Credit Portfolio Risk and PD Confidence Sets through the Business Cycle

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Abstract

Transition matrices are an important determinant for risk management and VaR calculations in credit portfolios. It is well known that rating migration behavior is not constant through time. It shows cyclical behavior and significant changes over the years. We investigate the effect of changes in migration matrices on credit portfolio risk in terms of Expected Loss and Value-at-Risk figures for exemplary loan portfolios. The estimates are based on historical transition matrices for different time horizons and a continuous-time simulation procedure. We further determine confidence sets for the probability of default (PD) in different rating classes by a bootstrapping methodology. Our findings are substantial changes in VaR as well as for the width of estimated PD confidence intervals.

Keywords: Credit VaR, Transition Matrices, Rating Migration, Business Cycle, Continuous-time Modeling, PD estimation

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1 Introduction

It is widely accepted that rating migrations and default probabilities show significant changes through time that have impact on the loss distribution of a credit portfolio. However, despite this observed relationship the literature is rather sparse on modeling this issue. In many existing models default and transition probabilities are modeled by using an average historical migration matrix without including a variable determining the state of the economy. Also for the second major determinant of credit risk, recovery rates, most models assume them to be constant through time, dependent only on the seniority grade of the issue. In the last decade, especially rating based models in credit risk management have become very popular. These systems use the rating of a company as the decisive variable to evaluate the default risk of a bond or loan. The popularity is due to the straightforwardness of the approach but also to the upcoming new capital accord (Basel II) of the Basel committee on banking supervision (5), a regulatory body under the bank of international settlements. Basel II allows banks to base their capital requirements on internal as well as external rating systems. Thus, sophisticated credit risk models are being developed or demanded by banks to assess the risk of their credit portfolio better by recognizing the different underlying sources of risk. As a consequence, default probabilities for certain rating categories but also the probabilities for moving from one rating state to another are important issues in such models for risk management and pricing.

Systematic changes in migration matrices have substantial effects on credit Value-at-Risk (VaR) of a portfolio or prices of credit derivatives like Collaterized Debt Obligations (CDOs). Macroeconomic conditions and the business cycle may be a reason for such systematic changes. Still, despite the obvious importance of recognizing the impact of business cycles on rating transitions, the literature is rather sparse on this issue. Helwege and Kleiman (23) as well as Alessandrini (1) have shown respectively that default rates and credit spreads clearly depend on the stage of the business cycle. An extensive study on differences in migration matrices was conducted by Bangia et al (4). Examining the stability of Standard & Poor's migration matrices through the business cycle they found time inhomogeneity and second order Markov behavior. Further, by separating the economy into two states or regimes, expansion and contraction, and conditioning the migration matrix on these states, Bangia et al. the authors show significant differences in the loss distribution of credit portfolios.

	Ba	B	C
Ba	1.0000	0.7477	0.4980
B	0.7477	1.0000	0.5865
C	0.4980	0.5865	1.0000

Table 1: Correlation between Default Frequencies for Speculative Grade Rating Classes Ba-C, 1984-2001

In this paper we investigate the effect of changes in migration behavior on Value-at-Risk and PD confidence sets for credit portfolios. We also investigate the link between the state of the economy and default risk or migration behavior in the industry. Intuition gives the following view: When the economy worsens both downgrades as well as defaults will increase. The contrary should be true when the economy becomes stronger. Figures 1 shows the Moody's historical default frequencies for non-investment grade bonds of rating class B for the years from 1984 to 2001. Clearly there is a high deviation from the average. For B rated bonds the default frequencies range from 2.5% in 1993 in high market times to more than 14% in 1991 where there was a deep recession in the American economy. We conclude that taking average default probabilities of a longer time horizon as estimators for future default probabilities might not give correct risk estimates for a portfolio.

Investigating the question of default correlation for the speculative rating classes we also get significant results. Correlations are between 0.5 and 0.74, indicating that speculative grade issuers tend to show similarities in default behavior. This default correlation could be interpreted as a concerted reaction of sub-investment grade bonds and loans to changes to the macroeconomic situation. Clearly this will affect the risk of a portfolio, since defaults in speculative grade loans or bonds have a tendency to be clustered.

For investment grade issuers variations in PDs are less significant. This is illustrated by figure 2 where Moody's historical default frequencies for rating class A issues are reported. We find that only for five years of the considered time horizon defaults could be observed. Empirical studies (28) have shown that the link between defaults or changes in migration between investment grade issuers and the cycle is less obvious than for speculative grade issuers.

In this paper, we want to investigate the substantial effects of changes in migration behavior on expected loss, VaR and also on confidence inter-

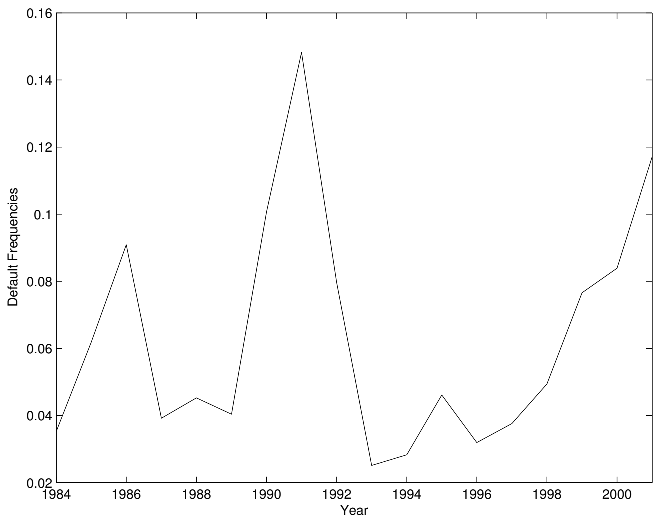


Figure 1: Moody's historical defaults rates for rating class B and time horizon 1984-2001.

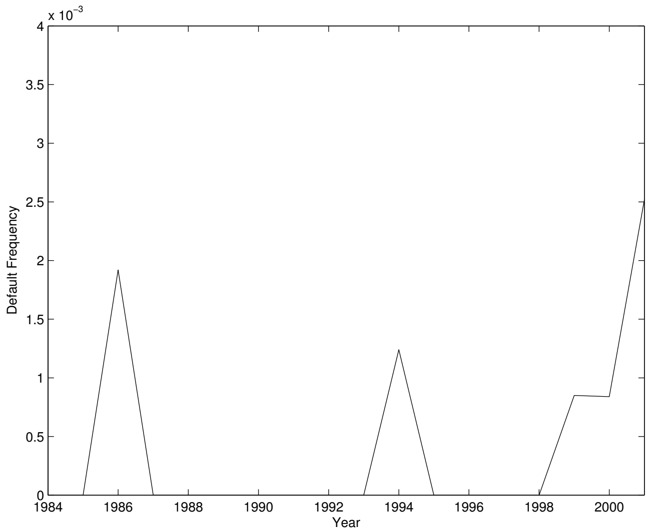


Figure 2: Moody's historical defaults rates for rating class A and time horizon 1984-2001.

vals for PDs. We will consider the influence of such changes in migration behavior on capital requirements in terms of expected losses and VaR figures for a exemplary Credit Portfolio. We provide evidence that for a considered exemplary portfolio these numbers vary substantially and that the effect of different migration behavior through the cycle should not be ignored in credit risk management.

We also address the issue of how to estimate the probability of default with publicly available credit ratings and compare the behavior of confidence intervals for such default probabilities through time. Especially for investment grade rating classes, since default is a very rare event, PDs are rather noisy through time and it is difficult to obtain confidence levels for PDs. Using a bootstrap method suggested by Lando and Christensen (30) we are able to get tighter confidence intervals than with the standard Wald estimator.

Section two gives a brief review rating based approach to credit risk with focus on continuous-time modeling of rating migrations. Section three provides evidence on the substantial effect on different migration behavior for credit VaR. Section four illustrates how confidence sets for rare PD events can be determined using a bootstrap methodology and investigates changes in PD volatility estimates thorough time. Section five concludes.

2 Modeling of Rating Migrations

In later sections we will deal with continuous-time Markov chains in order to determine the Value-at-Risk of a credit portfolio or bootstrapping of PD confidence sets. Therefore this section is dedicated to rating-based modeling with focus on continuous-time modeling of rating transitions and generator matrices.

2.1 Discrete versus Continuous-Time Modeling

Jarrow, Lando and Turnbull (JLT) model default and transition probabilities by using a discrete, time-homogeneous Markov chain on a finite state space $S = \{1, ..., K\}$ (27). The state space S represents the different rating classes. While state $S = 1$ denotes the best credit rating, state K represents the default case. Hence, the (KxK) one-period transition matrix looks as follows:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{12} & \cdots & p_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ p_{K-1,1} & p_{K-1,2} & \cdots & p_{K-1,K} \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{2.1}$$

where $p_{ij} \geq 0$ for all $i, j, i \neq j$, and $p_{ii} \equiv 1 - \sum_{\substack{j=1 \\ j \neq i}}^{K} p_{ij}$ for all i. The variable p_{ij} represents the actual probability of going to state j from initial rating state i in one time step.

Thus, rating based models can be seen as a special case of the so-called 'intensity model framework' (17) where randomness in the default arrival is simply modeled via a Markov chain. For practical purposes they recommend to use the continuous-time approach for rating migrations.

A continuous-time time-homogeneous Markov chain is specified via a KxK generator matrix of the following form:

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1,K} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.2}$$

where $\lambda_{ij} \geq 0$, for all i, j and $\lambda_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{K} \lambda_{ij}$, for $i = 1, ...K$. The off-diagonal elements represent the intensities of jumping from rating i to rating j. The default K is an absorbing state.

Definition 2.1 *Noris (35): A generator of a time-continuous Markov chain is given by a matrix $\Lambda = (\lambda_{ij})_{1 \leq i,j \leq k}$ satisfying the following properties:*

1. $\sum_{j=1}^{8} \lambda_{ij} = 0$ *for every* $i = 1, \cdots, K$;
2. $0 \leq -\lambda_{ii} \leq \infty$ *for every* $i = 1, \cdots, K$;
3. $\lambda_{ij} \geq 0$ *for all* $i, j = 1, \cdots, K$ *with* $i \neq j$.

Further, see Noris (35), the following theorem holds:

Theorem 2.2 *The following two properties are equivalent for matrix $\Lambda \in \mathbb{R}^{k \times k}$ satisfying the following properties:*

1. *Λ satisfies the Properties in Definition 2.1.*
2. *$exp(t\Lambda)$ is a transition matrix for every $t \geq 0$.*

Hence, the KxK t-period transition matrix is then given by:

$$P(t) = e^{t\Lambda} = \sum_{k=0}^{\infty} \frac{(t\Lambda)^k}{k!} = \mathcal{I} + (t\Lambda) + \frac{(t\Lambda)^2}{2!} + \frac{(t\Lambda^3)}{3!} + \cdots \tag{2.3}$$

For example consider the transition matrix P

P	A	B	D
A	0.90	0.08	0.02
B	0.1	0.80	0.1
D	0	0	1

where the corresponding generator matrix is of the form:

Λ	A	B	D
A	-0.1107	0.0946	0.0162
B	0.1182	-0.2289	0.1107
D	0	0	0

The use of generator matrices in credit risk is manifold. A main issue is for example the construction of so-called credit curves, giving information about cumulative default rates, see e.g. (27). For a given generator matrix Λ the cumulative default rate PD_t^i for rating class i is given by the K-th entry of the vector:

$$p_t^i = exp(t\Lambda)x_i^t \tag{2.4}$$

where x_i^t denotes the row of the corresponding transition matrix to the given rating R. Figure 2.1 shows a chart of the credit curves to the corresponding matrix P on our example.

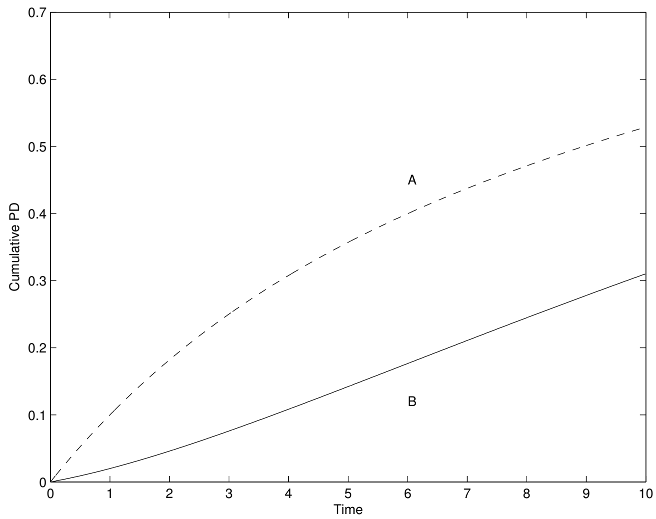


Figure 3: Credit curves for exemplary transition matrix with three rating states. Cumulative PDs for rating class A (dashed) and B.

Lando and Skodeberg (32) as well as Lando and Christensen (30) focus on the advantages of the continuous-time modeling over the discrete-time approach used by rating agencies in order to analyze rating transition data. Generally, rating agencies estimate transition probabilities using the multinomial method by computing

$$\hat{p}_{ij} = \frac{N_{ij}}{N_i} \tag{2.5}$$

for $j \neq i$. Where N_i is the number of firms in rating class i at the beginning of the year and N_{ij} is the number of firms that migrated from class i to rating class j.

The authors argue that these transition probabilities do not capture rare events such as a transition from rating AAA to default as they may not be observed. However, it is possible that a firm reaches default through subsequent downgrades from AAA - even within one year and the probability of moving from AAA to default must be non-zero. Following Küchler and Sorensen (29) maximum-likelihood estimator for the continuous-time method is proposed:

$$\hat{\lambda}_{ij} = \frac{N_{ij}(T)}{\int_0^T Y_i(s)ds}. \tag{2.6}$$

The variable $Y_i(s)$ denotes the number of firms in rating class i at time s and $N_{ij}(T)$ is the total number of transitions over the period from i to j, where $i \neq j$. Under the assumption of time-homogeneity over the considered period the transition matrix for a time interval t can be computed by the formula $P(t) = e^{t\Lambda}$.

Lando and Skodeberg point out that with the continuous-time approach one obtains strictly positive default probabilities also for rating classes where no direct default observation could be observed in the considered periods. However, this makes sense as for example default may happen through a consecutive downgrade of a company usually rated in the investment grade area. Thus, also for investment grade issuers default probabilities should be non-zero.

Summarizing the key advantages of the continuous-time approach we find that we get more realistic non-zero estimates for probabilities of rare events, whereas the multinomial method leads to estimates that are zero. Further, using generator matrices it is also possible to obtain transition matrices for arbitrary time horizons. We will see in the sequel that the continuous-time framework also permits to generate confidence sets for default probabilities in higher rating classes. Finally, in the continuous-time approach we do not have to worry which yearly periods we consider. Using a discrete-time approach may lead to quite different results depending on the starting point of our consideration.

However, the last issue is also a critical point for the estimation of generator matrices. In internal rating systems it is often the case that rating changes are reported only once a year and that the exact time of the change is not provided. Then it is not appropriate to use the maximum-likelihood or the Nelson-Aalen estimator for estimation of the transition matrix.

2.2 Applications of Generator Matrices

So far we have described the basic ideas of rating based credit risk evaluation methods and the advantages of continuous-time transition modeling over the discrete-time case. Despite these advantages of continuous-time modeling, there are also some problems to deal with, like the existence, uniqueness or adjustment of the generator matrix to the corresponding discrete transition matrix.

In many cases, in the internal rating system of a bank only discrete-time historical transition matrices are reported. To benefit from the advantages of continuous-time modeling we might still be interested in finding the correspondent generator matrix. In this case an important issue is whether for a given discrete one-year transition matrix a so-called 'true' generator exists. For some discrete transition matrices there is no generator matrix at all while for some there exists a generator that has negative off-diagonal elements. Examining the question of existence of a true generators or finding approximations of such matrices we briefly review some results obtained byIsrael, Rosenthal and Wei (25).

Given the one-year $N \times N$ transition matrix P we are interested in finding a generator matrix Λ such that:

$$P = e^{\Lambda} = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} = \mathcal{I} + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots \tag{2.7}$$

Dealing with the question if there exists a generator matrix we can use the following theorem (35):

Theorem 2.3 *If a migration matrix $P = (p_{ij}) \quad i, j = 1, \cdots, K$ is strictly diagonal dominant, i.e. $p_{ii} > 0.5$ for every i, then the log-expansion*

$$\Lambda_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{(P-I)^k}{k} \quad (n \in \mathbb{N}) \tag{2.8}$$

converges to a matrix $\Lambda = (\lambda_{ij}) i, j = 1, \cdots, K$ satisfying

1. $\sum_{j=1}^{8} \lambda_{ij} = 0$ *for every* $i = 1, \cdots, K$;
2. $exp(\Lambda) = P$.

The convergence $\Lambda_n \to \Lambda$ is geometrically fast and denotes a $N \times N$ matrix having row-sums of zero and satisfying $P = e^{\Lambda^*}$ exactly. For the proof, see (25). We point out that even if the series Λ^* does not converge or converges to a matrix that cannot be a true generator, P may still have a true generator (25).

However, often there remains another problem: The main disadvantage of series (2.8) is that Λ_n may converge but does not have to be a true generator matrix in economic sense, particularly it is possible that some off-diagonal elements are negative. From an economic viewpoint this is not acceptable because a negative entry in the generator for short time intervals may lead to negative transition probabilities for very short time intervals. Israel et al. show that it is also possible that there exists more than one generator. They provide conditions for the existence or non-existence of a valid generator matrix and further a numerical algorithm for finding this matrix.

Investigating the existence or non-existence of a valid generator matrix with only positive off-diagonal elements we start with another result obtained by Singer and Spilerman (40):

Proposition 2.4 *Let P be a transition matrix that has real distinct eigenvalues.*

> *If all eigenvalues of P are positive, then the matrix obtained by (2.8) is the only real matrix Λ such that $exp(\Lambda) = P$.*
>
> *If P has any negative eigenvalues, then there exists no real matrix Λ such that $exp(\Lambda) = P$.*

Using the conditions above we can conclude for the non-existence of a valid generator.

Proposition 2.5 *Let P be a transition matrix such that the series (2.8)converges to a matrix Λ with negative off-diagonal elements. If at least one of the following three conditions hold*

> *$det(P) > 1/2$ and $|P - I| < 1/2$ or*
>
> *P has distinct eigenvalues and $det(P) > e^{-\pi}$ or*
>
> *P has distinct real eigenvalues.*

then there does not exist a valid generator for P.

Israel et al. also provide a search algorithm for a valid generator if the series (2.8) fails to converge or converges to a matrix that has some off-diagonal terms but it not unique. For a further description we refer to (25) since in the case of non-existence of a true generator we used some approximation methods.

We conclude that despite the manifold advantages of continuous-time transition modeling the non-existence of a valid generator matrix to a given discrete-time transition may lead to some difficulties in practical implementations. We will see in later sections that especially for matrices having rows with several zeros (e.g. no transitions to default states) no valid generator matrix exists. In this case, some approximation methods can be used to determine an adequate generator matrix.

Let us now consider the historical migration matrices from 1982-2001 provided my Moody's that we will use for our empirical analysis. The figures for Value-at-Risk and PDs are based on a continuous-time simulation procedure allowing for several rating changes within a period. Unfortunately our data does not contain the exact date in terms of day or month of the rating changes for the considered time horizon. So based on historical one-year transition matrices from Moody's we will have to calculate the corresponding generators. Unfortunately, for several of the considered migration matrices, the series 2.8 converges to a generator with negative off-diagonal elements and for many transition matrices this is the only valid generator. For example, considering the historical transition matrix P_{1996} we get

$$P_{1996} = \begin{pmatrix} 0.9492 & 0.0457 & 0.0051 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0019 & 0.9437 & 0.0544 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0300 & 0.9497 & 0.0193 & 0.0010 & 0.0000 & 0.0000 & 0.0000 \\ 0.0015 & 0.0000 & 0.0596 & 0.9190 & 0.0185 & 0.0015 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0091 & 0.0793 & 0.8503 & 0.0476 & 0.0068 & 0.0068 \\ 0.0000 & 0.0000 & 0.0027 & 0.0053 & 0.0904 & 0.8538 & 0.0159 & 0.0319 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0870 & 0.1304 & 0.7391 & 0.0435 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

and for the corresponding generator matrix Λ_{1996}:

$\Lambda_{1996} =$

$$\begin{pmatrix} -0.052 & 0.048 & 0.004 & -0.000 & -0.000 & 0.000 & 0.000 & 0.000 \\ 0.002 & -0.059 & 0.058 & -0.001 & -0.000 & 0.000 & 0.000 & 0.000 \\ -0.000 & 0.032 & -0.053 & 0.021 & 0.001 & -0.000 & -0.000 & -0.000 \\ 0.002 & -0.001 & 0.064 & -0.086 & 0.021 & 0.001 & -0.000 & -0.000 \\ -0.000 & -0.000 & 0.007 & 0.090 & -0.167 & 0.055 & 0.008 & 0.006 \\ 0.000 & -0.000 & 0.003 & 0.001 & 0.105 & -0.163 & 0.020 & 0.034 \\ 0.000 & 0.000 & -0.001 & -0.005 & 0.101 & 0.161 & -0.304 & 0.047 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}.$$

Obviously, in row seven there are negative off-diagonal entries in the generator that may lead to negative transition probabilities for very short time intervals. Thus, we will have to use some approximation methods to determine a valid generator. The result may lead to a generator not providing exactly $P = e^{\Lambda^*}$ but only an approximation of the original transition matrix P. Instead the necessary condition from an economic viewpoint will be guaranteed and all off-diagonal row entries in the generator are non-negative.

The literature, suggests different methods to deal with this problem if calculating Λ, e.g. Jarrow et al. (27) or Israel et al. (25). Comparing the suggested methods, Trück and Özturkmen (42) find that the latter give better adjustments to the original migration matrix. Therefore, following Israel et al.(25), we used the following method for determining an approximation of the generator matrix. First, for each year by using (2.8) for the original migration matrix the associated generator is calculated. If there are negative off-diagonal elements the generator is changed according to the following method:

> Replace the negative entries by zero and add the appropriate value back into *all* entries of the corresponding row proportional to their absolute values. Let G_i be the sum of the absolute values of the diagonal and nonnegative off-diagonal elements and B_i the sum of the absolute values of the negative off-diagonal elements:
>
> $$G_i = |\lambda_{ii}| + \sum_{j \neq i} max(\lambda_{ij}, 0); \quad B_i = \sum_{j \neq i} max(-\lambda_{ij}, 0)$$
>
> Then set the modified entries
>
> $$\lambda_{ij} = \begin{cases} 0, & i \neq j \text{ and } \lambda_{ij} < 0 \\ \lambda_{ij} - \frac{B_i|\lambda_{ij}|}{G_i} & \text{otherwise if } G_i > 0 \\ \lambda_{ij}, & \text{otherwise if } G_i = 0 \end{cases}$$

In our example where the associated generator was Λ_{1996} applying this method, e.g. in the seventh row we have to set λ_{73} and λ_{74} to zero and then 'redistribute' -0.006 to the positive entries λ_{75}, λ_{76}, λ_{78} and to the diagonal element λ_{77}. This gives us the adjusted generator $\Lambda_{1996,approx} =$

$$\begin{pmatrix} -0.052 & 0.048 & 0.004 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.002 & -0.059 & 0.057 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.032 & -0.053 & 0.021 & 0.001 & 0.000 & 0.000 & 0.000 \\ 0.002 & 0.000 & 0.063 & -0.087 & 0.021 & 0.001 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.007 & 0.090 & -0.167 & 0.055 & 0.008 & 0.006 \\ 0.000 & 0.000 & 0.003 & 0.001 & 0.105 & -0.163 & 0.020 & 0.034 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.100 & 0.160 & -0.307 & 0.047 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

and the associated one-year transition matrix $P_{1996,approx} =$

$$\begin{pmatrix} 0.9492 & 0.0457 & 0.0051 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0019 & 0.9434 & 0.0541 & 0.0006 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0300 & 0.9496 & 0.0193 & 0.0010 & 0.0000 & 0.0000 & 0.0000 \\ 0.0015 & 0.0010 & 0.0592 & 0.9184 & 0.0183 & 0.0015 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.0091 & 0.0793 & 0.8502 & 0.0476 & 0.0068 & 0.0068 \\ 0.0000 & 0.0000 & 0.0027 & 0.0053 & 0.0903 & 0.8538 & 0.0159 & 0.0319 \\ 0.0000 & 0.0000 & 0.0006 & 0.0040 & 0.0861 & 0.1290 & 0.7371 & 0.0431 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

Obviously, the obtained approximate migration matrix is very close to the original one. To compare the goodness of our approximations more thoroughly, we would have to use some distance measures for the distances from the original transition matrix to the calculated approximations. For an extensive discussion on such distance measures see (26) and (44). In the following we will consider the approximations to Moody's historical transition matrices as 'close enough' for our purpose.

Ensuring that only valid generator matrices are used, we will now illustrate the simulation procedure that is used for estimating risk figures and confidence intervals for PDs for an exemplary portfolio of an international operating bank. The procedure for simulation follows a method suggested by Lando and Christensen (30). Based on historical observed transition matrices and the corresponding generator matrix we investigate the effect on risk capital for the credit portfolio. Technically, for each year we sample the time of a credit event using the generator matrix of the Markov process. If a credit event takes place we further sample the nature of the event - migration to another state or default - also based on the generator matrix. In the event of default, we calculate the recovery payment according to the expected recovery rate. The simulation procedure will briefly be described in this section.

Recall that a continuous-time, time-homogeneous Markov chain is specified via the (KxK) generator matrix:

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \lambda_{K-1,K} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.9}$$

where $\lambda_{ij} \geq 0$, for all i, j and $\lambda_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{K} \lambda_{ij}$, for $i = 1, ...K$. Recall also that the off-diagonal elements represent the intensities of jumping to rating j from rating i. The default state K is again absorbing.

As the waiting time for leaving state i has an exponential distribution with the mean $\frac{1}{-\lambda_{ii}}$ we draw an exponentially-distributed random variable t_1 with the density function

$$f(t_1) = -\lambda_{ii} e^{\lambda_{ii} t_1}$$

for each company with initial rating i. Depending on the considered time horizon T for $t_1 > T$, the company stays in its current class during the entire period T. If we get $t_1 < T$, we have to determine to which rating class the company migrates.

Hence, the interval [0,1] is divided into sub-intervals according to the migration intensities calculated via $\frac{\lambda_{ij}}{-\lambda_{ii}}$. Then a a uniform distributed random variable between 0 and 1 is drawn. Depending on which sub-interval the random variable lies in we determine the new rating class j the company migrates to. Then we draw again from an exponentially-distributed random variable t_2 - this time with parameter λ_{jj} from the generator matrix. If we find that $t_1 + t_2 > T$ the considered company stays in the new rating class and the simulation is completed for this firm. If $t_1 + t_2 < T$ we have to determine the new rating class. The procedure is repeated until we get $\sum t_k > T$ or the

Rating	Aaa	Aa	A	Baa	Ba	B	Caa
No.	11	106	260	299	241	95	148
Average Exposure (Mio. Euro)	20	15	15	10	10	5	5

Table 2: Ratings and exposures for the considered Credit Portfolio

company migrates to default state. As it was mentioned above the default state is considered to be absorbing. So after default a company will remain in the default state for the rest of the considered time period.

This simulation procedure is conducted for every company in the portfolio. Thus, each company has a simulated rating history including all rating changes and dates for the considered time period. The results can be used for determining the number and size of losses for the considered portfolio for different time horizons. Following Lando and Christensen (30) using equation (2.6) and the simulated rating histories for all companies in the portfolio we can also calculate the estimator for the generator matrix Λ. We will see in section 4 that these estimates can also be used to calculate so-called 'bootstrap' confidence intervals for the different rating classes.

3 Migration Matrices and Credit VaR

We consider an internal loan portfolio of an international operating major bank consisting of 1120 companies. The average exposure is dependent on its rating class. In the considered portfolio higher exposures could be observed in higher rating classes while companies with a non-investment grade rating Baa, B or Caa the average exposures were between 5 Mio. and 10 Mio. Euro. The distribution of ratings and average exposures in the considered rating classes of the loan portfolio is displayed in table 2.

We further make the following assumptions for the loans. For each of the simulated years we use the same rating distribution for the portfolio to keep the figures comparable. We also used an average yearly recovery rate of $R = 0.45$ for all companies. This is clearly a simplification of real recovery rates, however due to not having any information on the seniority of the considered loans this is an adequate assumptions for empirical recovery rates.

For the investigation we use credit rating histories from Moody's from

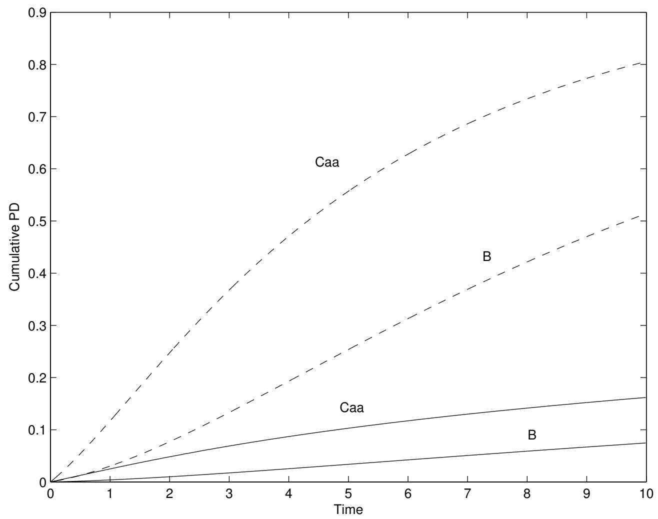


Figure 4: Credit curves for speculative grade issuers according to Moody's migration matrix 1997 and 2001 (dashed).

a twenty-year period from 1982-2001 and a continuous-time approach for determining VaR figures based on simulations. To illustrate the differences in migration behavior we show the credit curves of two years belonging to two different phases of the business cycle. The year 2001 was a year of economic turmoil with high default rates and many downgrades while in 1997 the macroeconomic situation was stable and the economy was growing. Lower default probabilities and more upgrade than downgrades were the consequences.

Both for investment grade issuers and speculative grade issuers we find completely different credit curves. The graphs were plotted for a ten year time horizon, Clearly it is rather unrealistic that the macroeconomic situation stays in a recession or expansion state for such a long time. However, in the sequel we will illustrate that even in a considered time horizon of one-year the effects can be substantial.

Further the effect of on risk figures was considered for different time horizons of six months, one year and a three year period. Two typical loss distributions for the years 1998 and 2001 are displayed in figure 6. The distributions have an expected loss of $\mu = 148.55$ Mio. with a standard deviation of $\sigma = 16.67$ Mio. for 1998 and $\mu = 223.15$ Mio. Euro $\sigma = 21.15$

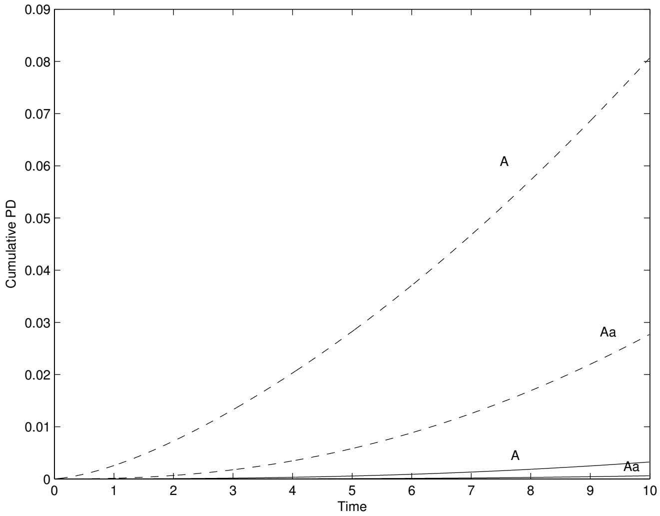
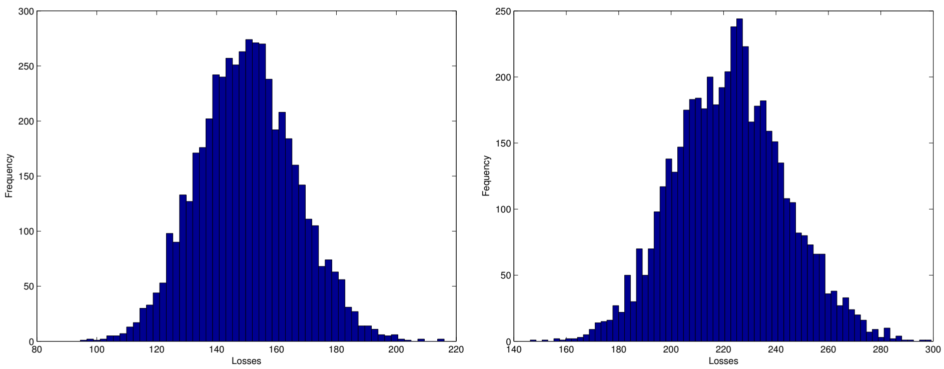


Figure 5: Credit curves for investment grade issuers according to Moody's migration matrix 1997 and 2001 (dashed).



(a) 1998

(b) 2001

Figure 6: Typical shape of simulated loss distributions for the years 1998 and 2001.

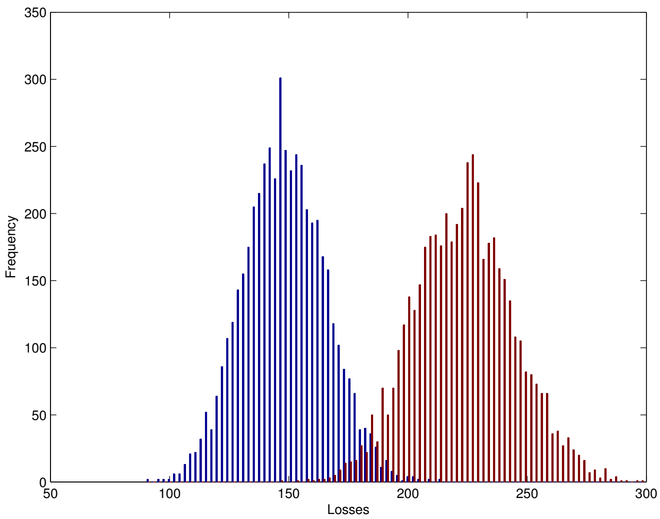


Figure 7: Simulated loss distributions for the years 2000 and 2001.

Mio. for 2001. Both distributions were slightly skewed to the right with $\gamma = 0.1217$ for the year 1998 and $\gamma = 0.1084$ for 2001. The kurtosis for the loss distributions is with $k = 2.99$ for 1998 and $k = 2.97$ for 2001 very close to the kurtosis of the normal distribution.

Comparing loss distributions for different years we find that in many cases the distributions do not even coincide. We plotted a comparison of the simulated loss distributions for the 2000 and 2001 in figure 8 and for the years with minimal (1996) and maximal (2001) portfolio risk in the considered period in figure 7. While for the subsequent years 2000 and 2001 the distributions at least coincide at very low (respectively high) quantiles, we find no intersection at all for the years 1996 and 2001. This points out the substantial effect of migration behavior on risk figures for a credit portfolio.

A closer picture of the significant changes in the Value-at-Risk for the considered credit portfolio through the consider period is provided in figure 9 and 10.

We find that simulated VaR and Expected Shortfall figures show great variation through the business cycle. While in the years 1983 and 1996 the average expected loss for the portfolio would be only 31.29 Mio. or 28.84 Mio. Euro in a one-year period, during the recession years 1991 and 2001, the simulated average loss for the portfolio would be 227.25 Mio. or 258.75

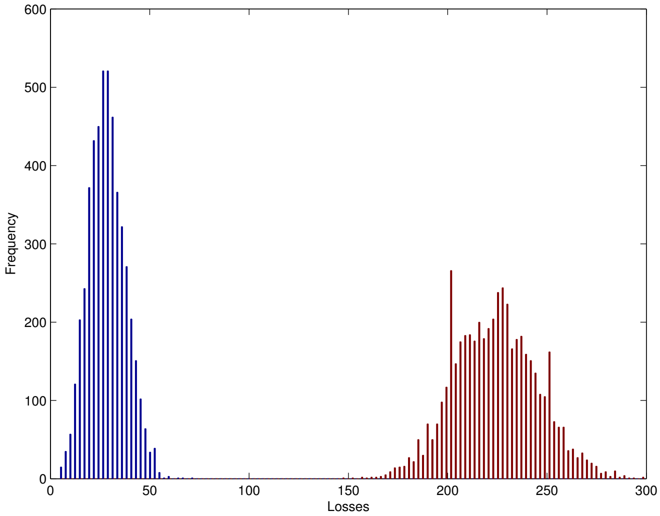


Figure 8: Simulated loss distributions for the years 1996 and 2001.

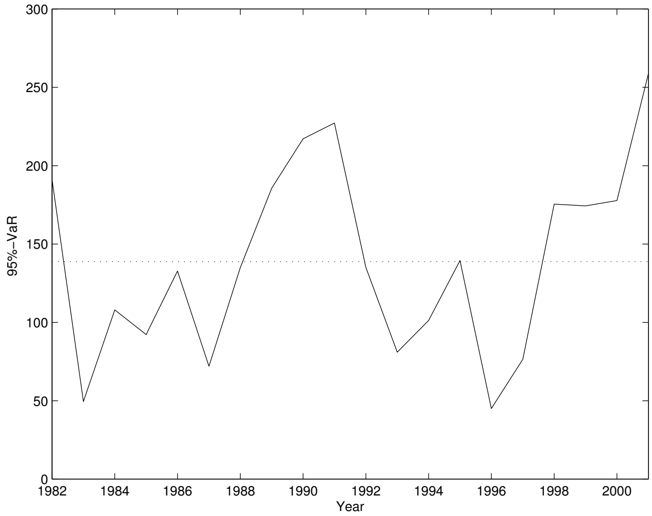


Figure 9: Simulated VaR alpha=0.95 for a one-year periods.

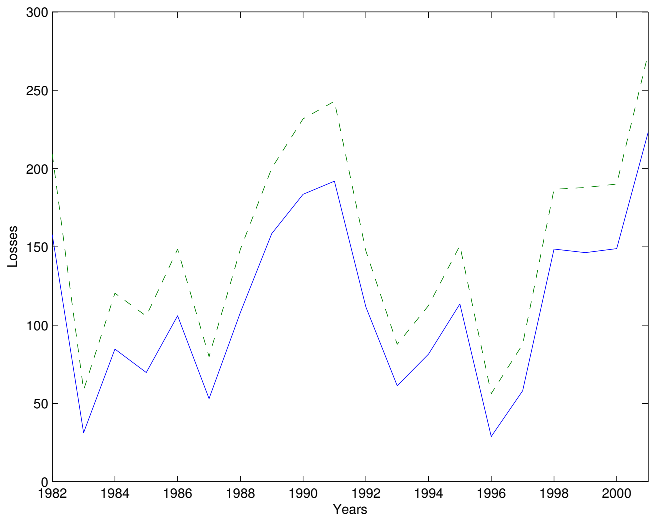


Figure 10: Simulated mean loss and VaR for alpha=0.99 for a considered one year period.

Mio. Euro. The maximum of the simulated average losses for the portfolio is about eight times higher than the minimum amount in the considered period. Similar numbers are obtained for considering Value-at-Risk or expected shortfall. The one-year 95%-VaR varies between 45 Mio. and 258.75 Mio. Euro, while the one-year 99%-VaR lies between a minimum of 56.25 Mio. in 1996 and 273.37 Mio. in the year 2001. This illustrates the enormous effect the business cycle might have on migration behavior and thus, ob the risk of a credit portfolio. Ignoring these effects may lead to completely wrong estimates of credit VaR and capital requirements for a loan or bond portfolio. This points out the necessity to use credit models that include variables measuring the state of the business cycle or use conditional migration matrices. In the next section we will further investigate the effect of the macroeconomic behavior on confidence sets for PDs.

4 Confidence Sets for Default Probabilities

Another main issue of credit risk modeling is the behavior of probabilities of default (PDs). In the internal rating based approach of the new Basel capital accord, PDs are the main input variables for determining the risk and the necessary regulatory capital for a portfolio. Of course, regulators are not the only constituency interested in the properties of PD estimates. PDs are inputs to the pricing of credit assets, from bonds and loans to more sophisticated instruments such as credit derivatives. However, especially for companies with an investment grade rating default is a rare event. Often high credit quality firms make up the bulk of the large corporate segment in any large bank's portfolio. But with only little information on actual defaulted companies in an internal credit portfolio, observed PDs for the investment grade categories are likely to be very noisy. The question rises how one should go about estimating reliable confidence intervals for PDs. This is of particular importance, since similar to the VaR or expected shortfall of a credit portfolio, PDs may also vary systematically with the business cycle. Thus, also investment grade ratings PDs are rather unlikely to be stable over time. Therefore, in this section we tackle the question of obtaining reliable estimates for PDs also in the investment grade sector and compare these PDs for the considered time period from 1982-2001.

Lando and Christensen (30) estimate transition and default probabilities and set confidence intervals for default probabilities for each rating class by using the continuous-time approach of the previous section. They find that a continuous-time bootstrap method can be more appropriate than using the estimates based on actual default observations. This is especially true for higher rating classes where defaults are very rare events.

To illustrate the advantages of the bootstrap method let us first consider a binomial random variable $X \sim B(p_i, n_i)$ where p_i denotes the probability of default for rating class i and n_i the number of companies in the rating class. Now assume that there is a investment grade rating class in the internal rating system where no actual defaults was observed in the considered time-period. Clearly, the corresponding estimator for the PD in this rating class is then $p_i = 0$. However, for VaR calculations a bank is also interested in confidence intervals for PDs of the investment grade rating classes. Based on the considered binomial distribution one could compute the largest default probability for a given confidence level α that cannot be rejected by solving the following equation:

$$(1 - p_i)^{n_i} = \alpha$$

Therefore, the corresponding upper value p_{max} of a confidence interval for

	n_i	X_i	$KI_{\alpha=0.05}$	$KI_{\alpha=0.01}$
Aaa	50	0	$[0, 0.0582]$	$[0, 0.0880]$
Aa	500	0	$[0, 0.0060]$	$[0, 0.0092]$

Table 3: Example for PD confidence interval estimated based on the binomial distribution.

a rating class with no observed defaults is:

$$\hat{p}_{max}(n_i, \alpha) = 1 - \sqrt[n_i]{\alpha}$$

The main disadvantage of the binomial method becomes obvious immediately. The confidence intervals are dependent on the number of firms n_i in the considered rating class. The lower n_i, the wider becomes the confidence interval what is illustrated in table 4 for an exemplary portfolio with 50 companies in the rating class *Aaa* and 500 companies in rating class *Aa*. We further assume that for both rating classes in the considered period no defaults could be observed. We find that using the binomial distribution to estimate 95% and 99% confidence intervals, the intervals for rating class Aaa are about ten times wider than those for Aa. From an economic point of view this is questionable and simply a consequence of the fact that more companies were assigned with the lower rating.

Of course, the binomial distribution can also be used for calculating two-sided confidence intervals for lower rating classes where also transition to defaults were observed. What is needed is the total number of firms with certain rating i at the beginning of the period and the number of firms among them that defaulted until the end of the considered period. Then, for a given confidence level α the standard Wald confidence interval is

$$\hat{p}_{i,max/min} = \hat{p}_i \pm q_\alpha \sqrt{\frac{\hat{p}_i(1-\hat{p}_i)}{n_i}} \tag{4.1}$$

where n_i is the total number of firms in rating class i and q_α is the α-quantile of the standard normal distribution. Unfortunately the estimates for confidence intervals obtained by the Wald estimator are not very tight, see e.g. (39). Christensen et al. (14) as well as Schürmann and Hanson (39) point out that the obtained confidence sets by a so-called continuous-time

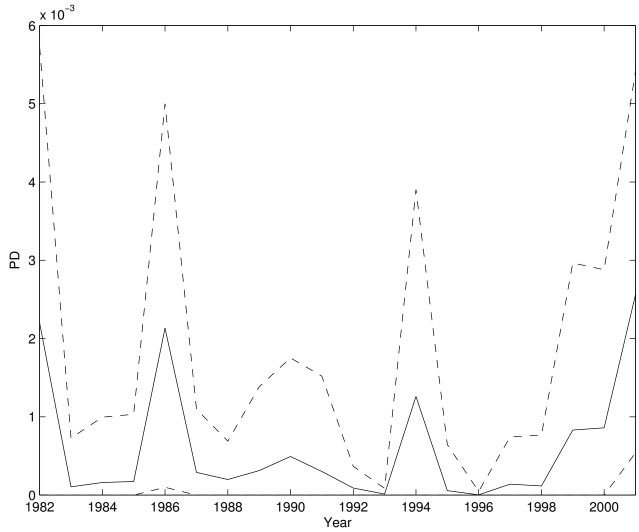


Figure 11: Mean and 95% Confidence Levels for Rating Class A from 1982-2001.

bootstrap method are much tighter than those obtained by the standard Wald estimator. Christensen et al. (14) state that the only advantage in the binomial case is that using this method one is able to derive genuine confidence sets, i.e. to analyze the set of parameters which an associated test would not reject based on the given observations.

To compare confidence intervals through the business cycle we therefore used the bootstrap method described in (14). An introduction to bootstrapping can be found in Efron and Tibshirani (18), so we will only briefly describe the idea of the bootstrap and our simulation algorithm. The same simulation procedure as in section 2.2 is used to obtain histories for each of the considered companies. We simulate $N = 5000$ fake datasets for each time window. We used a fake dataset with a number of 1000 issuers in each rating category. Then the issuers history background Markov process is simulated using the observed historical transition matrix for each year. The simulated rating changes are translated into a history of observed rating transitions. For each replication the generator matrix of the hidden Markov chain model is then reestimated, using the fake dataset history and the maximum-likelihood estimator

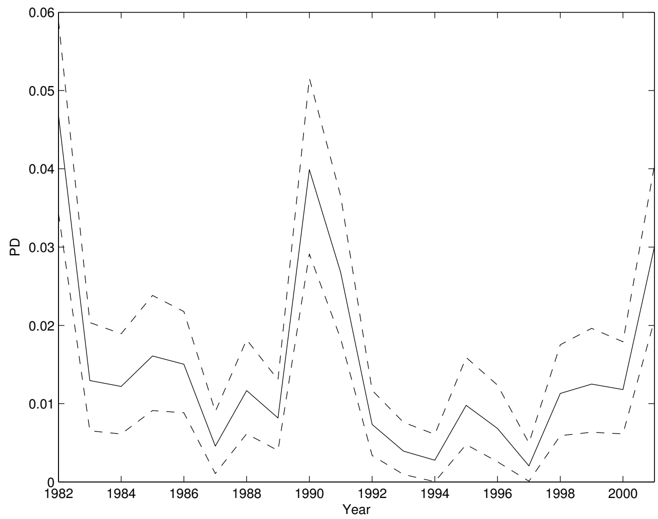


Figure 12: Mean and 95% Confidence Levels for Rating Class Ba from 1982-2001.

$$\hat{\lambda}_{ij} = \frac{N_{ij}(T)}{\int_0^T Y_i(s)ds}.$$

From the estimated transition structure we calculate the one-year default probability for each true state. Exponentiating this matrix gives an estimator of the one-year migration matrix and the last column of the transition matrix provides the vector of estimated default probabilities for each replication. We end up with N of these vectors. Thus, for each year or following (30) - each true state of the background process - we have N one-year default probabilities.

Another possibility to find confidence sets would have been to develop asymptotic expressions for the distribution of test statistics in the continuous-time formulation and use those for building approximate confidence sets. However, in practice the bootstrap method seems both easier to understand and to implement. Note that the maximum-likelihood estimator does not have a simple closed form expression for its variance/covariance matrix. This makes it difficult to provide information about the confidence sets for estimated parameters. In fact, we would need to use asymptotics twice. First to find the variance of the estimated generator $\hat{\Lambda}$ and additionally finding an

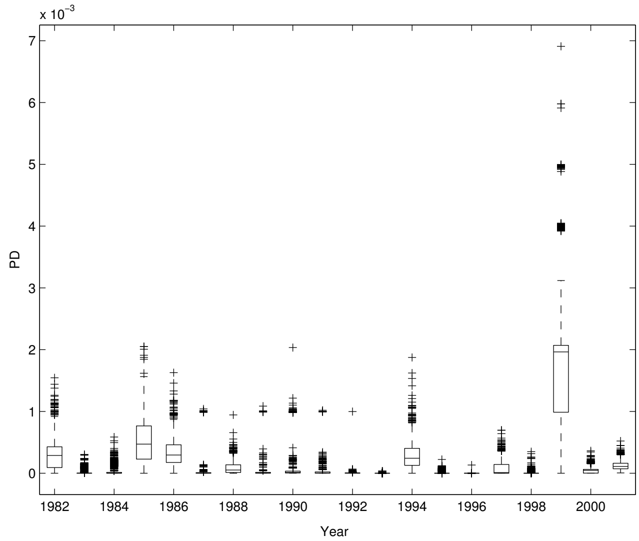


Figure 13: Boxplot for Bootstrapped Confidence Intervals for Rating Class Aa from 1982-2001.

expression for the variance of $\exp(\hat{\Lambda})$. The second step again only seems feasible using an asymptotic argument. Unfortunately the asymptotic variance of $\hat{\Lambda}$ is hardly a good estimator, since many types of transitions occur only rarely in the data set. Thus, the bootstrap method provides tighter intervals and is also more understandable.

Based on all replications for each year 1982-2001 the relevant quantiles and the distribution of the PDs is obtained. Efron and Tibshirani (18) suggest for confidence intervals to use bootstrap replications of 1000. To be on the safe side, for each rating class and each considered year from 1982-2001 we ran 5000 replications. Thus, for each year and for each rating class we obtained a distribution based on 5000 simulated PDs. The results for investment grade rating classes Aa and A can be found in figure 13 and 14 where we plotted boxplots of the PDs for the whole considered period.

It becomes obvious that confidence intervals vary substantially through time. This includes not only the level of the mean of the bootstrapped PDs but also the width of the confidence interval. Comparing for example the 95% interval for rating class A we find that the interval in 1993 $KI_{A,1993} = [0.000, 0.0001]$ compared to the 2001 interval $KI_{A,2001} = [0.0005, 0.0054]$ is

Rating	Aaa	Aa	A	Baa	Ba	B	Caa
$mean$	0.0001	0.0006	0.0019	0.0049	0.0126	0.0262	0.0473
σ	0.0002	0.0009	0.0017	0.0030	0.0053	0.0065	0.0140
min	0.0000	0.0000	0.0000	0.0002	0.0049	0.0172	0.0021
max	0.0007	0.0040	0.0057	0.0095	0.0246	0.0424	0.0605
$v = \frac{\sigma}{mean}$	1.5994	1.6162	0.9361	0.6125	0.4210	0.2493	0.2955

Table 4: Descriptive Statistics of the width of confidence intervals for different rating classes.

about 50 times wider. The variation of the lower and upper boundary of the intervals is illustrated for rating classes Ba and A in the figures 11 and 12. We find that for investment grade ratings with the level also the width of an estimated confidence set for the PD increases substantially. Histograms for the same rating class Ba but for different periods - 1991 and 1996 - can be found in figure 15. Obviously the plotted histograms for the two periods do not even coincide.

For non-investment grade ratings the variations in the level of average PDs is also extreme. However, as it can be seen in table 4, we find that the width of the intervals does not show the extreme variations as for the investment grade ratings. This is best illustrated by the coefficient of variation $v = \frac{\sigma}{mean}$ comparing the standard deviation of the width of the confidence intervals to its mean. We find a decreasing coefficient of variation for lower rating classes. Thus, we conclude that the fraction PD to the volatility of PD decreases with an increasing PD. This could be an interesting finding for credit derivative modeling where also the volatility of PDs is an important input variable.

5 Conclusion

This paper investigated the effect of different migration matrices on an exemplary portfolio through the business cycle. We also calculated confidence intervals based on a bootstrap method introduced in Lando and Christensen (30) and 20-year history of Moody's migration matrices. To determine Value-at-Risk for the portfolio as well as PD confidence sets, a continuous-time simulation and bootstrap method was used. Therefore, following Israel et al (25) we illustrated how from an original discrete transition matrix the corresponding generator can be derived. Then using an approach by Christensen

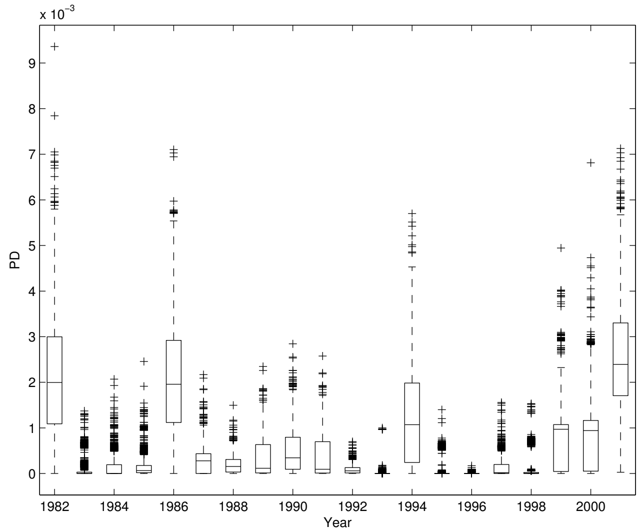


Figure 14: Boxplot for Bootstrapped Confidence Intervals for Rating Class A from 1982-2001.

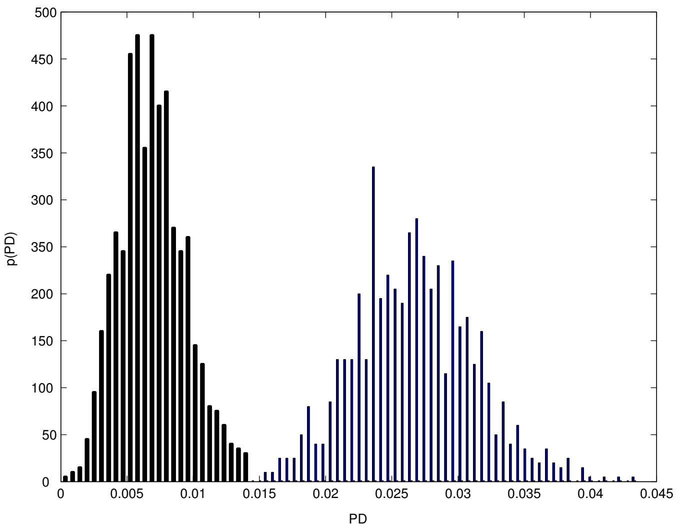


Figure 15: Histogram of bootstrap PDs for Rating Class Ba in 1991 (left) and 1996 (right).

and Lando (14) credit VaR and PD confidence intervals were determined. The results point out the substantial effect of variations in the economy on the expected loss, VaR or expected shortfall for a credit portfolio if a rating based credit risk system is used. The estimated one-year-VaR of the considered portfolio was about six times higher for the dramatic recession period in 2001 than for example in 1996 and more than twice of the average one-year VaR. The effect on confidence sets for default probabilities is even more dramatic. Variations in the width and level of confidence intervals for investment grade rating classes were significant and did not even coincide in periods of economic expansion or recession. We further found a decreasing coefficient of variation with increasing PD what could be an interesting finding for credit derivative modeling where also the volatility of PDs is an important determinant. We conclude that it cannot be considered as an appropriate approach to use average transition matrix as an input for rating based credit risk modeling. The effect of the business cycle on changes of migration behavior and therefore also on Value-at-Risk and PDs is too eminent to be neglected. Migration behavior and PDs in credit risk models should be adjusted or forecasted with respect to the macroeconomic situation.

6 Appendix

Year	Aaa	Aa	A	Baa
1982	$[0.0000, 0.0005]$	$[0.0000, 0.0010]$	$[0.0000, 0.0057]$	$[0.0019, 0.0062]$
1983	$[0.0000, 0.0000]$	$[0.0000, 0.0001]$	$[0.0000, 0.0007]$	$[0.0004, 0.0073]$
1984	$[0.0000, 0.0001]$	$[0.0000, 0.0003]$	$[0.0000, 0.0010]$	$[0.0026, 0.0121]$
1985	$[0.0000, 0.0001]$	$[0.0000, 0.0014]$	$[0.0000, 0.0010]$	$[0.0008, 0.0034]$
1986	$[0.0000, 0.0001]$	$[0.0000, 0.0010]$	$[0.0001, 0.0050]$	$[0.0014, 0.0069]$
1987	$[0.0000, 0.0000]$	$[0.0000, 0.0000]$	$[0.0000, 0.0011]$	$[0.0001, 0.0015]$
1988	$[0.0000, 0.0003]$	$[0.0000, 0.0004]$	$[0.0000, 0.0007]$	$[0.0008, 0.0032]$
1989	$[0.0000, 0.0000]$	$[0.0000, 0.0001]$	$[0.0000, 0.0014]$	$[0.0026, 0.0106]$
1990	$[0.0000, 0.0000]$	$[0.0000, 0.0010]$	$[0.0000, 0.0018]$	$[0.0023, 0.0101]$
1991	$[0.0000, 0.0000]$	$[0.0000, 0.0002]$	$[0.0000, 0.0015]$	$[0.0042, 0.0128]$
1992	$[0.0000, 0.0000]$	$[0.0000, 0.0000]$	$[0.0000, 0.0004]$	$[0.0002, 0.0015]$
1993	$[0.0000, 0.0000]$	$[0.0000, 0.0000]$	$[0.0000, 0.0001]$	$[0.0001, 0.0010]$
1994	$[0.0000, 0.0002]$	$[0.0000, 0.0009]$	$[0.0000, 0.0039]$	$[0.0000, 0.0006]$
1995	$[0.0000, 0.0000]$	$[0.0000, 0.0001]$	$[0.0000, 0.0006]$	$[0.0002, 0.0074]$
1996	$[0.0000, 0.0000]$	$[0.0000, 0.0000]$	$[0.0000, 0.0000]$	$[0.0000, 0.0002]$
1997	$[0.0000, 0.0000]$	$[0.0000, 0.0004]$	$[0.0000, 0.0007]$	$[0.0002, 0.0058]$
1998	$[0.0000, 0.0002]$	$[0.0000, 0.0001]$	$[0.0000, 0.0008]$	$[0.0012, 0.0090]$
1999	$[0.0000, 0.0007]$	$[0.0000, 0.0040]$	$[0.0000, 0.0030]$	$[0.0002, 0.0052]$
2000	$[0.0000, 0.0000]$	$[0.0000, 0.0002]$	$[0.0000, 0.0029]$	$[0.0008, 0.0072]$
2001	$[0.0000, 0.0000]$	$[0.0000, 0.0003]$	$[0.0005, 0.0054]$	$[0.0024, 0.0075]$

Table 5: Bootstrap-95%-confidence intervals for invstment grade ratings. Figures based on Moody's Historical Transition Matrices 1982-2001.

Year	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
$Mean, 6months$	83.48	14.46	44.05	34.66	52.50	27.07	57.42	87.58	97.30	102.21
$Mean, 1year$	157.83	31.29	84.67	69.71	105.96	53.03	108.13	158.47	183.59	191.93
$Mean, 3years$	343.50	88.95	198.83	178.30	275.23	127.73	233.95	314.05	413.67	411.20
Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
$Mean, 6months$	60.05	35.13	42.50	61.57	14.11	30.66	85.91	79.89	79.56	121.17
$Mean, 1year$	111.60	61.30	81.54	113.48	28.84	58.05	148.55	146.30	148.84	223.15
$Mean, 3years$	240.16	113.84	176.37	235.14	68.33	129.68	280.51	305.39	312.74	477.26

Year	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
$VaR_{0.95,6months}$	108.00	27.00	63.00	51.75	72.00	40.50	76.50	110.25	123.75	128.25
$VaR_{0.95,1year}$	191.25	49.50	108.00	92.25	132.75	72.00	135.00	185.62	217.12	227.25
$VaR_{0.95,3years}$	393.75	119.25	234.00	216.00	321.75	157.50	272.25	353.25	463.50	461.25
Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
$VaR_{0.95,6months}$	78.75	50.62	58.50	81.00	24.75	45.00	108.00	101.25	101.25	148.50
$VaR_{0.95,1year}$	135.00	81.00	101.25	139.50	45.00	76.50	175.50	174.37	177.75	258.75
$VaR_{0.95,3years}$	274.50	139.50	207.00	270.00	90.00	157.50	317.25	343.12	351.00	531.00

Year	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
$VaR_{0.99,6months}$	120.37	33.75	74.25	58.50	83.25	47.25	84.37	119.25	132.75	139.50
$VaR_{0.99,1year}$	209.25	58.50	120.37	105.75	148.50	79.87	148.50	200.25	231.75	243.00
$VaR_{0.99,3years}$	421.87	132.75	253.12	236.25	338.62	171.00	285.75	370.12	487.12	482.62
Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
$VaR_{0.99,6months}$	85.50	56.25	65.25	90.00	31.50	51.75	115.87	112.50	110.25	164.25
$VaR_{0.99,1year}$	147.37	87.75	112.50	150.75	56.25	87.75	186.75	187.87	190.12	273.37
$VaR_{0.99,3years}$	288.00	153.00	218.25	283.50	99.00	172.12	333.00	357.75	371.25	553.50

Table 6: Simulated average loss, 95%- and 99%-VaR for the exemplary portfolio for 1982-2001. Considered was a 6-month, 1-year and 3-year time horizon.

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