Portfolio Optimization: Distributional Approach

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Abstract[1]

This paper analyses the stable distributional approach for portfolio optimisation. We consider a portfolio optimization problem under the assumption of normal (Gaussian) and stable (non-Gaussian) distributed asset returns. We compare the results of portfolio allocations in normal and stable cases.

1. Introduction

The purpose of this paper is to describe a portfolio optimisation problem under the assumption of stable distributed asset returns. We first consider an asset allocation model consistent with the maximization of the expected utility. We also examine the coincidence of the model optimal allocation with the empirical one, where the government allocation is taken as an effective decision. We analyse and compare the results obtained under the alternative hypothesis of stable and normal distribution of asset returns.

It is well-known that the standard statistical methods of estimating, forecasting and taking decisions are not the most precise for studying financial markets. Standard methods, ranging from the simple to the advanced, do not provide reliable predictions or estimations of income. The traditional approach to financial market investigation supposes the existence of some probability distribution, which characterizes the market instruments behaviour, or, rather, their returns. Traditionally, a Gaussian distribution is assumed.

Empirical investigations of the financial returns distributions have shown that the properties of the market plots reject the normal distribution assumption. The real bar graphs appear to be asymmetric, have high peaks and also change greatly along the time. All these features have lead to the normal distribution rejection and it was proposed to apply the stable distribution as a statistical model for asset returns.

The stable distribution assumption is relatively recently in use in finance, therefore the exact enough methods for measuring market risks with stable distributions are not yet developed.

2. Stable Distributions

Financial investment decisions are based almost exclusively on the expected returns and risk associated with the initial investment possibilities. The distributive character of the returns is of great importance for the theoretical and empirical analysis in economics and finance. For instance, the portfolio and the option pricing theory are based on distribution assumptions.

The early theories proposed considered normal distribution assumptions. However, in the works of Mandelbrot [5] and Fama [4] they doubted in the suitability of the normal distribution. In the capital

Proceedings of the 7th International Workshop on Computer Science and Information Technologies CSIT'2005
Ufa, Russia, 2005

returns research, the additional kurtosis, i.e. high peaks of the density function were found, as well as the property of heavy tails which rejected the normal distribution. As the alternative the stable distribution was suggested. Further research has confirmed this assumption [6].

A detailed description of the properties of stable distributions is given in the monographs of Samorodnitsky and Taqqu [7] and Janicki and Weron [8]. They also describe the numerical methods and computer simulations generated from random stable distributed variables and processes.

Compared with the normal one, stable distributions possess features such as heavy tails and high peaks of the density function relatively to the center. Unlike the normal distribution, a stable one can be asymmetric.

A stable distribution is characterized by the following parameters:

- α – index of stability or, characteristic parameter;
- β – skewness parameter of the density function;
- σ – scale parameter;
- μ – location parameter.

The parameter α is "responsible" for the decreasing behavior of the distribution tails. Regarding this parameter, the normal distribution can be considered as a specific case of stable distribution when $\alpha=2$.

Skewness (asymmetry) parameter $\beta \in [-1, 1]$ in reflects the extent of distribution asymmetry. If $\beta=0$, then the density function of distribution is symmetric. If $\beta>0$, the density function is skewed to the left. The extent of the skewness increases as β approaches 1. In the case when $\beta<0$, skewness is to the right.

Parameter σ is the scale parameter. In case of normal distribution, ($\alpha=2$) $DX=2\sigma^2$.

Parameter μ is location parameter such that when $\alpha>1$, the mathematical expectation is $E|X|<\infty$ and the mean is $\mu=EX$.

Summarizing the section, it is worth noting that stable distributions possess two attractive features:

1. Stable distributions have a domain of attraction in the sense that in this area the distributions of the observed values are also stable, or, their characterizing features are close to the stable distribution features. It speaks for the fact that if the empirical data is fit to the "idealized" stable distribution, it does not break the empirical distribution.
2. Each of the stable distributions has an index of stability, which does not depend on the observed time horizon. Index of stability can be considered as a general parameter, which can be used in decision making or establishing conclusions.

Thus, stable distributions and stable processes are of significant importance as natural and likely candidates within construction of probability models of distributions and financial indexes evolution (e.g. currency rates, asset prices etc.). Application of the stable distributions in portfolio optimisation is not a new idea. Theoretical and computational difficulties and theoretical failure in adjusting the empirical data to the models have recognized the importance of this approach. Hence, theoretical research and computer technique development in recent years have stimulated the intensive use of the stable distributions.

3. The Model

We analyse a portfolio consisting of two assets: the Dow Jones Industrial index is taken as a risky asset, and the Treasury bill – as a relatively risk-free asset.

The investor aims to optimise his portfolio allocation so that he could get maximum utility. At the same time, it is necessary to invest all the money he has in the assets. The share (weight) of each asset in the portfolio should be non-zero.

In the present work this problem is solved in a discrete statement, i.e. when there is a final set of admissible portfolios among which the investor makes his choice. The solution in this case can be received by full search of all portfolios in each definite case.

Except for a problem of utility function maximization there is a problem of optimal risk aversion coefficient definition which describes the investor's attitude to risk. A large value of the coefficient means the investor's aversion to risk.

The research is carried out assuming normal and stable asset returns distributions. The aim of this research is to define, for which type of asset returns distribution (normal or stable), the decision on asset allocation will be closer to the empirical data.

The aim of the investigation is to show the general regularities and drawing up recommendations.

Let us indicate the following parameters:

p – number of the portfolio, p=1,...,100;

$z_{i,t}$ – price of the asset i at moment t;

$r_{i,t}$ – return of the asset i at moment t;

$x_{i,p}$ – simulated weight of the asset i in portfolio p;

Rp – return of portfolio p;

$E(Rp)$ – mathematical expectation of portfolio return;

$CVaR_\varepsilon(Rp)$ – measure of portfolio risk under the error level ε;

ε – error level, (1-ε) – confidence level;

c – coefficient of relative risk aversion;

$U(Rp)$ – portfolio utility function.

Formally, the problem can be represented in the following way:

$$\max U(Rp_t) = E[Rp_t] - c \cdot CVaR_\varepsilon(Rp_t) \qquad (1),$$

where the mathematical expectation of the portfolio return at moment t is calculated as follows:

$$E[Rp_t] = \frac{1}{N}\sum_{n=1}^{N} Rp_{n,t} \ . \qquad (2)$$

Portfolio return is defined as

$$Rp_t = \sum_{i=1}^{d} x_{i,t} \cdot r_{i,t} \ , \qquad (3)$$

where d is the number of the assets in the portfolio.

Portfolio risk or, the portfolio loss level is defined by the measure of risk $CVaR_\varepsilon(Rp)$ [1].

The admissible values of the coefficient of relative risk aversion are $c > 0$.

Since the values of $E[Rp_t]$ and $CVaR_\varepsilon(Rp_t)$ do not depend on the coefficient of relative risk aversion, thus the utility function of every portfolio as a function of risk aversion coefficient will be linear:

$$U(c) = k \cdot c + b \ , \qquad (4)$$

where $U(c) = U(Rp_t)$; $\quad b = E[Rp_t]$;
$k = -CVaR_\varepsilon(Rp_t)\, k$, i.e. the plot of the utility function as a function of the risk aversion coefficient is represented as a straight line. Therefore, the plot of the optimal (maximum) utility function is the upper boundary of a set of these lines, i.e. is formed by the sections of these lines, where the corresponding line is the highest.

4. Results

The plots of the density functions histograms for simulated and empirical asset returns have showed that:

- the density functions of the empirical asset returns possess the feature of heavy tails, while the simulated normal asset returns do not. At the same time the simulated stable asset returns are closer to the empirical data in the sense that the density function in stable case also has heavy tails.
- the peak of the density function of the empirical asset returns is higher than in the normal case, while the peak of the density function in stable case is also stretched in the area of central values.

The utility function dependence on risk aversion coefficient was investigated by means of constructing the following linear regression function, which characterizes the correspondence between the difference in values of the utility functions in normal and stable cases, $U_{normal}(c)$ and $U_{stable}(c)$, respectively, and the risk aversion coefficient c:

$$U_{stable}(c) - U_{normal}(c) = b_0 + b_1 \cdot c + \xi_1 \ . \qquad (5)$$

The obtained numerical results showed that the predicted value of the utility functions difference will have the following view:

$$U_{stable}(c) - U_{normal}(c) = 0.0087 + 0.1206 \cdot c \ , \qquad (6)$$

where the *p-level* (or, statistical significance) is close to zero (0.037), which indicates that there is a small probability of error of assuming that the calculation result is "true".

Finally, we also investigated the ratio of the risk aversion coefficients in normal and stable cases, where the optimal risk aversion coefficient (under condition of maximum utility and closeness of the simulated weights to the empirical weights) in stable case is represented as a function of the corresponding risk aversion coefficient in normal case. The regression equation is the following:

$$c_{stable}^{opt} = b_0 + b_1 \cdot c_{normal}^{opt} + \xi_2 \ . \qquad (7)$$

The estimation of the regression coefficients gives the following result:

$$c_{stable}^{opt} = 0.0014 + 0.9907 \cdot c_{normal}^{opt} \ , \qquad (8)$$

where the *p-level* = 0.0063, which also speaks for the statistical significance of the estimated values and, therefore, of the prediction.

Besides, the received calculation results have indicated that the risk aversion coefficient value in stable case is significantly lower than in the normal one, i.e. $c_{stable}^{opt} << c_{normal}^{opt}$. The difference is the more visible, the higher is the value of the risk aversion coefficient. Thus, by the results of the problem we can conclude that supposing a stable portfolio asset returns distribution leads to decreasing the risk aversion coefficient compared with the case of normal distribution.

5. Conclusion

In this paper, we first briefly describe the stable distributions as an alternative approach to financial market investigation and its advantages compared with the normal distribution.

We propose a model for portfolio optimisation under the assumption that portfolio asset returns are subject to the stable distribution. The comparison made between the stable and the normal approach in terms of the optimal allocation problem has indicated that taking the assumption about stable distributed asset returns allows preserving risk, as compared with the normal distribution. The stable approach considers that the risk is somehow due to heavy tails. Thus the heavy tail behaviour of the stable distributed asset returns, compared with the normal case, can greatly affect decision making in asset allocation. Since the stable distribution reflects better the financial market reality, the better results in portfolio optimisation can be obtained under the stable distribution assumption.

Finally, we investigate the relationship between the maximum expected utility functions of an investor in stable and normal cases. Furthermore, we analyse the optimal risk aversion coefficients ratio obtained in normal and stable cases. The analysis showed significant differences in the stable and normal approaches, where the assumption of the stable distributed asset returns allowed for improvement on performance of the model.

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