr{H}^2$ is a Hilbert space. To see this, let $(M^n)$ be a Cauchy sequence for $\|\cdot\|_{H^2}$. Then the sequence $(M^n_\infty)$ is Cauchy in $L^2(\Omega, \mathscr{F}_{\infty-}, P)$; let $M_\infty$ be its $L^2$ limit. If $M$ is the (unique) martingale with terminal variable $M_\infty$, then it belongs to $\mathscr{H}^2$ and $\|M^n - M\|_{H^2} \to 0$.

Note that the previous theorem implies $(M, N)_{H^2} = \mathrm{E}(\langle M, N\rangle_\infty) + \mathrm{E}(M_0 N_0)$.

**Corollary 1.2** *The set of all continuous elements of $\mathscr{H}^2$ is a closed subspace of the Hilbert space $\mathscr{H}^2$.*

Example (The Wiener process).

a) A *Wiener process* on $(\Omega, \mathscr{F}, \mathbf{F}, P)$ (or, relative to $\mathbf{F}$) is a continuous adapted process $W$ such that $W_0 = 0$ and

   (i) $\mathrm{E}(W_t^2) < \infty$ for each $t \in \mathbb{R}_+$, and $\mathrm{E}(W_t) = 0$ for each $t \in \mathbb{R}_+$;

   (ii) $W_t - W_s$ is independent of the $\sigma$-field $\mathscr{F}_s$ for all $0 \le s \le t$.

b) The function $\sigma^2(t) = \mathrm{E}(W_t^2)$ is called the *variance function* of $W$. If $\sigma^2(t) = t$, we say that $W$ is a *standard Wiener process*.

**Proposition 1.8** *A Wiener process $W$ is a continuous martingale, and its angle bracket $\langle W, W\rangle$ is $\langle W, W\rangle_t(\omega) = \sigma^2(t)$.*

We now turn to decompositions of a local martingale.

Two local martingales $M$ and $N$ are called *orthogonal* if their product $MN$ is a local martingale. A local martingale $X$ is called a *purely discontinuous local martingale* if $X_0 = 0$ and if it is orthogonal to all continuous local martingales.

**Lemma 1.3** *A local martingale that belongs to $\mathscr{V}$ is purely discontinuous.*

**Proposition 1.9** *Let $M, N \in \mathscr{H}^2$. The following are equivalent:*

*a)* *$M$ and $N$ are orthogonal.*

*b)* $\langle M, N\rangle = 0$.

*c)* *For all stopping times $T$, $M^T$ and $N - N_0$ are orthogonal in the Hilbert space $\mathscr{H}^2$.*

**Corollary 1.3** *The set $\mathscr{H}^{2,d}$ of all purely discontinuous martingales in $\mathscr{H}^2$ is the orthogonal subspace, in the Hilbert space $\mathscr{H}^2$, of the set $\mathscr{H}^{2,c}$ of all continuous elements of $\mathscr{H}^2$.*

**Proposition 1.10** *Let $a > 0$. Any local martingale $M$ admits a (non-unique) decomposition $M = M_0 + M' + M''$, where $M'$ and $M''$ are local martingales with $M'_0 = M''_0 = 0$, $M'$ has finite variation, and $|\Delta M''| \leq a$ (hence $M'' \in \mathscr{H}^2_{\text{loc}}$).*

Since the local martinagale $M'$ in Proposition 1.10 is of finite variation then it is purely discontinuous by Lemma 1.3. Note also that by Corollary 1.3 $M'' \in \mathscr{H}_{\text{loc}}$ is the sum of purely discontinuous and continuous components. We thus obtain the following decomposition:

**Theorem 1.8** *Any local martingale $M$ admits a unique (up to indistinguishability) decomposition*

$$M = M_0 + M^c + M^d,$$

*where $M^c = 0$, $M^d = 0$, $M^c$ is a continuous local martingale, and $M^d$ is a purely discontinuous local martingale.*

$M^c$ is called the *continuous part* of $M$, and $M^d$ its *purely discontinuous part*. We denote by $\mathscr{L}$ the set of all local martingales $M$ such that $M_0 = 0$.

## 1.4 Semimartingales, Stochastic Integrals

Here we introduce the class of semimartingales, which is our basic modeling tool. Hence we are interested in developing a calculus with respect to semimartingales. In this section we give meaning to the notion of a stochastic integral and state some of its properties.

A *semimartingale* is a process $X$ of the form

$$X = X_0 + M + A, \quad M \in \mathscr{L}, \quad A \in \mathscr{V}, \tag{2}$$

where $X_0$ is finite-valued and $\mathscr{F}_0$-measurable. We denote by $\mathscr{S}$ the space of all semimartingales. A *special semimartingale* is a semimartingale $X$ which admits a decomposition (2) where the process $A$ is predictable. In this case, $A$ is unique (up to an evanescent set). We denote by $\mathscr{S}_p$ the set of all special semimartingales.

If $X \in \mathscr{S}_p$, the unique decomposition $X = X_0 + M + A$ such that $M \in \mathscr{L}$ and that $A$ is a predictable element of $\mathscr{V}$ is called the *canonical decomposition of* $X$.

**Proposition 1.11** *Let $X$ be a semimartingale. The following are equivalent:*

*(i)* *$X$ is a special semimartingale;*

*(ii)* *there exists a decomposition* $X = X_0 + M + A$ *where* $A \in \mathscr{A}_{loc}$;

*(iii)* *all decompositions* $X = X_0 + M + A$ *satisfy* $A \in \mathscr{A}_{loc}$;

*(iv)* *the process* $Y_t = \sup_{s \le t} |X_s - X_0|$ *belongs to* $\mathscr{A}^+_{loc}$.

**Lemma 1.4** *If a semimartingale* $X$ *satisfies* $|\Delta X| \le a$, *then it is special and its canonical decomposition* $X = X_0 + M + A$ *satisfies* $|\Delta A| \le a$ *and* $|\Delta M| \le 2a$. *In particular, if* $X$ *is continuous then* $M$ *and* $A$ *are continuous.*

**Proposition 1.12** *Let* $X$ *be a semimartingale. There is a unique (up to indistinguishability) continuous local martingale* $X^c$ *with* $X_0^c = 0$, *such that any decomposition of type (2) meets* $M^c = X^c$ *(up to indistinguishability).* $X^c$ *is called the* continuous martingale part *of* $X$.

We will now construct the integral process $H \cdot X$ where $H$ is a locally bounded predictable process and $X$ is a semimartingale. Denote by $\mathscr{E}$ the set of all processes of the following forms:

$$\begin{aligned} H &= Y 1_{[\![0]\!]},\ Y \text{ is bounded } \mathscr{F}_0\text{-measurable, and} \\ H &= Y 1_{]\!]r,s]\!]},\ r < s,\ Y \text{ is bounded } \mathscr{F}_r\text{-measurable.} \end{aligned}$$

For $H \in \mathscr{E}$ and $X$ a semimartingale, the integral process $H \cdot X_t$ is defined by

$$H \cdot X_t = \begin{cases} 0 & \text{if } H = Y 1_{[\![0]\!]}, \\ Y(X_{s\wedge t} - X_{r \wedge t}) & \text{if } H = Y 1_{]\!]r,s]\!]}. \end{cases} \tag{3}$$

**Theorem 1.9** *Let* $X$ *be a semimartingale. The map* $H \to H \cdot X$ *defined on* $\mathscr{E}$ *by (3) has an extension, still denoted by* $H \to H \cdot X$ *to the space of all locally bounded predictable processes* $H$, *with the following properties:*

*i)* $H \cdot X$ *is a càdlàg adapted process;*

*ii)* $H \to H \cdot X$ *is linear, up to evanescence (i.e.* $(aH + K) \cdot X$ *and* $aH \cdot X + K \cdot X$ *are indistinguishable);*

*iii)* *if a sequence* $(H^n)$ *of predictable processes converges pointwise to a limit* $H$, *and if* $|H^n| \le K$ *where* $K$ *is a locally bounded predictable process, then* $H^n \cdot X_t \to H \cdot X_t$ *in measure for all* $t \in \mathbb{R}_+$.

*Moreover, this extension is unique, up to evanescence (i.e. if* $H \to \alpha(H)$ *is another extension with the same properties, then* $\alpha(H)$ *and* $H \cdot X$ *are indistinguishable), and in iii) above* $H^n \cdot X$ *converges to* $H \cdot X$ *in measure, uniformly on finite intervals:* $\sup_{s \le t} |H^n \cdot X_s - H \cdot X_s| \xrightarrow{P} 0$.

We now state some elementary properties of stochastic integrals. Here $X$ is a semimartingale, and $H, K$ are locally bounded predictable process. All statements are up to evanescence.

(P1) $X \to H \cdot X$ is linear.

(P2) (a) $H \cdot X$ is a semimartingale;

(b) if $X$ is a local martingale then so is $H \cdot X$;

(c) if $X \in \mathscr{V}$ then $H \cdot X \in \mathscr{V}$ and $H \cdot X$ coincides with the process defined in (1) (Stieltjes integral process).

(P3) $(H \cdot X)_0 = 0$ and $H \cdot X = H \cdot (X - X_0)$.

(P4) $\Delta(H \cdot X) = H \Delta X$.

(P5) $X^T = X_0 + 1_{[\![0,T]\!]} \cdot X$ and $(H \cdot X)^T = (H 1_{[\![0,T]\!]}) \cdot X$ for all stopping times $T$; more generally, $K \cdot (H \cdot X) = (HK) \cdot X$.

For $\mathscr{H}^2_{\mathrm{loc}}$, we denote by $L^2(X)$ (resp. $L^2_{\mathrm{loc}}(X)$) the set of all predictable processes $H$ such that the process $H^2 \cdot \langle X, X\rangle$ is integrable (resp. locally integrable). Since $\langle X, X\rangle \in \mathscr{A}^+_{\mathrm{loc}}$, all locally bounded predictable processes belong to $L^2_{\mathrm{loc}}(X)$.

**Theorem 1.10** *Let $X \in \mathscr{H}^2_{loc}$. The map $H \to H \cdot X$ (defined either on $\mathscr{E}$ by (3) or for all locally bounded predictable $H$ by Theorem 1.9 has a further extension to the set $L^2_{loc}(X)$, still denoted by $H \to H \cdot X$, which meets i), ii) in Theorem 1.9, and if a sequence $(H^n)$ of predictable processes converges pointwise to a limit $H$, and $|H^n| \leq K$ for some $K \in L^2_{loc}(X)$, then $sup_{s\leq t}|H^n \cdot X_s - H \cdot X_s| \xrightarrow{P} 0$ for all $t \in \mathbb{R}_+$. Moreover this extension is unique (up to evanescence), and we have:*

1. *$H \cdot X \in \mathscr{H}^2_{loc}$.*
2. *$H \cdot X \in \mathscr{H}^2$ if and only if $H \in L^2(X)$.*
3. *Properties (P1), (P3), (P4), (P5) ( for $H \in L^2_{loc}(X)$ and $K \in L^2_{loc}(H \cdot X)$) hold.*
4. *If $X, Y \in \mathscr{H}^2_{loc}$ and $H \in L^2_{loc}(X)$ and $K \in L^2_{loc}(Y)$, then*
$$\langle H \cdot X, K \cdot Y\rangle = (HK) \cdot \langle X, Y\rangle.$$

We will now show that the stochastic integral of a predictable process that is càg may be approximated by Riemann sums. We call an *adapted subdivision* any sequence $\tau = (T_n)_{n\in\mathbb{N}}$ of stopping times with $T_0 = 0$, $\sup_n T_n < \infty$, and $T_n < T_{n+1}$ on $\{T < \infty\}$ (a *deterministic* subdivision if all the $T_n$'s are constant). The *$\tau$-Riemann approximant* of $H \cdot X$ is the process $\tau(H \cdot X)$ defined by
$$\tau(H \cdot X)_t = \sum_{n\in\mathbb{N}} H_{T_n}(X_{T_{n+1}\wedge t} - X_{T_n\wedge t}).$$
A sequence $(\tau_n = (T(n, m))_{m\in\mathbb{N}})_{n\in\mathbb{N}}$ of adapted subdivisions is called a *Riemann sequence* if $\sup_{m\in\mathbb{N}}[T(n, m+1) \wedge t - T(n, m) \wedge t] \to 0$ for all $t \in \mathbb{R}_+$ (that is, if the mesh of the restriction of the subdivisions $\tau_n$ to each interval $[0, t]$ tends to 0).

**Proposition 1.13** *Let $X$ be a semimartingale, $H$ be a càg adapted process, and $(\tau_n)$ a Riemann sequence of adapted subdivisions. Then the $\tau_n$-Riemann approximants $\tau_n(H \cdot X)$ converge to $H \cdot X$ in measure, uniformly on each compact interval.*

## 1.5 Quadratic Variation, Ito's Formula

In this section we state Ito's formula for semimartingales. The following result gives an explicit solution to a "stochastic differential equation" and describes some of its properties.

Let $X$ and $Y$ be two semimartingales. The *quadratic co- variation* of $X$ and $Y$, denoted by $[X, Y]$ (the *quadratic variation* of $X$, when $Y = X$) is the process defined by:
$$[X, Y] = XY - X_0Y_0 - X_- \cdot Y - Y_- \cdot X$$
(it is defined uniquely, up to an evanescent set). Note that the following properties hold:
$$[X, Y]_0 = 0, \quad [X, Y] = [X - X_0, Y - Y_0],$$
$$[X, Y] = \frac{1}{4}([X + Y, X + Y] - [X - Y, X - Y]). \tag{4}$$

**Theorem 1.11** *Let $X$ and $Y$ be two semimartingales.*

*a) For any Riemann sequence $\{\tau_n = (T(n,m))_{m\in\mathbb{N}}\}_{n\in\mathbb{N}}$ of adapted subdivisions, the processes $S_{\tau_n}(X,Y)$ defined by*

$$S_{\tau_n}(X,Y)_t = \sum_{m\geq 1}(X_{T(n,m+1)\wedge t} - X_{T(n,m)\wedge t})(Y_{T(n,m+1)\wedge t} - Y_{T(n,m)\wedge t})$$

*converge to the process $[X,Y]$ in measure, uniformly on every compact interval.*

*b) $[X,Y] \in \mathscr{V}$ and $[X,X] \in \mathscr{V}^+$.*

*c) $\Delta[X,Y] = \Delta X \Delta Y$.*

Let $X \in \mathscr{H}^2$. It follows from the definition of $[X,Y]$ that $X^2 - X_0^2 - [X,X] \in \mathscr{L}$. By Theorem 1.7, we have that $X^2 - X_0^2 - \langle X,X\rangle \in \mathscr{L}$. It follows that $[X,X] - \langle X,X\rangle \in \mathscr{L}\cap\mathscr{V}$ so that $[X,X] \in \mathscr{A}_{\text{loc}}$ and $\langle X,X\rangle$ is the compensator of $[X,X]$. The argument extends to $[X,Y]$ with $Y \neq X$ by (4). We formalize this result as follows:

**Proposition 1.14** *Let $X$ and $Y$ be two local martingales.*

*a) $XY - X_0Y_0 - [X,Y]$ is a local martingale.*

*b) If $X,Y \in \mathscr{H}^2_{loc}$ then $[X,Y] \in \mathscr{A}_{loc}$ and its compensator is $\langle X,Y\rangle$; if moreover $X,Y \in \mathscr{H}^2$, then $XY - [X,Y] \in \mathscr{M}$.*

*c) $X$ belongs to $\mathscr{H}^2$ (resp. $\mathscr{H}^2_{loc}$) if and only if $[X,X]$ belongs to $\mathscr{A}$ (resp. $\mathscr{A}_{loc}$) and $X_0$ is square-integrable.*

*d) $X = X_0$ a.s. if and only if $[X,X] = 0$.*

**Theorem 1.12** *Let $X$ and $Y$ be two semimartingales and denote by $X^c$, $Y^c$ their continuous martingale parts. Then*

$$[X,Y]_t = \langle X^c, Y^c\rangle_t + \sum_{s\leq t}\Delta X_s \Delta Y_s. \tag{5}$$

**Theorem 1.13 (Ito's formula)** *Let $X = (X^1,\dots,X^d)$ be a $d$-dimensional semimartingale, and let $f$ be a class $C^2$ function on $\mathbb{R}^d$. Then $f(X)$ is a semimartingale and we have:*

$$\begin{aligned} f(X_t) = f(X_0) &+ \sum_{i\leq d} D_i f(X_-)\cdot X^i + \frac{1}{2}\sum_{i,j\leq d} D_{ij}f(X_-)\cdot\langle X^{i,c}, X^{j,c}\rangle \\ &+ \sum_{s\leq t}\left[f(X_s) - f(X_{s-}) - \sum_{i\leq d} D_i f(X_{s-})\Delta X^i_s\right], \end{aligned} \tag{6}$$

*where $D_i f$ and $D_{ij}f$ denote the partial derivatives $\partial f/\partial x^i$ and $\partial^2 f/\partial x^i\partial x^j$.*

As an application of Ito's formula, we will study the equation

$$Y = 1 + Y_- \cdot X \quad (\text{or: } dY = Y_- dX \text{ and } Y_0 = 1)$$

where $X$ is a given semimartingale and $Y$ is an unknown càdlàg adapted process. We will consider $X$ to be a complex-valued semimartingale, that is, $X = X' + iX''$ with $X'$ and $X''$ two real-valued semimartingales. Then (6) is read as a system of equations with real-valued terms as follows:

$$\begin{aligned} Y' &= 1 + Y'_-\cdot X' - Y''_-\cdot X'' \\ Y'' &= \phantom{1 +} Y''_-\cdot X' - Y'_-\cdot X'' \end{aligned}$$

and $Y = Y' + iY''$.

**Theorem 1.14** *Let $X = X' + iX''$ be a complex-valued semimartingale. Then equation (6) has one and only one (up to indistinguishability) càdlàg adapted solution. This solution is a semimartingale, is denoted by $\mathscr{E}(X)$, and is given by*

$$\mathscr{E}(X)_t = \{\exp(X_t - X_0 - \frac{1}{2}\langle X'^c, X'^c\rangle_t + \frac{1}{2}\langle X''^c, X''^c\rangle_t - i\langle X'^c, X''^c\rangle_t)\} \times \prod_{s\leq t}[(1+\Delta X_s)e^{-\Delta X_s}]$$

*where the (possibly infinite) product is absolutely convergent. Furthermore,*

*a) If $X$ has finite variation, then so has $\mathscr{E}(X)$.*

*b) If $X$ is a local martingale, then so is $\mathscr{E}(X)$.*

*c) Let $T = \inf(t\colon \Delta X_t = -1)$. Then $\mathscr{E}(X) \neq 0$ on the interval $[\![0,T[\![$, and $\mathscr{E}(X)_- \neq 0$ on the interval $[\![0,T]\!]$, and $\mathscr{E}(X) = 0$ on the interval $[\![T,\infty[\![$.*

*In particular, (5) implies that when $X$ has finite variation, then*

$$\mathscr{E}(X)_t = e^{X_t - X_0}\prod_{s\leq t}(1+\Delta X_s)e^{-\Delta X_s}.$$

*When $X$ is a real-valued semimartingale, then*

$$\mathscr{E}(X)_t = e^{X_t - X_0 - \frac{1}{2}\langle X^c, X^c\rangle_t}\prod_{s\leq t}(1+\Delta X_s)e^{-\Delta X_s}.$$

# 2 Characteristics of Semimartingales

Our goal in this section is to describe a convenient way of representing semimartingales. To this end, we define a set of "characteristics" associated with a semimartingale $X$. We then define a "canonical representation", which expresses $X$ in terms of its characteristics. Section 2.1 is an outline of the results concerning random measures that will be needed.

Let $\mathscr{B} = (\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t\geq 0}, P)$ be a stochastic basis. In addition, let $(E, \mathscr{E})$ be an auxiliary measurable space which we stipulate to be a *Blackwell space*. In this paper, however, we will assume the special case $E = \mathbb{R}^d$ with its Borel $\sigma$-field.

## 2.1 Random Measures

In order to characterize the behavior of the jumps of the processes of interest, we introduce the concept of an integer-valued random measure and its compensator. We also define an integral with respect to these. Then, after introducing some necessary notation, we define an integral with respect to the "compensated" random measure. This important result is closely related to the purely discontinuous part of a local martingale which was introduced in the previous section.

A *random measure* on $\mathbb{R}_+ \times E$ is a family $\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$ of nonnegative measures on $(\mathbb{R}_+ \times E, \mathscr{R}_+ \otimes \mathscr{E})$ such that $\mu(\omega; \{0\} \times E) = 0$ identically.

We will be using the following notation:

$\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times E$, with $\sigma$-fields $\tilde{\mathscr{O}} = \mathscr{O} \otimes \mathscr{E}$ and $\tilde{\mathscr{P}} = \mathscr{P} \otimes \mathscr{E}$. Here $\mathscr{O}$ and $\mathscr{P}$ are the optional and predictable $\sigma$-fields on $(\Omega \times \mathbb{R}_+)$, respectively.

A function $W$ on $\tilde{\Omega}$ that is $\tilde{\mathscr{O}}$- measurable(resp. $\tilde{\mathscr{P}}$-measurable) is called an *optional*(resp. a *predictable*) function.

Let $\mu$ be a random measure and $W$ an optional function on $\tilde{\Omega}$. Define the *integral process* $W * \mu$ by

$$W * \mu(\omega) = \int_{[0,t]\times E} W(\omega, s, x)\mu(\omega; ds, dx)$$

if $\int_{[0,t]\times E} |W(\omega, s, x)|\mu(\omega; ds, dx) < \infty$.

A random measure $\mu$ is called *optional*(resp. *predictable*) if the process $W * \mu$ is optional(resp. predictable) for every optional(resp. predictable) function $W$.

An optional measure $\mu$ is called *integrable* if the random variable $1 * \mu_\infty = \mu(\cdot, \mathbb{R}_+ \times E)$ is integrable (or equivalently, if $1 * \mu \in \mathscr{A}^+$).

An optional random measure $\mu$ is called $\tilde{\mathscr{P}}$-$\sigma$- *finite* if there exists a strictly positive predictable function $V$ on $\tilde{\Omega}$ such that the random variable $V * \mu_\infty$ is integrable(or equivalently, if $V * \mu \in \mathscr{A}^+$). This property is equivalent to the existence of a $\tilde{\mathscr{P}}$-measurable partition $(A_n)$ of $\tilde{\Omega}$ such that each $(1_{A_n} * \mu)_\infty$ is integrable.

**Theorem 2.1** *Let $\mu$ be an optional $\tilde{\mathscr{P}}$-$\sigma$-finite random measure. There exists a random measure, called the* compensator *of $\mu$ and denoted by $\mu^p$, which is unique up to a $P$-null set, and which is characterized as being a predictable random measure satisfying either one of the two following equivalent properties:*

1. *$E(W * \mu^p_\infty) = E(W * \mu_\infty)$ for every nonnegative $\tilde{\mathscr{P}}$-measurable function $W$ on $\tilde{\Omega}$.*
2. *For every $\tilde{\mathscr{P}}$-measurable function $W$ on $\tilde{\Omega}$ such that $|W| * \mu \in \mathscr{A}^+_{loc}$, $|W| * \mu^p \in \mathscr{A}^+_{loc}$, and $W * \mu - W * \mu^p$ is a local martingale.*

An *integer-valued random measure* is a random measure that satisfies:

1. $\mu(\omega; \{t\} \times E) \leq 1$ identically,
2. for each $A \in \mathscr{R}_+ \otimes \mathscr{E}$, $\mu(\cdot, A)$ takes its values in $\overline{\mathbb{N}}$;
3. $\mu$ is optional and $\tilde{\mathscr{P}}$-$\sigma$-finite.

**Proposition 2.1** *If $\mu$ is an integer-valued random measure, there exists a thin random set $D$ and an $E$-valued optional process $\beta$ such that*

$$\mu(\omega; dt, dx) = \sum_{s\geq 0} 1_D(\omega, s)\varepsilon_{(s,\beta_s(\omega))}(dt, dx), \tag{7}$$

*where $\varepsilon_a$ denotes the Dirac measure at the point $a$.*

Note that if $(T_n)$ is a sequence of stopping times that exhausts the thin set $D$, then

$$\begin{aligned} W * \mu_t &= \sum_n W(T_n, \beta_{T_n}) 1_{\{T_n \leq t\}} \\ &= \sum_{0<s\leq t} W(s, \beta_s) 1_D(s). \end{aligned}$$

where $W$ is any nonnegative optional function.

**Proposition 2.2** *Let $X$ be an adapted càdlàg $\mathbb{R}^d$-valued process. Then*

$$\mu^X(\omega; dt, dx) = \sum_s 1_{\{\Delta X_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta X_s(\omega))}(dt, dx) \tag{8}$$

*defines an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. (In the representation (7) we have $D = \{\Delta X \neq 0\}$ and $\beta = \Delta X$). We call $\mu^X$ the random measure* associated with the jumps *of $X$.*

**Proposition 2.3** *Let $\mu$ be an integer-valued random measure, $\nu = \mu^p$ its compensator, and $J = \{(\omega, t) : \nu(\omega; \{t\} \times E) > 0\}$.*

*i) $J$ is the* predictable support $\{^p(1_D)\}$ *of the set $D$ in (7), and for all predictable times $T$ and nonnegative predictable $W$:*

$$\int_E W(T, x) \nu(\{T\} \times dx) = E[W(T, \beta_T) 1_D(T) | \mathscr{F}_{T-}] \text{ on } \{T < \infty\}.$$

*ii) There is a version of $\nu$ such that $\nu(\omega; \{t\} \times E) \leq 1$ identically and that the thin set $J$ is exhausted by a sequence of predictable times.*

Example (Poisson Measures). An *extended Poisson measure* on $\mathbb{R}_+ \times E$, relative to the filtration $\mathbf{F}$, is an integer-valued random measure $\mu$ such that

(i) the positive measure $m$ on $\mathbb{R}_+ \times E$ defined by $m(A) = \mathrm{E}[\mu(A)]$ is $\sigma$-finite;

(ii) for every $s \in \mathbb{R}_+$ and every $A \in \mathscr{R}_+ \otimes \mathscr{E}$ such that $A \subset (s, \infty) \times \mathscr{E}$ and that $m(A) < \infty$, the variable $\mu(\cdot, A)$ is independent of the $\sigma$-field $\mathscr{F}_s$.

The measure $m$ is called the *intensity measure* of $\mu$. If $m$ satisfies $m(\{t\} \times E) = 0$ for each $t \in \mathbb{R}_+$, then $\mu$ is called a *Poisson measure*; if $m$ has the form $m(dt, dx) = dt \times F(dx)$, where $F$ is a positive $\sigma$-finite measure on $(E, \mathscr{E})$, then $\mu$ is called a *homogeneous Poisson measure*.

**Proposition 2.4** *Let $\mu$ be an extended Poisson measure on $\mathbb{R}_+ \times E$, relative to the filtration* $\mathbf{F}$*, with intensity measure $m$. Then its compensator is*
$\mu^p(\omega; \cdot) = m(\cdot)$.

Let $\mu$ be an integer-valued random measure on $\mathbb{R}_+ \times E$. Let $\nu$ be a version of the dual predictable projection given in Proposition 2.3. We will now define stochastic integrals with respect to the "compensated" integer-valued random measure $(\mu - \nu)$. We will use the following notation:

$$\begin{aligned} a_t(\omega) &= \nu(\omega; \{t\} \times E), \\ J &= \{a > 0\}, \text{ exhausted by the sequence } (T_n) \\ &\qquad\qquad \text{of predictable times,} \\ \nu^c(\omega; dt, dx) &= 1_{J^c}(\omega, t). \end{aligned}$$

For any measurable function $W$ on $\tilde{\Omega}$ we define the process $\hat{W}$ by

$$\hat{W}_t(\omega) = \int_E W(\omega, t, x) \nu(\omega; \{t\} \times dx)$$

if $\int_E |W(\omega, t, x)| \nu(\omega; \{t\} \times dx) < \infty$.

**Lemma 2.1** *If $W$ is $\tilde{\mathscr{P}}$-measurable, then $\hat{W}$ is predictable and it is a version of the predictable projection of the process $(\omega, t) \to W(\omega, t, \beta_t(\omega)) 1_D(\omega, t)$. In particular, for all predictable times $T$,*

$$\hat{W}_t = E[W(T, \beta_T) 1_D(T) | \mathscr{F}_{T-}] \ \text{ on } \{T < \infty\}.$$

We denote by $G_{\text{loc}}(\mu)$ the set of all $\tilde{\mathscr{P}}$-measurable real-valued functions $W$ on $\tilde{\Omega}$ such that the process $\tilde{W}$ defined by

$$\tilde{W}_t(\omega) = W(\omega, t, \beta_t(\omega)) 1_D(\omega, t) - \hat{W}_t(\omega)$$

satisfies $\left[\sum_{s \leq \cdot} (\tilde{W}_s)^2\right]^{1/2} \in \mathscr{A}^+_{\text{loc}}$.

If $W \in G_{\text{loc}}(\mu)$ we call *stochastic integral of $W$ with respect to $\mu - \nu$*, denoted by $W * (\mu - \nu)$, any purely discontinuous local martingale $X$ such that $\Delta X$ and $\tilde{W}$ are indistinguishable.

**Proposition 2.5** *Let $W$ be a predictable function on $\tilde{\Omega}$ such that $|W| * \mu \in \mathscr{A}^+_{loc}$ (or equivalently, $|W| * \nu \in \mathscr{A}^+_{loc}$). Then $W \in G_{loc}(\mu)$ and*

$$W * (\mu - \nu) = W * \mu - W * \nu.$$

We now characterize the property $W \in G_{\text{loc}}(\mu)$ by the integrability of a suitable increasing predictable process. To this end, we associate to any predictable function $W$ on $\tilde{\Omega}$ two increasing (possibly infinite) predictable processes as follows:

$$\begin{aligned} C(W)_t &= (W - \hat{W})^2 * \nu_t + \sum_{s \leq t} (1 - a_s)(\hat{W}_s)^2, \\ \overline{C}(W)_t &= |W - \hat{W}| * \nu_t + \sum_{s \leq t} (1 - a_s)|\hat{W}_s|. \end{aligned}$$

**Theorem 2.2** *Let $W$ be a predictable function on $\tilde{\Omega}$.*

*a) $W$ belongs to $G_{loc}(\mu)$ and $W * (\mu - \nu)$ belongs to $\mathscr{H}^2$ (resp. $\mathscr{H}^2_{loc}$) if and only if $C(W)$ belongs to $\mathscr{A}^+$ (resp. $\mathscr{A}^+_{loc}$), in which case*

$$\langle W * (\mu - \nu), W * (\mu - \nu) \rangle = C(W).$$

*b) $W$ belongs to $G_{loc}(\mu)$ and $W * (\mu - \nu)$ belongs to $\mathscr{A}$ (resp. $\mathscr{A}_{loc}$) if and only if $\overline{C}(W)$ belongs to $\mathscr{A}^+$ (resp. $\mathscr{A}^+_{loc}$).*

*c) $W$ belongs to $G_{loc}(\mu)$ if and only if $C(W') + \overline{C}(W'')$ belongs to $\mathscr{A}_{loc}$, where*

$$\begin{aligned} W' &= (W - \hat{W}) 1_{\{|W - \hat{W}| \leq 1\}} + \hat{W} 1_{\{|\hat{W}| \leq 1\}}, \\ W'' &= (W - \hat{W}) 1_{\{|W - \hat{W}| > 1\}} + \hat{W} 1_{\{|\hat{W}| > 1\}}. \end{aligned}$$

*d) Assume in addition that $\tilde{W} \geq -1$ identically. Then $\hat{W} \leq 1$ on $\{a < 1\}$ up to an evanescent set, and $W$ belongs to $G_{loc}(\mu)$ if and only if the increasing predictable process $C'(W)$ defined by*

$$C'(W) = \left(1 - \sqrt{1 + W - \hat{W}}\right)^2 * \nu_t + \sum_{s \leq t} (1 - a_s) \left(1 - \sqrt{1 - \hat{W}}\right)^2$$

*belongs to $\mathscr{A}^+_{loc}$.*

## 2.2 Definition of the Characteristics, Canonical Representation

The characteristics of a semimartingale which we present in this section can be interpreted as an extension of the terms that characterize the distribution of a process with independent increments. Frequently, statements about semimartingales will be stated in terms of their characteristics.

Consider a $d$-dimensional semimartingale $X = (X^1, \ldots, X^d)$ defined on the stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$. Denote by $\mathscr{C}_t^d$ the class of functions $h : \mathbb{R}^d \to \mathbb{R}^d$ which are bounded, have compact support, and satisfy $h(x) = x$ in a neighborhood of 0. In everything that follows, it may be assumed that $h(x) = x1_{\{|x|\leq 1\}}$.

Fix a truncation function $h \in \mathscr{C}_t^d$. The process $X(h)$ defined by

$$X(h) = X - \sum_{s\leq t}[\Delta X_s - h(\Delta X_s)]$$

is a special semimartingale, and we consider its canonical decomposition

$$X(h) = X_0 + M(h) + B(h), \quad M(h) \in \mathscr{L}^d, \quad B(h) \in \mathscr{V}^d \text{ predictable.} \tag{9}$$

The *characteristics of $X$ associated with $h$* is the triplet $(B, C, \nu)$ consisting in:

1. $B = (B^i)_{i\leq d}$ is the predictable process $B = B(h) \in \mathscr{V}^d$ appearing in (9) above.
2. $C = (C^{ij})_{i,j\leq d}$ is the continuous process $C^{ij} = \langle X^{i,c}, X^{j,c}\rangle \in \mathscr{V}^{d\times d}$.
3. $\nu$ is the compensator of the random measure $\mu^X$ associated with the jumps of $X$.

**Proposition 2.6** *One can find a version of the characteristics $(B, C, \nu)$ of $X$ which is of the form:*

$$\begin{aligned} B^i &= b^i \cdot A \\ C^{ij} &= c^{ij} \cdot A \\ \nu(\omega; dt, dx) &= dA_t(\omega)K_{\omega,t}(dx) \end{aligned}$$

*where:*

1. *$A$ is a predictable process in $\mathscr{A}^+_{loc}$;*
2. *$b = (b^i)_{i\leq d}$ is a $d$-dimensional predictable process;*
3. *$c = (c^{ij})_{i,j\leq d}$ is a predictable process with values in the set of all symmetric nonegative $d \times d$ matrices;*
4. *$K_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{R}^d)$ which satisfies:*

$$K_{\omega,t}(\{0\}) = 0, \quad \int K_{\omega,t}(dx)(|x|^2 \wedge 1) \leq 1,$$

$$\Delta A_t(\omega) > 0 \implies b_t(\omega) = \int K_{\omega,t}(dx)h(x),$$

$$\textit{and } \Delta A_t(\omega)K_{\omega,t}(\mathbb{R}^d) \leq 1.$$

The following result characterizes semimartingales with independent increments in terms of the characteristics.

**Theorem 2.3** *Let $X$ be a $d$-dimensional semimartingale with $X_0 = 0$. Then it is a process with independent increments if and only if there is a version $(B, C, \nu)$ of its characteristics that is deterministic.*

**Proposition 2.7** *Let $X$ be a semimartingale with characteristics $(B(h), C, \nu)$ relative to a truncation function $h$. $X$ is a special semimartingale if and only if $(|x|^2 \wedge |x|) * \nu \in \mathscr{A}_{loc}$. In this case, the canonical decomposition $X = X_0 + N + A$ satisfies $A = B(h) + (x - h(x)) * \nu$.*

**Theorem 2.4** *Let $X$ be a $d$-dimensional semimartingale, with characteristics $(B, C, \nu)$ relative to a truncation function $h \in \mathscr{C}_t^d$, and with the measure $\mu^X$ associated to its jumps by (8). Then $W^i(\omega, t, x) = h^i(x)$ belongs to $G_{loc}(\mu^X)$ for all $i \leq d$, and the following canonical representation of $X$ holds:*

$$X = X_0 + X^c + h * (\mu^X - \nu) + (x - h(x)) * \mu^X + B.$$

The following corollary follows from the last two results and Proposition 2.5. It implies that in the case of a special semimartingale we can take $h(x) = x$.

**Corollary 2.1** *Let $X$ be a $d$-dimensional special semimartingale with characteristics $(B, C, \nu)$ and $\mu^X$ the measure associated to its jumps. Then*
*$W^i(\omega, t, x) = x^i$ belongs to $G_{loc}(\mu^X)$, and if $X = X_0 + N + A$ is its canonical decomposition, then*

$$X = X_0 + X^c + x * (\mu^X - \nu) + A.$$

# 3 Martingale Problems, Diffusion Processes

In this section we present diffusion processes, which will be used to construct a term structure model in Section 6. First we introduce the notion of a martingale problem, which describes a useful framework in which to characterize the set of probability measures under which a suitable process is a semimartingale. The connection between the two concepts is established in Theorem 3.2.

We will be working in the following setting. Let $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t\geq 0})$ be a filtered space, and let $\mathscr{H}$ be a sub-$\sigma$-field of $\mathscr{F}_0$, called the *initial $\sigma$-field*. Let $P_H$ be an *initial condition*, that is, a probability measure on $(\Omega, \mathscr{H})$.

## 3.1 Martingale Problems

Let $X = (X^i)_{i\leq d}$ be a $d$-dimensional càdlàg adapted process on $(\Omega, \mathscr{F}, \mathbf{F})$. $X$ will be a candidate for a semimartingale, so we introduce the following in relation to $X$.

i) $h \in \mathscr{C}_t^d$, a truncation function;

ii) A triplet $(B, C, \nu)$ such that

  (a) $B = (B^i)_{i\leq d}$ is $\mathbf{F}$-predictable, with finite variation over finite intervals, and $B_0 = 0$;

  (b) $C = (C^{ij})_{i,j\leq d}$ is $\mathbf{F}$-predictable, continuous, $C_0 = 0$, and $C_t - C_s$ is a non-negative symmetric $d \times d$ matrix for $s \leq t$;

  (c) $\nu$ is an $\mathbf{F}$-predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, which charges neither $\mathbb{R}_+ \times \{0\}$ nor $\{0\} \times \mathbb{R}^d$, and such that

$$(|x|^2 \wedge 1) * \nu_t(\omega) < \infty, \qquad \int \nu(\omega; \{t\} \times dx) h(x) = \Delta B_t(\omega),$$

  and $\nu(\omega; \{t\} \times \mathbb{R}^d) \leq 1$ identically.

A *solution to the martingale problem* associated with $(\mathscr{H}, X)$ and $(P_H; B, C, \nu)$ is a probability measure $P$ on $(\Omega, \mathscr{F})$ such that:

1. the restriction $P_{|\mathscr{H}}$ of $P$ to $\mathscr{H}$ equals $P_H$;
2. $X$ is a semimartingale on the basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$, with characteristics $(B, C, \nu)$ relative to the truncation function $h$.

The set of all solutions $P$ will be denoted by $\mathfrak{s}(\mathscr{H}, X|P_H; B, C, \nu)$.

Sometimes we will impose additional structure on $(\Omega, \mathscr{F}, \mathbf{F})$ as follows:

1. $\mathbf{F}$ is *generated* by $X$ and $\mathscr{H}$, by which we mean:
   (a) $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s^0$ and $\mathscr{F}_s^0 = \mathscr{H} \vee \sigma(X_r : r \leq s)$ (i.e. $\mathbf{F}$ is the smallest filtration such that $X$ is adapted and $\mathscr{H} \subset \mathscr{F}_0$);
   (b) $\mathscr{F} = \mathscr{F}_{\infty-}$ $(= \vee_t \mathscr{F}_0)$.
2. (*The canonical setting*) $\Omega$ is the *canonical space* of càdlàg functions $\omega\colon \mathbb{R}_+ \to \mathbb{R}^d$; $X$ is the *canonical process* defined by $X_t(\omega) = \omega(t)$; $\mathscr{H} = \sigma(X_0)$; $\mathbf{F}$ is generated by $X$ and $\mathscr{H}$ in the sense of 1. above.

When $\mathscr{H} = \sigma(X_0)$, we can identify the initial measure $P_H$ with the distribution of $X_0$ as follows: If $\eta$ is a probability measure on $\mathbb{R}^d$, we also denote by $\eta$ the measure on $(\Omega, \mathscr{H})$ defined by $\eta(X_0 \in A) = \eta(A)$.

The next result partly justifies the restrictions introduced above.

**Theorem 3.1** *Let $(B, C, \nu)$ meet ii) above and be deterministic.*

*a) If $\mathscr{F}$ is generated by $X$ and $\mathscr{H}$ then $\mathfrak{s}(\mathscr{H}, X|P_H; B, C, \nu)$ contains at most one element $P$.*

*b) Under the canonical setting, for any probability measure $\eta$ on $\mathbb{R}^d$, $\mathfrak{s}(\mathscr{H}, X|\eta; B, C, \nu)$ contains one and only one solution.*

## 3.2 Diffusion Processes

We now assume the canonical setting defined above. Let $P$ be a probability measure on $(\Omega, \mathscr{F})$. $X$ is called a *diffusion process with jumps* on $(\Omega, \mathscr{F}, \mathbf{F}, P)$ if it is a semimartingale with the following characteristics (the truncation function $h$ is fixed):

$$\begin{aligned} B_t^i(\omega) &= \int_0^t b^i(s, X_s(\omega))ds \quad (= +\infty \text{ if integral diverges}) \\ C_t^{ij}(\omega) &= \int_0^t c^{ij}(s, X_s(\omega))ds \quad (= +\infty \text{ if integral diverges}) \\ \nu(\omega; dt \times dx) &= dt \times K_t(X_t(\omega), dx), \end{aligned} \tag{10}$$

where

$$\begin{aligned} &b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ is Borel} \\ &c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \text{ is Borel, } c(s,x) \text{ is symmetric nonnegative} \\ &K_t(x, dy) \text{ is a Borel transition kernel from } \mathbb{R}_+ \times \mathbb{R}^d \text{ into } \mathbb{R}^d, \\ &\qquad \text{with } K_t(x, \{0\}) = 0. \end{aligned}$$

Moreover,
a) if $\nu = 0$, $X$ is called a *diffusion* (it is then a.s. continuous);
b) if $b(s,x)$, $c(s,x)$, $K_s(x, dy)$ do not depend upon $s$, $X$ is called a *homogeneous diffusion* (with jumps).

We now introduce some notation in order to define a stochastic differential equation related to a diffusion. Let $\mathscr{B}' = (\Omega', \mathscr{F}', \mathbf{F}', P')$ be another stochastic basis endowed with the following *driving terms*:

1. $W = (W^i)_{i \le m}$, an $m$-dimensional standard Wiener process
2. $\mathfrak{p}$, a Poisson random measure on $\mathbb{R}_+ \times E$ with intensity measure $q(dt, dx) = dt \otimes F(dx)$. Here $(E, \mathscr{E})$ is a measurable space as in Section 2 (recall that we assume $E = \mathbb{R}^d$), and $F$ is a positive $\sigma$-finite measure on $(E, \mathscr{E})$.

We assume that the following coefficients are given:

$$\begin{aligned} \beta &= (\beta^i)_{i \le d}, \text{ a Borel function: } \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \\ \gamma &= (\gamma^{ij})_{i \le d, j \le m}, \text{ a Borel function: } \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m \\ \delta &= (\delta^i)_{i \le d}, \text{ a Borel function: } \mathbb{R}_+ \times \mathbb{R}^d \times E \to \mathbb{R}^d. \end{aligned} \tag{11}$$

We also let $\xi$ be a given $\mathscr{F}'_0$-measurable $\mathbb{R}^d$- valued random variable which we call the *initial variable*.

Define a stochastic differential equation as follows:

$$\begin{aligned} dY_t =& \beta(t, Y_t)dt + \gamma(t, Y_t)dW_t + h \circ \delta(t, Y_{t-}, z)(\mathfrak{p}(dt, dz) - q(dt, dz)) \\ &+ h' \circ \delta(t, Y_{t-}, z)\mathfrak{p}(dt, dz) \end{aligned} \tag{12}$$

with $Y_0 = \xi$. Here $h$ is a truncaton function and $h'(x) = x - h(x)$.

A *solution-process*(*strong solution*) to (12), on the basis $\mathscr{B}'$ and relative to the driving terms $(W, \mathfrak{p})$, is a càdlàg adapted process $Y$ such that for each $i \le d$,

$$\begin{aligned} Y^i = \xi^i + & \beta^i(Y) \cdot t + \sum_{j \le m} \gamma^{ij}(Y_-) \cdot W^j + h^i \circ \delta(Y_-) * (\mathfrak{p} - q) \\ & + h'^i \circ \delta(Y_-) * \mathfrak{p}. \end{aligned} \tag{13}$$

A *solution-measure* (or *weak solution*) to (12) with initial condition $\eta$ (a probability measure on $\mathbb{R}^d$) is a probability measure $P$ on $(\Omega, \mathscr{F})$(the canonical space) with the following property: there exists a stochastic basis $\mathscr{B}'$ with driving terms $(W, \mathfrak{p})$ and with a $\mathscr{F}'_0$-measurable variable $\xi$ meeting $\mathcal{L}(\xi) = \eta$, and a solution-process $Y$ on $\mathscr{B}'$, such that $P$ be the law of $Y$.

Note that if $Y$ is a solution-process then the above expression gives the canonical representation of $Y$:

$$Y_0 = \xi, \qquad Y^{i,c} = \sum_{j \le m} \gamma^{ij}(Y_-) \cdot W^j, \qquad h * (\mu^Y - \nu) = h \circ \delta(Y_-) * (\mathfrak{p} - q)$$

$$(x - h(x)) * \mu^Y = h' \circ \delta(Y_-) * \mathfrak{p}, \qquad B = \beta(Y) \cdot t.$$

**Theorem 3.2** *Let $\eta$ be an initial condition (a probability on $\mathbb{R}^d$), and $\beta$, $\gamma$, $\delta$ be coefficients as in (11). The set of all solution-measures to (12) with initial condition $\eta$ is the set $\mathfrak{s}(\mathscr{H}, X|\eta; B, C, \nu)$ of all solutions to a martingale problem on the canonical space, where $(B, C, \nu)$ are given by (10) with*

$$b = \beta, \qquad c = \gamma\gamma^T \qquad \left( i.e. \ c^{ij} = \sum_{1 \le k \le m} \gamma^{ik}\gamma^{jk} \right)$$

$$K_t(y, A) = \int 1_{A \setminus \{0\}} \left(\delta(t, y, z)\right) F(dz).$$

**Theorem 3.3** *Assume the following two conditions:*

*1.* Local Lipshitz coefficients. *For each $n \in \mathbb{N}^*$ there is a constant $\theta_n$ and a function $\rho_n : E \to \mathbb{R}_+$ with $\int \rho_n(z)^2 F(dz) < \infty$, such that for $t \le n$, $|y| \le n$, $|y'| \le n$:*

$$|\beta(s,y) - \beta(s,y')| \le \theta_n |y - y'|, \quad |\gamma(s,y) - \gamma(s,y')| \le \theta_n |y - y'|,$$

$$|h \circ \delta(s,y,z) - h \circ \delta(s,y',z)| \le \rho_n(z) |y - y'|,$$

$$|h' \circ \delta(s,y,z) - h' \circ \delta(s,y',z)| \le \rho_n(z)^2 |y - y'|.$$

*2.* Linear growth. *For each $n \in \mathbb{N}^*$ there are $\theta_n$ and $\rho_n$ as above, such that for all $t \le n$ and all $y \in \mathbb{R}^d$:*

$$|\beta(s,y)| \le \theta_n (1 + |y|), \quad |\gamma(s,y)| \le \theta_n (1 + |y|),$$

$$|h \circ \delta(s,y,z)| \le \rho_n(z) |y - y'|,$$

$$|h' \circ \delta(s,y,z)| \le [\rho_n(z)^2 \wedge \rho_n(z)^4](1 + |y|).$$

*Then (12) has a solution-process $Y$, and only one (up to indistinguishability) on any stochastic basis $\mathscr{B}'$ supporting driving terms $(W, \mathfrak{p})$.*

**Theorem 3.4** *Suppose that on any stochastic basis $\mathscr{B}'$ supporting driving terms $(W, \mathfrak{p})$, there is at most one solution-process (up to indistinguishability). Then, if there is a solution-measure, with a given initial condition, this solution- measure is unique.*

# 4 Changes of Measures

Our goal in this section is to characterize the behavior of a semimartingale after a change of probability measure. In particular, assume $X$ is a $P$-semimartingale and $P'$ is another probability measure such that $P' \overset{\text{loc}}{\ll} P$. Then Girsanov's theorem states that $X$ is also a $P'$-semimartingale and gives its $P'$-characteristics in terms of its $P$-characteristics. These results have important applications in finance as will be seen in Section 6.

The setting is the same as in Section 2. measurable space $(E, \mathscr{E})$. To every random measure $\mu$ on $\mathbb{R}_+ \times E$ defined on the stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ we associate the following:

$M^P_\mu$ is the positive measure on $(\tilde{\Omega}, \mathscr{F} \otimes \mathscr{R}_+ \otimes \mathscr{E})$ defined by $M^P_\mu(W) = \mathrm{E}(W * \mu_\infty)$ for all measurable nonnegative functions $W$.

Assume that $\mu$ is $\tilde{\mathscr{P}}$-$\sigma$-finite on $(\Omega, \mathscr{F}, \mathbf{F}, P)$. For every nonnegative measurable function $W$ we call the *conditional expectation relative to* $M^P_\mu$ with respect to $\tilde{\mathscr{P}}$ the $M^P_\mu$- a.e unique $\tilde{\mathscr{P}}$-measurable function $W' = M^P_\mu(W|\tilde{\mathscr{P}})$ such that

$$M^P_\mu(WU) = M^P_\mu(W'U)$$

for all nonnegative $\tilde{\mathscr{P}}$-measurable $U$.

**Theorem 4.1 (Girsanov's Theorem for Random Measures)** *Assume that $P' \overset{loc}{\ll} P$ and let $Z$ be the density process. Let $\mu$ be an integer-valued random measure on $\mathbb{R}_+ \times E$ defined on the stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ (this implies in particular that it is $\tilde{\mathscr{P}}$- $\sigma$-finite relative to $P$), and denote by $\nu$ its $P$-compensator.*

*a)* *$\mu$ is also $\tilde{\mathscr{P}}$-$\sigma$-finite relative to $P'$.*

*b)* *Let $Y$ be a $\tilde{\mathscr{P}}$-measurable nonnegative function on $\tilde{\Omega}$. Let $\nu'$ be a version of the $P'$-compensator of $\mu$. There is equivalence between:*

*i)* $\nu' = Y \cdot \nu$ $P'$-a.s. (where $Y \cdot \nu(\omega; dt, dx) = \nu(\omega; dt, dx) Y(\omega; dt, dx)$);

*ii)* $1_{\{Z_0>0\}} \cdot \nu' = Y 1_{\{Z_0>0\}} \cdot \nu$ $P$-a.s.;

*iii)* $YZ_-$ is a version of the conditional expectation $M^P_\mu(Z|\tilde{\mathscr{P}})$.

*Moreover, any nonnegative version $Y$ of $M^P_\mu\left(\frac{Z}{Z_-} 1_{\{Z_0>0\}}|\tilde{\mathscr{P}}\right)$ has the above properties.*

*c)* *There is a version of $\nu'$ that meets identically $\nu' = Y \cdot \nu$ for some $\tilde{\mathscr{P}}$-measurable nonnegative function $Y$.*

**Theorem 4.2 (Girsanov's Theorem for Semimartingales)** *Assume that $P' \overset{loc}{\ll} P$. There exist a $\tilde{\mathscr{P}}$-measurable nonnegative function $Y$ and a predictable process $\beta = (\beta^i)_{i \le d}$ satisfying*

$$|h(x)(Y-1)| * \nu_t < \infty \quad P'\text{-a.s. for } t \in \mathbb{R}_+,$$

$$\left|\sum_{j\le d} c^{ij}\beta^j\right| \cdot A < \infty, \text{ and } \left(\sum_{j,k\le d} \beta^j c^{jk} \beta^k\right) \cdot A_t < \infty \quad P'\text{-a.s. for } t \in \mathbb{R}_+,$$

*and such that a version of the characteristics of $X$ relative to $P'$ are*

$$\begin{aligned} B'^i &= B^i + \left(\sum_{j\le d} c^{ij}\beta^j\right) \cdot A + h(x)(Y-1) * \nu \\ C' &= C \\ \nu' &= Y \cdot \nu \end{aligned}$$

*Moreover, $Y$ and $\beta$ meet all of the above conditions, if and only if*

$$YZ_- = M^P_{\mu^X}(Z|\tilde{\mathscr{P}}) \tag{14}$$

$$\langle Z^c, X^{i,c}\rangle = \left(\sum_{j\le d} c^{ij}\beta^j Z_-\right) \cdot A \tag{15}$$

*up to a $P$-null set, where $Z$ is the density process, $Z^c$ is its continuous martingale part relative to $P$, and $\langle Z^c, X^{i,c}\rangle$ is the bracket relative to $P$.*

# 5 The Representation Property, Fundamental Representation Theorem

Let $(\Omega, \mathscr{F}, \mathbf{F}, P)$ be a stochastic basis supporting a semimartingale $X = (X^i)_{i\le d}$ with characteristics $(B, C, \nu)$ relative to a truncation function $h$. $X^c$ denotes the continuous martingale part of $X$, and $\mu = \mu^X$ is the random measure associated with the jumps of $X$.

We have seen that any local martingale can be decomposed in terms of a continuous part and a purely discontinuous part. In this section we look at the structure of the decomposition of a local martingale in terms of the given semimartingale $X$. We first state two related results for future reference.

**Proposition 5.1** *Let $M = X^c$ and $Z$ be an arbitrary local martingale.*

a) *There is a predictable process $H = (H^i)_{i \le d}$ such that*

$$[Z, M^i] = \langle Z^c, M^i \rangle = \left( \sum_{j \le d} c^{ij} H^j \right) \cdot A. \tag{16}$$

b) *Any predictable process meeting (16) belongs to $L^2_{loc}(M)$, and the stochastic integral $H \cdot M$ does not depend upon the chosen version of $H$, and $Y = Z - H \cdot M$ is orthogonal to all components of $M$ and*

$$[Y, M^i] = \langle Y^c, M^i \rangle = 0. \tag{17}$$

**Proposition 5.2** *Let $N$ be a local martingale, and $U = M^P_\mu(\Delta N \mid \tilde{\mathscr{P}})$ (here, $\Delta N$ is considered as defined on $\tilde{\Omega}$ by $\Delta N(\omega, t, x) = \Delta N_t(\omega)$).*

a) *There is a version of $U$ such that $\{a = 1\} \subset \{\hat{U} = 0\}$.*

b) *Let $W = U + \frac{\hat{U}}{1-a} 1_{\{a<1\}}$. Then $W \in G_{loc}(\mu)$, and if $Y = W * (\mu - \nu)$ and $Z = N - Y$, then we have $M^P_\mu(\Delta Z \mid \tilde{\mathscr{P}}) = 0$.*

$M^P_\mu(\Delta Z \mid \tilde{\mathscr{P}}) = 0$ may be interpreted as $Z$ being "orthogonal" to $\mu$, so $Y$ is a sort of *projection* of $N$ on $\mu$ (or, rather, on the space of all integrals of the form $V * (\mu - \nu)$).

We say that a local martingale $M$ has the *representation property relative to $X$* if it has the form

$$M = M_0 + H \cdot X^c + W * (\mu - \nu),$$

where $H = (H^i)_{i \le d}$ belongs to $L^2_{\mathrm{loc}}(X^c)$ and $W \in G_{\mathrm{loc}}(\mu)$.

**Lemma 5.1** *Every local martingale $M$ has a decomposition*

$$M = H \cdot X^c + W * (\mu - \nu) + N, \tag{18}$$

*where $H \in L^2_{loc}(X^c)$, $W \in G_{loc}(\mu)$, and*

$$\langle N^c, (X^c)^i \rangle = 0 \ \ \forall i \le d, \qquad M^P_\mu(\Delta N | \tilde{\mathscr{P}}) = 0. \tag{19}$$

*Moreover, this decomposition is unique, up to indistinguishability.*

**Corollary 5.1** *The following statements are equivalent:*

i) *All local martingales have the representation property.*

i) *All local martingales satisfying (19) are trivial (a local martingale $N$ is called* trivial *if $N_t = N_0$ a.s. for all $t \in \mathbb{R}_+$).*

iii) *All bounded martingales satisfying (19) are trivial.*

We now relate the representation property with the martingale problem. To this end, we introduce an initial condition: $\mathscr{H}$ is a sub $\sigma$-field of $\mathscr{F}_0$, and $P_H = P_{|\mathscr{H}}$ is the restriction of $P$ to $\mathscr{H}$. Note that $P \in \mathfrak{s}(\mathscr{H}, X | P_H; B, C, \nu)$. The following result implies that if all local martingales have the representation property then we can say something about the uniqueness of $P$.

**Theorem 5.1 (Fundamental Representation Theorem)** *In addition to the above, assume that $\mathscr{F} = \mathscr{F}_{\infty-}$. Then the following are equivalent:*

*(i) All local martingales have the representation property relative to $X$; moreover $\mathscr{F}_0$ is contained in the $\sigma$-field generated by $\mathscr{H}$ and the $P$-null sets of $\mathscr{F}$.*

*(ii) If $P' \in \mathfrak{s}(\mathscr{H}, X|P_H; B, C, \nu)$ and $P' \stackrel{loc}{\ll} P$, then $P' = P$.*

*(iii) If $P' \in \mathfrak{s}(\mathscr{H}, X|P_H; B, C, \nu)$ and $P' \ll P$, then $P' = P$.*

Note that if $\mathscr{H} = \mathscr{F}_0$ then the second condition in (i) is met and we have the following:

**Corollary 5.2** *Denote by $P_0$ the restriction of $P$ to $\mathscr{F}_0$. If $\mathscr{F} = \mathscr{F}_{\infty-}$ then the following are equivalent:*

*(i) All local martingales have the representation property relative to $X$.*

*(ii) If $P' \stackrel{loc}{\ll} P$ and $P'_0 = P_0$ and $X$ admits $(B, C, \nu)$ for $P'$-characteristics, then $P' = P$.*

*(iii) If $P' \ll P$ and $P'_0 = P_0$ and $X$ admits $(B, C, \nu)$ for $P'$- characteristics, then $P' = P$.*

# 6 Term Structure Models

Following Björk, Kabanov and Runggaldier [2] we will outline a model for the bond price dynamics in the context of a diffusion with jumps described above.

## 6.1 Introduction

We assume the canonical setting. Let $\mathsf{P}(t, T)$ be the price at time $t$ of a bond which matures at time $T$. It is assumed that for each $T > 0$, $(\mathsf{P}(t, T))_{0 \le t \le T}$ is an optional, $(\mathscr{F}_t)$-adapted process, and for each $t$, $\mathsf{P}(t, T)$ is $P$-a.s. continuously differentiable in the $T$ variable. Let $f(t, T)$ denote the *$T$-forward rate* at time $t$, defined by $f(t, T) = -\frac{\partial}{\partial T}\mathsf{P}(t, T)$. The *short rate* $r$ is defined by $r_t = f(t, t)$, and the *money account process* $\mathsf{B}$ is defined by

$$\mathsf{B}_t = \exp\left(\int_0^t r_s ds\right).$$

In order to model the bond price dynamics we could start with a description of the forward rate or short rate dynamics. Alternatively, we could follow a direct approach, obtaining $\mathsf{P}(t, T)$ as the solution of a stochastic differential equation. Therefore, we are interested in studying dynamics of the following forms:

$$dr_t = a_t dt + b_t dW_t + \int_E q(t, x)\mu(dt, dx) \tag{20}$$

$$d\mathsf{P}(t, T) = \mathsf{P}(t-, T)\left\{m(t, T)dt + v(t, T)dW_t + \int_E n(t, x, T)\mu(dt, dx)\right\} \tag{21}$$

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t + \int_E \delta(t, x, T)\mu(dt, dx). \tag{22}$$

The coefficients $b(t, T)$, $v(t, T)$, and $\sigma(t, T)$ are assumed to be $m$- dimensional row vector processes. The following technical assumptions will be needed:

Assumption

1. For any fixed $T > 0$, $n(t, x, T)$ and $\delta(t, x, T)$ are uniformly bounded. Furthermore, for each $t$,
$$\int_0^t \int_E h'(n(s, x, T))F(dx)ds < \infty,$$
where $h'(z) = |z|^2 \wedge |z|$ for $z \in \mathbb{R}$.

2. For each fixed $\omega$, $t$, and (in appropriate cases) $x$, all the objects $m(t,T)$, $v(t,T)$, $n(t,x,T)$, $\alpha(t,T)$, $\sigma(t,T)$ and $\delta(t,x,T)$ are assumed to be continuously differentiable in the $T$-variable. This partial $T$-derivative is denoted $m_T(t,T)$, etc.
3. All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.
4. For any $t$ the price curves $\mathsf{P}(\omega,t,T)$ are bounded functions for almost every $\omega$.

**Proposition 6.1** *If $f(t,T)$ satisfies (22), then $\mathsf{P}(t,T)$ satisfies*

$$d\mathsf{P}(t,T) = \mathsf{P}(t-,T)\Big[\Big(r_t + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\Big)dt + S(t,T)dW_t + \int_E \left(e^{D(t,x,T)} - 1\right)\mu(dt,dx)\Big],$$

*where*

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)ds \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \\ D(t,x,T) &= -\int_t^T \delta(t,x,s)ds. \end{aligned} \tag{23}$$

We could also introduce the models above by specifying the characteristics as in Section 3. By way of example, let $F$ be a Lévy measure on $E$ and define the set of characteristics $(B, C, \nu)$ by

$$\begin{aligned} B_t &= \int_0^t \mathsf{P}(s,T)\left(m(s,T) + \int_E n(s,x,T)F(dx)\right)ds \\ C_t &= \int_0^t \mathsf{P}(s,T)^2 v(s,T)v(s,T)^{\mathrm{T}} ds \\ \nu(dt,dx) &= \mathsf{P}(t,T)n(t,x,T)F(dx)dt. \end{aligned}$$

In order to express the corresponding stochastic differential equation, let $J(dt,dx)$ be the integer valued random measure constructed as in Section 2.1. We then have

$$\frac{d\mathsf{P}(t,T)}{\mathsf{P}(t-,T)} = m(t,T)dt + v(t,T)dW_t + \int_E n(t,x,T)J(dt,dx).$$

## 6.2 Example: Stable Driving Process

Let $X$ be a Lévy process on $\mathbb{R}$ with characteristics $(tb, tc, t\nu)$, where $b \in \mathbb{R}$, $c \geq 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$.

Let $T$ be an increasing Lévy process on $\mathbb{R}$ with characteristics $(t\beta, 0, t\rho)$. Here $\beta \geq 0$ and $\int_{(0,\infty)}(1 \wedge \xi)\rho(d\xi) < \infty$. We call the process $T$ a *subordinator*.

We now introduce the process $Y_t(\omega) = X_{T_t(\omega)}(\omega)$, $t > 0$, obtained by the *subordination* of $X$ by the subordinator $T$. A process identical in law to $Y$ is said to be a subordinate of $X$. It is a Lévy process, (see Sato [9]) with characteristics $(tb', tc', t\nu')$ where

$$\begin{aligned} b' &= \beta b + \int_{(0,\infty)} \rho(ds) \int_{|x|\leq 1} x P_{X_s}(dx) \\ c' &= \beta c \\ \nu'(dx) &= \beta\nu(dx) + \int_{(0,\infty)} P_{X_s}(dx)\rho(ds). \end{aligned} \tag{24}$$

In financial applications the subordinator $T$ can be interpreted as the market operational time. For example, it can be used to model the arrival of news which would imply changes in market activity.

Following Hurst, Platen and Rachev(1999) we present an example of this approach using a stable subordinator. Let $W$ be a Wiener process and let $T$ be an $\alpha/2$-stable Lévy process such that

$$T_{t+s} - T_t \sim S_{\alpha/2}(cs^{\alpha/2}, 1, 0), \quad s, t \geq 0.$$

The parameter $\alpha/2$ denotes the index of stability. The other three parameters represent scale, skewness and location, respectively. (See Samorodnitsky and Taqqu [8]) In terms of our notation this means that the set of characteristics of $T$ is $(0, 0, t\rho)$ where

$$\rho(d\xi) = c^{\alpha/2} 1_{(0,\infty)}(\xi) \frac{d\xi}{\xi^{1+\alpha/2}}.$$

It follows from (24) that the set of characteristics of the subordinated process $W_T$ is $(0, 0, t\nu')$ where

$$\nu'(dx) = \frac{(2c)^{\alpha/2}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha+1}{2}\right) \frac{dx}{|x|^{\alpha+1}}.$$

## 6.3 Bond Markets, Arbitrage

We now present the framework (Björk, Kabanov and Runggaldier [2]) in which we will state results concerning the absence of arbitrage in a model of bond prices. It will be assumed throughout that the filtration $\mathbf{F}$ is the natural filtration generated by $W$ and $\mu$.

A *portfolio* in the bond market is a pair $(g, h)$, where

1. $g$ is a predictable process.
2. For each $\omega$, $t$, $h_t(\omega, \cdot)$ is a signed finite Borel measure on $[t, \infty)$.
3. For each Borel set $A$ the process $h_t(A)$ is predictable.

The *discounted bond prices* $\overline{\mathsf{P}}(t, T)$ are defined by

$$\overline{\mathsf{P}}(t, T) = \frac{\mathsf{P}(t, T)}{\mathsf{B}_t}.$$

A portfolio $(g, h)$ is said to be *feasible* if the following conditions hold for every $t$:

$$\int_0^t |g_s| ds, \qquad \int_0^t \int_s^\infty |m(s, T)||h_s(dT)| ds < \infty,$$

$$\int_0^t \int_s^\infty \int_E |n(s, x, T)||h_s(dT)| \nu(ds, dx) < \infty,$$

$$\text{and} \quad \int_0^t \left\{ \int_s^\infty |v(s, T)||h_s(dT)| \right\}^2 ds < \infty.$$

The *value process* corresponding to a feasible portfolio $\pi = (g, h)$ is defined by

$$V_t^\pi = g_t \mathsf{B}_t + \int_t^\infty \mathsf{P}(t, T) h_t(dT).$$

The *discounted value process* is

$$\overline{V_t^\pi} = \mathsf{B}_t^{-1} V_t^\pi.$$

A feasible portfolio is said to be *admissible* if there is a number $a \geq 0$ such that $V_t^\pi \geq -a$ $P$-a.s. for all $t$.

A feasible portfolio is said to be *self-financing* if the corresponding value process satisfies

$$\begin{aligned}V_t^\pi = V_0^\pi &+ \int_0^t g_s d\mathsf{B}_s + \int_0^t \int_s^\infty m(s,t)\mathsf{P}(s,t)h_s(dT)ds \\ &+ \int_0^t \int_s^\infty v(s,t)\mathsf{P}(s,t)h_s(dT)dW_s \\ &+ \int_0^t \int_s^\infty \int_E n(s,x,T)\mathsf{P}(s-,t)h_s(dT)\mu(ds,dx).\end{aligned}$$

The preceding relation can be interpreted formally as follows:

$$dV_t^\pi = g_t d\mathsf{B}_t + \int_s^\infty h_t(dT)d\mathsf{P}(t,T).$$

A *contingent T-claim* is a random variable $X \in L_+^0(\mathscr{F}_T, P)$. An *arbitrage portfolio* is an admissible self-financing portfolio $\pi = (g,h)$ such that the corresponding value process satisfies

1. $V_0^\pi = 0$
2. $V_T^\pi \in L_+^0(\mathscr{F}_T, P)$ with $P(V_T^\pi > 0) > 0$.

If no arbitrage portfolios exist for any $T > 0$ we say that the model is *arbitrage-free*.

Take the measure $P$ as given. We say that a positive martingale $M = (M_t)_{t \geq 0}$ with $\mathrm{E}^P[M_t] = 1$ is a *martingale density* if for every $T > 0$ the process $(\overline{\mathsf{P}}(t,T)M_t)_{0 \leq t \leq T}$ ia a $P$- local martingale. If, moreover, $M_t > 0$ for all $t > 0$ we say that $M$ is a *strict martingale density*.

We say that that a probability measure $Q$ on $(\Omega, \mathscr{F})$ is a *martingale measure* if $Q_t \sim P_t$ and the process $(\overline{\mathsf{P}}(t,T))_{0 \leq t \leq T}$ is a $Q$-local martingale for every $T > 0$. Here $Q_t$, $P_t$ are the restrictions $Q_{|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$, respectively.

**Proposition 6.2** *Suppose that there exists a strict martingale density. Then the bond market model is arbitrage-free.*

We will make the following simplifying assumption:

Assumption For any positive martingale $N = (N)_t$ with $\mathrm{E}^P(N_t) = 1$ there exists a probability measure $Q$ on $\bigcup_{t \geq 0} \mathscr{F}_t$ such that $N_t = dQ_t/dP_t$.

The following results relate the coefficients in (21) and (22) with a model free of arbitrage.

**Theorem 6.1** *Let the bond price dynamics be given by (21). There exists a martingale measure if and only if the following conditions hold:*

*(i) There exists a predictable process $\phi$ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega,t,x)$ with $Y > 0$ satisfying*

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx)ds < \infty.$$

*and such that $\mathrm{E}^P(\mathscr{E}(L)_t) = 1$ for all finite $t$, where the process $L$ is defined by*

$$L = \phi \cdot W + (Y-1) * (\mu - \nu).$$

*(ii) For all $T > 0$, and $t \in [0, T]$ we have*

$$m(t,T) + \phi_t v(t,T)^T + \int_E Y(t,x) n(t,x,T) F(dx) = r_t. \tag{25}$$

The following theorem gives a similar result when we consider the forward rate dynamics.

**Theorem 6.2** *Let the forward rate dynamics be given by (22). There exists a martingale measure if and only if the following conditions hold:*

*(i) There exists a predictable process $\phi$ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega, t, x)$ with $Y > 0$ satisfying*

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx) ds < \infty.$$

*and such that $E^P(\mathscr{E}(L)_t) = 1$ for all finite $t$, where the process $L$ is defined by*

$$L = \phi \cdot W + (Y - 1) * (\mu - \nu).$$

*(ii) For all $T > 0$, and $t \in [0, T]$ we have*

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 + \phi_t S(t,T)^T + \int_E Y(t,x) \left(e^{D(t,x,T)} - 1\right) F(dx) = 0,$$

*where $A$, $S$ and $D$ are defined in (23).*

Since $n(\cdot, \cdot, T)$ in (21) is uniformly bounded, we can assume $h(x) = x$ in (13) to obtain the canonical representation

$$\begin{aligned}
\frac{d\mathsf{P}(t,T)}{\mathsf{P}(t-,T)} &= \left(m(t,T) + \int_E n(t,x,T) F(dx)\right) dt + v(t,T) dW_t \\
&\quad + \int_E n(t,x,T)(\mu(dt,dx) - \nu(dt,dx)).
\end{aligned}$$

By Theorem 3.2 the $P$ characteristics of $\mathsf{P}(\cdot, T)$ are

$$\begin{aligned}
B_t &= \int_0^t \mathsf{P}(s,T) \left(m(s,T) + \int_E n(s,x,T) F(dx)\right) ds \\
C_t &= \int_0^t \mathsf{P}(s,T)^2 v(s,T) v(s,T)^{\mathrm{T}} ds \\
w(dt,dx) &= \mathsf{P}(t,T) n(t,x,T) F(dx) dt.
\end{aligned}$$

**Lemma 6.1** *Let $P'$ be a probability measure such that $P' \overset{loc}{\ll} P$.*

*a) $\overline{\mathsf{P}}(\cdot, T)$ is a $P'$ local martingale if and only if*

$$m(t,T) + \mathsf{P}(t,T) v(t,T) v(t,T)^T \beta_t + \int_E Y(\omega,t,x) n(t,x,T) F(dx) = r_t$$

*where $\beta$, $Y$ are given by Theorem 4.2.*

*b) Let $Z$ be the density process. Then if $\phi$ is a predictable process such that $Z^c = \mathscr{E}(\int_0^\cdot \phi_s dW_s)$ then the condition in a) is equivalent to*

$$m(t,T) + \phi_t v(t,T)^T + \int_E Y(\omega,t,x) n(t,x,T) F(dx) = r_t.$$

Proof of Lemma. Since $P' \overset{\text{loc}}{\ll} P$ then $\mathsf{P}$ is a $P'$ semimartingale with characteristics $(B', C', w')$, where

$$B' = \int_0^{\cdot} \mathsf{P}(t,T)\Big[m(t,T) + \int_E n(t,x,T)F(dx) + \mathsf{P}(t,T)v(t,T)v(t,T)^{\mathrm{T}}\beta_t \\ + \int_E (Y(\omega,t,x)-1)n(t,x,T)F(dx)\Big]dt. \tag{26}$$

The expression in brackets reduces to

$$m(t,T) + \mathsf{P}(t,T)v(t,T)v(t,T)^{\mathrm{T}}\beta_t + \int_E Y(\omega,t,x)n(t,x,T)F(dx).$$

Therefore, $\overline{\mathsf{P}}(\cdot,T)$ is a $P'$ local martingale if the last expression is (a.s.) equal to $r_t$.

Since $dZ^c = Z_-\phi \cdot W$ and $d\mathsf{P}^c = \mathsf{P}_- v \cdot W$ then by the definition of $\beta$,

$$\beta = \frac{d\langle Z^c, \mathsf{P}^c\rangle}{d\langle \mathsf{P}^c, \mathsf{P}^c\rangle}\frac{1}{Z_-} = \frac{\phi v^{\mathrm{T}}}{\mathsf{P}_- v v^{\mathrm{T}}}.$$

The condition in b) now follows from a).

Proof of Theorem 6.1. (Necessity) Since $P' \overset{\text{loc}}{\ll} P$, there is a predictable process $\phi$ and a non-negative $\tilde{\mathscr{P}}$-measurable function $Y$ such that

$$\langle Z^c, W\rangle = (\phi Z_-)\cdot A, \text{ and } YZ_- = M^P_\mu(Z|\tilde{\mathscr{P}}) \quad P\text{- a.s.}$$

Then $\phi Z_- \in L^2_{\text{loc}}(W)$ by Proposition 5.1. Let $\xi = Z - (\phi Z_-)\cdot W$. It follows that $Z = \xi + (\phi Z_-)\cdot W$ and $\langle \xi^c, W\rangle = 0$. We also have that $M^P_\mu(\Delta Z|\tilde{\mathscr{P}}) = YZ_- - Z_- = Z_-(Y-1)$. Since $(\phi Z_-)\cdot W$ is continuous,

$$\Delta\xi = (Z - (\phi Z_-)\cdot W) - (Z_- - (\phi Z_-)\cdot W) = Z - Z_- = \Delta Z$$

so that $M^P_\mu(\Delta\xi|\tilde{\mathscr{P}}) = M^P_\mu(\Delta Z|\tilde{\mathscr{P}}) = Z_-(Y-1)$. By Proposition 5.2,
$Z_-(Y-1) \in G_{\text{loc}}(\mu)$, and if we let $\eta = \xi - Z_-(Y-1) * (\mu - \nu)$ then $M^P_\mu(\Delta\eta|\tilde{\mathscr{P}}) = 0$.
In summary, we have

$$Z = \xi + (\phi Z_-)\cdot W$$

and

$$\xi = \eta + Z_-(Y-1) * (\mu - \nu),$$

so that

$$Z = (\phi Z_-)\cdot W + Z_-(Y-1) * (\mu-\nu) + \eta \tag{27}$$

where $\eta$ is a local martingale such that $M^P_\mu(\Delta\eta|\tilde{\mathscr{P}}) = 0$, and $\langle \eta^c, W\rangle = \langle \xi^c, W\rangle = 0$.

Let $R_n = \inf\left(t : Z < \frac{1}{n}\right)$ and define a process $H$ by

$$H_t = |\phi|^2 \cdot A_t + (1-\sqrt{Y})^2 * \nu_t.$$

Since $\frac{1}{Z_-}1_{[\![0,R_n]\!]} \le n$ by definition of $R_n$ we have that

$$\phi Z_- \in L^2_{\text{loc}}(W) \Rightarrow \phi 1_{[\![0,R_n]\!]} \in L^2_{\text{loc}}(W)$$

and

$$Z_-(Y-1) \in G_{\text{loc}}(\mu) \Rightarrow (Y-1)1_{[\![0,R_n]\!]} \in G_{\text{loc}}(\mu).$$

Using Theorem 2.2 (note $C'(Y-1) = (1-\sqrt{Y})^2 * \nu$), conclude that $H_{R_n \wedge t} < \infty$ $P$-a.s. for all $t$. Since $P'$, $P$ are locally equivalent by hypothesis, $Z > 0$ $P$-a.s. This implies that $R_n \to \infty$

a.s. as $n \to \infty$, hence $H < \infty$ a.s.

It follows from the definition of the driving terms $(W, \mu)$ and Theorem 2.3 that the characteristics are deterministic. We can then apply Theorem 3.1 and Corollary 5.2 to conclude that all local martingales have the representation property relative to $(W, \mu)$. Since the decomposition (27) was constructed so that (19) is satisfied then Corollary 5.1 implies that $\eta_t = Z_0 (= 1)$ a.s. for all $t$. It follows that $\mathrm{E}^P(\mathscr{E}(L)) = 1$. Since $\overline{\mathsf{P}}$ is a $P'$-local martingale by hypothesis, then the Lemma implies (25).

(Sufficiency) By hypothesis, we can define a probability measure $P'_t = Z_t P_t$ for each $t$, where $P_t = P_{|\mathscr{F}_t}$ and $Z = \mathscr{E}(L)$. The process $Z$ satisfies

$$\begin{aligned} Z &= 1 + (Z_-\phi) \cdot W + Z_-(Y-1) * (\mu - \nu), \\ \Delta Z &= Z_-(Y-1)\ M^P_\mu\text{-a.s.} \end{aligned}$$

Since the right hand side of the last expression is $\tilde{\mathscr{P}}$-measurable, then $M^P_\mu(\Delta Z | \tilde{\mathscr{P}}) = Z_-(Y-1)$ and hence (14) is satisfied. Denoting the $P'$- compensator of $\mu$ by $\nu' = \nu'(dt, dx)$, Theorem 4.1 implies that

$$\nu' = Y\nu \ \ P'\text{-a.s.}$$

Multiplying both sides by $\mathsf{P}(\cdot, T)n(\cdot, \cdot, T)$ gives $w' = Yw$, where $w' = w'(dt, dx)$ is the $P'$-compensator of $w(dt, dx) = \mathsf{P}(t-, T)n(t, x, T)\mu(dt, dx)$. Since $\mathsf{P}(\cdot, T)n(\cdot, \cdot, T)$ is uniformly bounded by hypothesis, we can apply Theorem 4.1 again to conclude that

$$YZ_- = M^P_w(Z | \tilde{\mathscr{P}}) \text{ a.s.}$$

We observe that (15) is satisfied, so $\mathsf{P}(\cdot, T)$ is a $P'$-semimartingale and its first $P'$-characteristic $B'$ is given by (26). The result then follows from the Lemma above.

## 6.4 Application: Defaultable Bonds

In this subsection we apply our results to derive sufficient conditions for absence of arbitrage in a market for a zero coupon bond subject to default risk. We follow Schönbucher [10]. In order to model the default times and their associated loss quotas, we introduce a random measure $\mu_d$, independent of $\mu$, with compensator $\nu_d$ defined by

$$\nu_d(dt, dq) = q\lambda_t K(dq)dt,$$

where the intensity $\lambda_t$ is continuous and finite for each $t$, and $K(\cdot)$ is a finite measure on $E_d = [0, 1]$.

Under the canonical setting, let the filtration $\mathbf{F}$ be generated by $(W, \mu, \mu_d)$. We begin with the dynamics of the *defaultable forward rates* $\overline{f}(t, T)$, with coefficients $\overline{\alpha}$, $\overline{\sigma}$, $\overline{\delta}$ defined as in the default-free case:

$$d\overline{f}(t, T) = \overline{\alpha}(t, T)dt + \overline{\sigma}(t, T)dW_t + \int_E \overline{\delta}(t, x, T)\mu(dt, dx). \tag{28}$$

We also define the *defaultable short rate* by $\overline{r}_t = \overline{f}(t, t)$. The defaultable zero coupon bond is defined as follows:

$$\mathsf{R}(t, T) = \left(1 - \int_0^1 q\mu_d(dt, dq)\right) \exp\left(-\int_t^T \overline{f}(t, s)ds\right).$$

As in the risk-free case, we obtain the defaultable bond dynamics

$$\begin{aligned} d\mathsf{R}(t, T) = \mathsf{R}(t-, T)\bigg[ &\left(\overline{r}_t + \overline{A}(t, T) + \frac{1}{2}\|\overline{S}(t, T)\|^2\right) dt + \overline{S}(t, T)dW_t \\ &+ \int_E \left(e^{\overline{D}(t,x,T)} - 1\right)\mu(dt, dx) - \int_{[0,1]} q\mu_d(dt, dq)\bigg], \end{aligned}$$

where

$$\begin{aligned}\overline{A}(t,T) &= -\int_t^T \overline{\alpha}(t,s)ds \\ \overline{S}(t,T) &= -\int_t^T \overline{\sigma}(t,s)ds \\ \overline{D}(t,x,T) &= -\int_t^T \overline{\delta}(t,x,s)ds.\end{aligned} \tag{29}$$

Since we assume the absence of arbitrage in the risk-free case, we obtain from the theorem the probability $P'$ under which $\frac{\mathsf{P}(\cdot,T)}{\mathsf{B}}$ is a local martingale. The set of $P'$-characteristics of $\mathsf{R}(\cdot,T)$ is

$$\begin{aligned}dB_t &= \mathsf{R}(t,T)\left(m(t,T)+\int_E Y(t,x)n(t,x,T)F(dx)-\int_{[0,1]} q\lambda_t K(dq)\right)dt \\ dC_t &= \mathsf{R}(t,T)^2\|\overline{S}(t,T)\|^2 dt \\ w(dt,dx,dq) &= \mathsf{R}(t,T)\Big(Y(t,x)n(t,x,T)F(dx)-q\lambda_t K(dq)\Big)dt,\end{aligned}$$

where

$$m(t,T)=\overline{r}_t+\overline{A}(t,T)+\frac{1}{2}\|\overline{S}(t,T)\|^2+\phi_t\overline{S}(t,T)^T,$$

and

$$n(t,x,T)=\left(e^{\overline{D}(t,x,T)}-1\right).$$

It follows from the proof of the theorem that for absence of arbitrage in this case we require the existence of an appropriate random variable $Y_d$ from which we can construct an equivalent martingale measure $P''$ using $\mathscr{E}(L_d)$ where

$$L_d=(Y_d-1)*(\mu_d-\nu_d). \tag{30}$$

We now seek conditions on the first $P''$-characteristic $B''$ of $\mathsf{R}$ so that $\frac{\mathsf{R}(\cdot,T)}{\mathsf{B}}$ is a local martingale. We have that

$$\begin{aligned}dB''_t = \mathsf{R}(t-,T)\Big[&\overline{r}_t+\overline{A}(t,T)+\frac{1}{2}\|\overline{S}(t,T)\|^2+\phi_t\overline{S}(t,T)^T \\ &+\int_E Y(t,x)\left(e^{\overline{D}(t,x,T)}-1\right)F(dx)-\int_{[0,1]} Y_d(t,q)q\lambda_t K(dq)\Big]dt.\end{aligned}$$

Hence we require the following conditions:

$$\begin{aligned}&\overline{A}(t,T)+\|\overline{S}(t,T)\|^2+\phi_t\overline{S}(t,T)^T \\ &\qquad +\int_E Y(t,x)\left(e^{\overline{D}(t,x,T)}-1\right)F(dx)=0\end{aligned} \tag{31}$$

and

$$0<\overline{r}_t-r_t=\int_{[0,1]} Y_d(t,q)q\lambda_t K(dq). \tag{32}$$

The last inequality is a formal relationship between between the default-free and defaultable models necessary for absence of arbitrage. The following result summarizes the above discussion.

**Theorem 6.3** *Let the defaultable forward rate dynamics be given by (28). There exists a martingale measure if the following conditions hold:*

*(i) The conditions in Theorem 6.2 hold for the default-free forward rates.*

*(ii) There exists a predictable function* $Y_d(t,q)$ *with* $Y_d > 0$ *satisfying*

$$\int_0^t \int_{[0,1]} |Y(s,q)| K(dq) ds < \infty, \quad \textit{and} \quad E^{P'}(\mathscr{E}(L_d)_t) = 1$$

*for all finite* $t$. *Here* $L_d$ *is given by (30).*

*(iii) The coefficients* $\overline{A}(t,T)$, $\overline{S}(t,T)$ *and* $\overline{D}(t,x,T)$ *satisfy (31) and (32).*

# 7 Concluding Remarks

We have described a mathematical framework in which to study term structure models. All processes considered are semimartingales, and we represent them in terms of their set of characteristics. In particular, the necessary and sufficient conditions for absence of arbitrage in a bond market are presented in terms of the characteristics of the price process. As an example, we presented a stable process as the underlying source of randomness in terms of its characteristics. We also illustrate how the general methodology can be extended to defaultable bonds. In future work we will seek the development of estimation and numerical procedures following this approach.

# References


[1] S. Babbs, N. Webber, "A Theory of the Term Structure with an official Short Rate", preprint, University of Warwick, (1994).

[2] T. Björk, Y. Kabanov, W. Runggaldier, "Bond Market Structure in the Presence of Marked Point Processes", *Math. Finance* 7, 211-239, (1997).

[3] L. El-Jahel, H. Lindberg, W. Perraudin, "Yield Curves with Jump Short Rates", preprint, (2001).

[4] D. Heath, R. Jarrow, A. Morton, "Bond Pricing and the Term Structure of Interest Rates", *Econometrica* 60(1), 77-106, (1992).

[5] J.L. Hernández, "A General Framework for Term Structure and Credit Risk Models Driven by Lévy Processes", Ph.D. dissertation, University of California, Santa Barbara (2003).

[6] S.R. Hurst, E. Platen, S.T. Rachev, "Option Pricing for a Logstable Asset Price Model", *Mathematical and Computer Modelling* 29, 105-119, (1999).

[7] R. Jarrow, D. Madan, "Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns", *Math. Finance* 5, 311-336, (1995).

[8] G. Samorodnitsky, M.S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York, (1994).

[9] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, (2000).

[10] P.J. Schönbucher, *Credit Risk Modelling and Credit Derivatives*, Ph.D. dissertation, University of Bonn, (2000).

[11] H. Shirakawa, "Interest Rate Option Pricing with Poisson-Gaussian Forward Rate Curve Processes", *Math. Finance* 1, 77-94, (1991).

[12] A.N. Shiryaev, J. Jacod, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, (1987).

[13] A.N. Shiryaev, *Essentials of Stochastic Finance: Facts, Models, Theory*, World Scientific, Singapore, (1999).