

Sensitivity of portfolio VaR and CVaR to portfolio return characteristics

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Abstract

Risk management through marginal rebalancing is important for institutional investors due to the size of their portfolios. We consider the problem of improving marginally portfolio VaR and CVaR through a marginal change in the portfolio return characteristics. We study the relative significance of standard deviation, mean, tail thickness, and skewness in a parametric setting assuming a Student's t or a stable distribution for portfolio returns. We also carry out an empirical study with the constituents of DAX30, CAC40, and SMI. Our analysis leads to practical implications for institutional investors and regulators.

Key words: value-at-risk, conditional value-at-risk, Student's t distribution, stable distributions, marginal rebalancing

JEL Classification: G11, G32

1 Introduction

In the literature, there has been a debate about the properties of various risk measures and which risk measure is best from a practical viewpoint. From a historical perspective, variance was suggested as a proxy for risk by Markowitz (1952) as a part of a framework for portfolio selection that is still widely used by practitioners. The disadvantages of variance as a measure of risk are well documented in the literature – variance is not a true risk measure because it penalizes symmetrically both profit and loss. Since risk is an asymmetric phenomenon, a true risk measure should focus on the downside only; the upside potential should be irrelevant from a risk management perspective.

A risk measure which has been widely accepted since the 1990s is the value-at-risk (VaR). It was first popularized by JP Morgan and later by Risk-Metrics Group in their risk management software. VaR became so popular that it was approved by bank regulators as a valid approach for calculating capital reserves needed to cover market risks. It was found out, however, that VaR has an important disadvantage: it is not always sub-additive. This means that VaR may be incapable of identifying diversification opportunities. There has been a good deal of criticism of VaR in the literature because of this shortcoming but it remains a widely used method for risk measurement by practitioners mainly because it has an intuitive interpretation and because it is required by regulation.

The fact that VaR may be unable to detect diversification opportunities raised an important debate as to whether it is possible to define a set of desirable properties that a risk measure should satisfy. This is, essentially, an axiomatic approach towards defining risk measures. A set of such properties was given by Artzner et al. (1998) who defined axiomatically the family of coherent risk measures. A representative of coherent risk measures which gained popularity is conditional value-at-risk (CVaR), also known as average value-at-risk or expected tail loss. CVaR is more informative than VaR about extreme losses and is always sub-additive, implying it can always identify diversification opportunities. Even though CVaR has been discussed a good deal in the academic literature, it is not as widely used as VaR.

The choice of risk measure is an important step towards building a realistic picture of portfolio risk. There is, however, another essential component represented by the model for asset returns. It has been pointed out in numerous empirical studies that asset returns exhibit autoregressive behavior, clustering of volatility, skewness, and fat-tails. These phenomena should be accounted for by the probabilistic model, otherwise the risk measure may be unable to take into account appropriately the probability of extreme events.

For example, if the multivariate Gaussian distribution is selected as an asset return model, any risk measure will underestimate extreme losses because the normal distribution cannot describe the fat tails and the skewness observed in historical data. The combination of a risk measure and a probabilistic model we call simply a *risk model*.

From a practical viewpoint, computing only portfolio risk, without any additional information, provides a static picture. Although this may be sufficient to calculate capital reserves, it is insufficient to make decisions about changing portfolio allocation in order to improve certain portfolio risk-return characteristics. In this paper, we consider the following two aspects of this problem for VaR and CVaR.

First, a portfolio manager may have to decide among a number of rebalancing strategies for which the impact on the portfolio return characteristics is known in advance. Some of these strategies may influence predominantly the mean, the standard deviation or the skewness of the portfolio return distribution. Thus, the portfolio manager can make a decision knowing which portfolio return characteristic has the largest impact on portfolio risk or even design rebalancing strategies based on this information. We explore the sensitivity of portfolio VaR or CVaR to return distribution characteristics assuming that portfolio return follows the classical Student's t distribution or a stable Paretian distribution. Both distributional assumptions allow for analytic expressions for CVaR which is crucial for comparing the relative impact of portfolio return characteristics. We compare the relative importance of the portfolio mean, scale, skewness, and degree of heavy-tailedness.

Second, a portfolio manager can make a decision based solely on the risk model. The usual approach is to calculate the marginal contribution to risk (MCTR) of each position and then decrease the weights of the assets with the largest MCTR by a small amount and increase the weights of the assets with the smallest MCTR. The reduction in portfolio risk depends on the size of the MCTR of the positions we choose to rebalance but there is no information how the portfolio return distribution changes as a result. Comparing directly the cumulative distribution function (cdf) of the initial portfolio and the cdf of the rebalanced portfolio is a difficult task. For this reason, it is reasonable to study how the portfolio return characteristics change as a result of a rebalancing based on MCTR.

In order to do that, we need an explicit multivariate assumption for the asset returns because the calculation of MCTR requires it. We choose to use the historical method which assumes that historical observations are a sample from the multivariate distribution of asset returns. The rationale is that (1) in spite of its deficiencies the historical method is widely used in practice and (2) any multivariate parametric hypothesis requires statistical tests for

validity which is a very difficult statistical problem itself. We provide empirical examples with the constituents of three major European stock indexes – the German Deutscher Aktien Index (DAX30), the French Cotation Assistée en Continu (CAC40), and the Swiss Market Index (SMI).

The impact of portfolio return characteristics on portfolio risk can be important for two other reasons. First, in any parametric probabilistic model, certain distribution parameters get estimated from historical data. The parameter estimators have a certain variability which means that changing the input sample will result in different parameter estimates which will lead to a different portfolio risk. Knowing which parameter has the greatest impact on portfolio risk is important for identifying key aspects of the risk model that may need attention on a regular basis.

Second, from a regulatory perspective, understanding the impact of portfolio return characteristics on portfolio risk can lead to particular recommendations as to which areas in the risk estimation process require careful surveillance.

2 A parametric approach

In analyzing the significance of portfolio return characteristics, we calculate derivatives of portfolio risk with respect to the corresponding characteristics. In this way, we can answer the following question. Suppose that the portfolio manager can change the portfolio distribution characteristics by a small amount. Which characteristic has the biggest impact on portfolio risk?

We take advantage of two properties valid for VaR, CVaR, and all coherent risk measures, which hold irrespective of the distributional assumption. These properties are known as translation invariance,

$$\rho(X + a) = \rho(X) - a, \quad a \in \mathbb{R}, \quad (1)$$

and positive homogeneity,

$$\rho(\sigma X) = \sigma \rho(X), \quad \sigma > 0. \quad (2)$$

where ρ denotes a measure of risk and X is a random variable describing portfolio return.

These properties indicate that the sensitivity of portfolio risk with respect to the location parameter, or the mean, is always one and the same and equals -1. If the portfolio mean increases by 1%, then portfolio risk decreases by the same amount. In a similar way, the derivative with respect to the scale

parameter equals the risk of the standardized distribution. This statement will be made more precise in the following sections.

We proceed with computing numerically the derivatives of VaR and CVaR assuming that X follows a Student's t and a stable distribution. Both parametric assumptions can take into account heavy tails and the stable distribution can deal with skewness. In this way, we can compare the relative importance of the tail thickness, scale, mean, and skewness where applicable. We chose the Student's t model because it is a common assumption. There are two reasons for choosing stable distributions. First, because their tails are thicker than the Student's t tails, we can see how the relative importance of the tail thickness changes. Second, stable distributions are a very heavy-tailed model. This implies that the relative importance of the tail thickness will be weaker for any other distributional model with tails decaying faster than the tails of stable laws.

2.1 VaR

VaR is defined as the minimum level of loss at a given, sufficiently high, confidence level for a predefined time horizon. Formally, the VaR at confidence level $(1 - \epsilon)$ (tail probability ϵ) is defined as the negative of the lower ϵ -quantile of the return distribution,

$$VaR_\epsilon(X) = -\inf_x \{x : P(X \leq x) \geq \epsilon\} = -F_X^{-1}(\epsilon) \quad (3)$$

where X is a random variable describing portfolio return, $\epsilon \in (0, 1)$ and $F_X^{-1}(p)$ is the inverse of the cdf of X . Thus, VaR at tail probability 0.01 is in fact VaR at a confidence level equal to 0.99.

2.1.1 Student's t distribution

The random variable X is said to have Student's t distribution, $X \in t(\nu, \sigma, \mu)$, with a scale parameter $\sigma > 0$ and mean $\mu \in \mathbb{R}$ if $X = \sigma Y + \mu$ in which Y is a random variable with a density

$$f_Y(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}$$

where ν is the degrees of freedom parameter. In the classical definition, the Student's t distribution has a unit scale and a zero mean. In financial applications, however, the scale and location parameters are crucial. The degrees of freedom parameter controls the tail thickness – the smaller ν is,

the thicker the tail. If $\nu > 60$, the Student's t distribution is considered indistinguishable from the Gaussian distribution.

The VaR of the Student's t distribution is defined through the inverse cdf,

$$\text{VaR}_\epsilon(X) = \text{VaR}_\epsilon(\nu, \sigma, \mu) = -F_X^{-1}(\epsilon, \nu, \sigma, \mu).$$

We introduce the distribution parameters in the notation to emphasize that the VaR of the Student's t distribution depends on them. The corresponding derivatives are provided in the next theorem.

Proposition 1. *The derivatives of VaR with respect to the distribution parameters of the Student's t distribution equal*

$$\frac{\partial \text{VaR}_\epsilon(\nu, \sigma, \mu)}{\partial \nu} = -\sigma \frac{\partial F_X^{-1}(\epsilon, \nu, 1, 0)}{\partial \nu} \quad (4)$$

$$\frac{\partial \text{VaR}_\epsilon(\nu, \sigma, \mu)}{\partial \sigma} = \text{VaR}_\epsilon(\nu, 1, 0) \quad (5)$$

$$\frac{\partial \text{VaR}_\epsilon(\nu, \sigma, \mu)}{\partial \mu} = -1 \quad (6)$$

Proof. The expressions are derived by taking advantage of (1), (2), and the definition in (3). \square

Since the derivative with respect to σ equals the VaR of the normalized distribution, it follows that (5) is a decreasing function of ϵ and ν . The relationship between (5) and (6) is easy to establish because of this monotonic behavior. A numerical calculation shows that $\text{VaR}_{0.15}(60, 1, 0) = 1.046$ which means that for any $\epsilon \leq 0.15$ and any $\nu \leq 60$, $|\partial \text{VaR}_\epsilon(\nu, \sigma, \mu)/\partial \sigma| > |\partial \text{VaR}_\epsilon(\nu, \sigma, \mu)/\partial \mu|$. As a consequence of this relationship, a change in portfolio σ is more effective than a change in portfolio μ in reducing portfolio VaR. This conclusion holds for typical choices for the tail probability, $\epsilon = 0.01$ and $\epsilon = 0.05$ and any $\nu > 0$.

A similar analysis for (4) seems harder because the derivative depends on the scale parameter. It is possible to gain insight into the typical values of σ by relating the scale parameter to the standard deviation of X . In fact,

$$\text{stdev}(X) = \sigma \frac{\nu}{\nu - 2}, \text{ for } \nu > 2.$$

Typical values for the standard deviation of daily stock returns range between 1% and 6%. Bearing in mind that $\nu > 3$ is a reasonable constraint, it follows that the typical values for σ are approximately in this range too.

While this reasoning may be useful for a particular portfolio or a universe of stocks, it is not necessary to obtain a relationship between (4) and the other derivatives. Notice that ν controls the tail behavior and, therefore, we would expect to see a larger derivative in absolute value the deeper we go into the tail, i.e. for smaller ϵ other things fixed. We would expect to see the same behavior with respect to ν .

Figure 1 confirms these expectations. It shows the derivative in (4) as a function of ϵ and ν for $\sigma = 1$. We chose the case $\sigma = 1$ because it illustrates an important point. Even for this large and unrealistic choice for σ , the value of the derivative is comparable to $\partial VaR_\epsilon(\nu, \sigma, \mu)/\partial \mu$ only for tail probabilities beyond 0.01 and $\nu < 4$. For realistic choices of σ , the impact of the tail behavior becomes one order of magnitude weaker than the impact of μ . This implies that the influence of the tail behavior on VaR is extremely weak even for tail probabilities beyond 0.01 and very small values of ν .

This analysis implies that if a portfolio manager hesitates which parameter to influence most by a rebalancing strategy, the ordering is as follows. For the typical choices $\epsilon = 0.01, 0.05$, first they should try to reduce the scale parameter. The second most important parameter is the mean, μ . The tail behavior comes last. Its relative importance is several orders of magnitude weaker than the effect of the scale or the mean.

2.1.2 Stable distributions

Stable distributions are defined through their characteristic function. With very few exceptions, no closed-form expressions are known for their densities and distribution functions. A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$, $\mu \in \mathbb{R}$ such that its characteristic function $\varphi_X(t) = Ee^{itX}$ has the following form

$$\varphi_X(t) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) + i\mu t\}, & \alpha \neq 1 \\ \exp\{-\sigma |t|(1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \ln(|t|)) + i\mu t\}, & \alpha = 1 \end{cases} \quad (7)$$

where $\frac{t}{|t|} = 0$ if $t = 0$. Zolotarev (1986) and Samorodnitsky and Taqqu (1994) provide further details on the properties of stable distributions. Rachev and Mitnik (2000) provide applications in finance.

The parameters appearing in equation (7) are the following: α is called the *index of stability* or the *tail exponent*, β is a skewness parameter, σ is a scale parameter, and μ is a location parameter. Since stable distributions are uniquely determined by the four parameters, the common notation is $S_\alpha(\sigma, \beta, \mu)$.

Both the Student's t and stable distributions are heavy-tailed models. The difference, however, is in the tail thickness. The standard deviation of Student's t distribution is finite if $\nu > 2$ and it is infinite for stable distributions irrespective of the tail index parameter. Therefore, the tail of the stable distribution is, generally, heavier than the tail of the Student's t distribution. It is interesting to see if the relative importance of the distribution parameters changes if the distribution has a heavier tail.

The VaR of stable distributions is expressed through the inverse cdf,

$$\text{VaR}_\epsilon(X) = \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu) = -F_X^{-1}(\epsilon, \alpha, \beta, \sigma, \mu).$$

where $X \in S_\alpha(\sigma, \beta, \mu)$. Since no closed-form expressions for the inverse cdf exist, VaR can be calculated only numerically. The derivatives with respect to the distribution parameters are provided in the next proposition.

Proposition 2. *The derivatives of VaR with respect to the distribution parameters of stable distributions equal,*

$$\frac{\partial \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \alpha} = -\sigma \frac{\partial F_X^{-1}(\epsilon, \alpha, \beta, 1, 0)}{\partial \alpha} \quad (8)$$

$$\frac{\partial \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \beta} = -\sigma \frac{\partial F_X^{-1}(\epsilon, \alpha, \beta, 1, 0)}{\partial \beta} \quad (9)$$

$$\frac{\partial \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \sigma} = \text{VaR}_\epsilon(\alpha, \beta, 1, 0) \quad (10)$$

$$\frac{\partial \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \mu} = -1 \quad (11)$$

Proof. We follow the arguments in Proposition 1. □

In comparing the derivatives given in the proposition, there is an additional complexity due to the skewness parameter. Thus, the derivative in (10) depends on three parameters: ϵ , α , and β . A numerical calculation shows that

$$\frac{\partial \text{VaR}_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \sigma} > 1$$

if $\alpha \geq 1.3$, $\epsilon \leq 0.1$, and for any β . Both inequalities do not represent any real limitation for daily stock returns, see Rachev et al. (2005) for a large-scale empirical analysis. As a result, the scale parameter appears relatively more important than the location parameter for the usual choices $\epsilon = 0.01, 0.05$.

In order to compare the derivatives with respect to α and β , however, we would need to know which values for σ are realistic. In order to see the range

of values that σ can take in practice, we carried out an empirical experiment on the constituents of three European indexes – DAX30, CAC40 and SMI. There are 80 stocks altogether in the three universes. We estimated the stable distribution parameters of each stock using the daily returns from September 21, 2007 to September 7, 2009. Table 1 contains the 95% confidence intervals of the fitted α , β , and σ . We choose 0.03 as a value for σ which, according to Table 1, is a relatively conservative choice.

The impact of the tail parameter is expected to be stronger for lower tail probabilities. For this reason, we compare the derivative with respect to the tail index to the other derivatives with $\epsilon = 0.01$. Figure 2 includes a comparison to the derivative with respect to σ . The upper plot shows the contour lines of (8) as a function of α and β and the lower plot shows the corresponding contour lines of (10). The numbers indicate the value of the corresponding derivative along the contour line. The comparison indicates that the effect of the tail parameter is almost one order of magnitude weaker than the effect of σ irrespective of α and β . In contrast to the Student's t case, the tail index becomes a more significant factor than the location parameter for $\alpha < 1.6$ and $\beta < -0.4$. For distributions which are relatively symmetric, however, the mean remains a more significant factor than the tail thickness.

The derivative with respect to the skewness parameter with the same choice for ϵ and σ is shown on Figure 3. The plot indicates that the skewness parameter is clearly the least important one. For relatively symmetric distributions, it is about one order of magnitude weaker than the mean.

Increasing the tail probability above 0.01 results in a less significant tail index. Therefore, $\epsilon = 0.01$ seems to be a threshold beyond which the tail index becomes the second most significant parameter for relatively heavy-tailed and quite skewed representatives. Nevertheless, we can generalize that even at $\epsilon = 0.01$, the scale parameter is most effective in reducing stable VaR.

2.2 CVaR

By definition, CVaR equals the average VaR beyond a given VaR level. Formally,

$$CVaR_\epsilon(X) = \frac{1}{\epsilon} \int_0^\epsilon VaR_p(X) dp \quad (12)$$

where ϵ denotes the tail probability and $VaR_p(X)$ is defined in (3). CVaR, being an average of high quantiles, is by definition more sensitive to the tail behavior of X . In contrast to VaR, which can be calculated through

the inverse cdf, CVaR cannot be calculated directly through the definition. Therefore, we need an expression for CVaR for any particular distributional assumption of X .

We study the relative importance of the distribution characteristics for CVaR when X follows a Student's t distribution or a stable distribution. For both assumptions, there are expressions for CVaR which are suitable for numerical work.

2.2.1 Student's t distribution

Dokov et al. (2008) derive a general expression for the CVaR of a skewed Student's t distribution. The formula for the symmetric Student's t case is

$$CVaR_\epsilon(X) = \begin{cases} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{\sigma\sqrt{\nu}}{(\nu-1)\epsilon\sqrt{\pi}} \left(1 + \frac{(VaR_\epsilon(X))^2}{\nu}\right)^{\frac{1-\nu}{2}} - \mu & , \quad \nu > 1 \\ \infty & , \quad \nu = 1 \end{cases}$$

The derivatives of CVaR with respect to the distribution parameters have a structure similar to the corresponding derivatives of VaR. Since there is no material difference in the derivation, we provide the formulae without a proof:

$$\frac{\partial CVaR_\epsilon(\nu, \sigma, \mu)}{\partial \nu} = -\sigma \frac{\partial CVaR_\epsilon(\nu, 1, 0)}{\partial \nu} \quad (13)$$

$$\frac{\partial CVaR_\epsilon(\nu, \sigma, \mu)}{\partial \sigma} = CVaR_\epsilon(\nu, 1, 0) \quad (14)$$

$$\frac{\partial CVaR_\epsilon(\nu, \sigma, \mu)}{\partial \mu} = -1 \quad (15)$$

The derivative in (14) can be easily compared to that in (15). $CVaR_\epsilon(\nu, 1, 0)$ is monotonic with respect to ν and ϵ . A numerical calculation shows that $CVaR_{0.385}(60, 1, 0) = 1$. Therefore,

$$\frac{\partial CVaR_\epsilon(\nu, \sigma, \mu)}{\partial \sigma} \geq 1$$

for $\epsilon \leq 0.385$ and practically any ν . This result implies that the scale parameter is more efficient in changing portfolio CVaR if the tail probability is below 0.385. In the case of VaR, we obtained a similar result, however, with a threshold of $\epsilon = 0.15$.

In comparing the derivative in (13), we adopt the same strategy as in the case of Student's t VaR. Figure 4 shows (13) as a function of ϵ for

three choices of ν and $\sigma = 1$. Notice that these derivatives are higher in absolute value than the corresponding derivatives in Figure 1. This is because CVaR by definition averages the quantiles in the tail which implies a higher sensitivity to tail behavior. Even though the derivatives are generally higher, the relative importance of the tail behavior becomes smaller than that of the mean if $\nu \geq 5$.

Bearing in mind that σ is at least an order of magnitude higher, we can conclude that the order of the distribution parameters by importance remains the same as in the Student's t VaR case. Therefore, a portfolio manager would choose σ as the most effective way to change portfolio CVaR. Second comes the location parameter μ . The least important is the tail behavior represented by ν .

2.2.2 Stable distributions

Stoyanov et al. (2006) derived the CVaR for stable distributions. If $\alpha > 1$ and $VaR_\epsilon(X) \neq 0$, then the CVaR can be represented as

$$CVaR_\epsilon(X) = \sigma A_{\epsilon, \alpha, \beta} - \mu$$

where the term $A_{\epsilon, \alpha, \beta}$ does not depend on the scale and the location parameters. Concerning the term $A_{\epsilon, \alpha, \beta}$,

$$A_{\epsilon, \alpha, \beta} = \frac{\alpha}{1 - \alpha} \frac{|VaR_\epsilon(X)|}{\pi \epsilon} \int_{-\bar{\theta}_0}^{\pi/2} g(\theta) \exp\left(-|VaR_\epsilon(X)|^{\frac{\alpha}{\alpha-1}} v(\theta)\right) d\theta$$

where

$$g(\theta) = \frac{\sin(\alpha(\bar{\theta}_0 + \theta) - 2\theta)}{\sin \alpha(\bar{\theta}_0 + \theta)} - \frac{\alpha \cos^2 \theta}{\sin^2 \alpha(\bar{\theta}_0 + \theta)},$$

$$v(\theta) = (\cos \alpha \bar{\theta}_0)^{\frac{1}{\alpha-1}} \left(\frac{\cos \theta}{\sin \alpha(\bar{\theta}_0 + \theta)} \right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha \bar{\theta}_0 + (\alpha - 1)\theta)}{\cos \theta},$$

in which $\bar{\theta}_0 = \frac{1}{\alpha} \arctan\left(\bar{\beta} \tan \frac{\pi\alpha}{2}\right)$, $\bar{\beta} = -\text{sign}(VaR_\epsilon(X))\beta$, and $VaR_\epsilon(X)$ is the VaR of the stable distribution at tail probability ϵ .

The derivatives of CVaR with respect to the four distribution parameters are provided below without proof:

$$\frac{\partial CVaR_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \alpha} = -\sigma \frac{\partial CVaR_\epsilon(\alpha, \beta, 1, 0)}{\partial \alpha} \quad (16)$$

$$\frac{\partial CVaR_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \beta} = -\sigma \frac{\partial CVaR_\epsilon(\alpha, \beta, 1, 0)}{\partial \beta} \quad (17)$$

$$\frac{\partial CVaR_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \sigma} = CVaR_\epsilon(\alpha, \beta, 1, 0) \quad (18)$$

$$\frac{\partial CVaR_\epsilon(\alpha, \beta, \sigma, \mu)}{\partial \mu} = -1 \quad (19)$$

Computing numerically the CVaR for different choices of α , β , and ϵ , we find out that $CVaR_\epsilon(\alpha, \beta, 1, 0) > 1$ for any $\epsilon \leq 0.5$. This result implies that the scale parameter is relatively more important than the location parameter for all practical purposes.

In order to compare the other derivatives, we use the information in Table 1 once again. We compute the derivatives in (16) and (17) with $\epsilon = 0.01$ and $\sigma = 0.03$. Figure 5 includes two plots. The upper plot shows the contour lines of the derivative with respect to the tail index as a function of α and β . The bottom plot shows the contour lines of the derivative with respect to the scale parameter as a function of α and β . From the two plots we can conclude that the derivative with respect to σ is larger in absolute value than the derivative with respect to α for any of the (α, β) pairs on the plots. This implies that the scale parameter is more effective in reducing portfolio CVaR.

The top plot in Figure 5 indicates that the derivative with respect to α can be below or above -1. The contour line passing through the (α, β) pairs for which the derivative equals -1 is provided on the plot. For example in the symmetric case $\beta = 0$, $\alpha = 1.75$ is approximately the threshold dividing the stable laws into two categories. If $\alpha < 1.75$, then the tail index is more effective than the mean in reducing portfolio CVaR. If $\alpha > 1.75$, then the mean is more effective. If $\beta \neq 0$, then the corresponding threshold for α changes.

Concerning the derivative with respect to the skewness parameter, Figure 6 indicates that it is least effective in reducing portfolio CVaR. The derivative with respect to the location parameter is larger in absolute value for all of the (α, β) pairs shown on the plot.

3 A non-parametric approach

In contrast to the parametric approach, it is not possible to obtain a formula for VaR or CVaR involving other portfolio return characteristics if there is no explicit distributional hypothesis. Nevertheless, the distributional characteristics, which describe the shape of the distribution to some extent, do influence VaR or CVaR. For example, a portfolio manager can rebalance the portfolio and change portfolio skewness. This means that the shape of the distribution is impacted and, therefore, portfolio VaR and CVaR can be affected. As a result, instead of expressing the change of VaR or CVaR in terms of the change in the cdf, we can loosely try to express it in terms of the change in the distribution parameters, in this case skewness.

A more formal motivation of this idea can be based on the notions of modified VaR (mVaR) and modified CVaR (mCVaR) which are derived through the Cornish-Fisher expansion (see Boudt et al. (2009) and Zangari (1996)). mVaR equals

$$mVaR_\epsilon(X) = \mu - \left(z_\epsilon + \frac{1}{6}(z_\epsilon^2 - 1)s + \frac{1}{24}(z_\epsilon^3 - 3z_\epsilon)k - \frac{1}{36}(2z_\epsilon^3 - 5z_\epsilon)s^2 \right) \sigma \quad (20)$$

where $z_\epsilon = \Phi^{-1}(\epsilon)$ denotes the standardized Gaussian ϵ -quantile, μ is the mean of X , σ is the standard deviation of X , s denotes the skewness of X , and k denotes the excess kurtosis of X . By the method of derivation, mVaR can be viewed as an approximation to the theoretical VaR in which the distribution parameters appear explicitly,

$$VaR_\epsilon(X) = mVaR_\epsilon(X) + R_\epsilon$$

where R_ϵ denotes the residual which can, in theory, be calculated or approximated by means of the Cornish-Fisher expansion if the series is convergent. The advantage of mVaR is in the explicit relationship with the sample characteristics. A detailed discussion of the advantages and disadvantages of mVaR is not within the scope of this study. We only mention that the Cornish-Fisher expansion, on which mVaR is based, works well for non-normal distributions that are “close” to the Gaussian distribution and for tail probabilities which are not too small.

The explicit connection with the sample characteristics in (20) allows us to quantify the impact of small changes of those characteristics on mVaR. The first-order approximation to mVaR has the following form

$$dmVaR = d\mu + bd\sigma + R$$

where b denotes a coefficient scaling the impact of σ and R is a residual. The higher moments are not included because they appear in a joint product with σ and, therefore, they exercise a second- or higher-order effect. Since mVaR is an approximation to VaR itself, we can consider a similar first-order approximation to VaR.

Similar conclusions hold for CVaR and mCVaR. Our goal, however, is not to study the exact form of these first-order approximations and the corresponding residuals but to find an answer to a practical problem. If a portfolio manager rebalances marginally a given portfolio according to MCTR, then what is the corresponding effect on the portfolio return characteristics and to what degree the change in risk can be attributed to the change in these characteristics. We approach this problem using CVaR as a risk measure in order to calculate MCTR. MCTR in this case is the gradient of CVaR with respect to portfolio weights,

$$MCTR_i = \partial CVaR_\epsilon(w'X)/\partial w_i$$

which we approximate with the expectation $-E(X_i|w'X \leq VaR_\epsilon(w'X))$, see, for example, Boudt et al. (2009). In these expressions, w denotes a vector of portfolio weights and $X = (X_1, \dots, X_d)$ is a random vector describing asset returns. We consider only CVaR because it is not possible to compute from empirical data MCTR for VaR.

Since we are looking for a statistical relationship between the change in CVaR and the change in the corresponding sample characteristics, we carry out the following empirical experiments on the universe of the constituents of the DAX30, CAC40, and SMI for the period from September 21, 2007 to September 7, 2009. We start from an initial portfolio of randomly selected 20 stocks out of the 80 stocks in the universe. The initial weights are either chosen at random or are calculated in a specific way. For each stock in the portfolio, we calculate its MCTR and then we rank the stocks by MCTR. The weights of the three stocks with the largest MCTR are reduced by 0.1% and the weights of the three stocks with the smallest MCTR are increased by 0.1%. Thus, we sell stocks for a total of 0.3% of the portfolio value and buy stocks for the same amount of capital. We consider this rebalancing marginal in the sense that portfolio weights do not change much.

We calculate the CVaR of the initial portfolio and also the four portfolio characteristics – mean, standard deviation, skewness, and kurtosis.¹ The same quantities are computed for the rebalanced portfolio as well. This process is repeated 500 times. Finally, we estimate the following regression,

¹Skewness and kurtosis are estimated in a robust way since the classical estimator is sensitive to outliers.

$$\Delta CVaR_\epsilon(X) = b_1 \Delta \mu + b_2 \Delta \sigma + b_3 \Delta s + b_4 \Delta k + \delta \quad (21)$$

where a_i and b_i denote regression coefficients, the Δ operator denotes the change in the corresponding quantity, and δ denotes the residual terms. We include the skewness and kurtosis because we would like to study the incremental benefit of including the two higher moments in addition to the mean and standard deviation. Besides estimating the multivariate regression, we also calculate the effect of each factor on a stand-alone basis.

We proceed with two versions of this empirical study. The first study is based on initial portfolios that are away from the efficient frontier. The second study is based on initial portfolios that are the corresponding minimum-variance portfolios.

The rationale for having two studies is the following one. On the basis of the theory concerning Student's t and stable distributions, we noticed that the scale parameter has the most significant impact on portfolio CVaR. The second most significant parameter is the mean, and in more heavy-tailed cases, the tail thickness. Even though most effective, the scale sometimes may be impossible to reduce further. For instance, if we hold the minimum-variance portfolio, its scale cannot be reduced further. In this case, the theoretical arguments suggest that the mean should appear as the most effective way to reduce portfolio CVaR.

In the non-parametric case, there can be other factors besides the mean, scale, skewness, and kurtosis and this is the reason we approach the problem statistically – the influence of the remaining factors will be aggregated in the residual in (21). Note that the influence of the sample skewness and kurtosis is also determined by the extent to which there are skewed and leptokurtic stock returns in the sample. For example, if all stock returns are Gaussian, then the sample portfolio skewness and kurtosis will appear to have no significance at all which would be, however, valid for that particular data set.

Table 2 provides information about the distribution of the sample skewness and the excess kurtosis. A normal distribution has a skewness and excess kurtosis of zero. Apparently, the stocks in the universe are leptokurtic and mainly skewed to the left.

3.1 Inefficient initial portfolios

We start with the general case in which the initial portfolios are random, long-only portfolios. Some of them may happen to be close to the corresponding efficient frontier but since 500 portfolios are generated, the results cannot be

influenced significantly by few of such cases. Therefore, the results in this experiment are based on inefficient initial portfolios.

Below we provide results for three choices of tail probability: 0.5, 0.05 and 0.01. The case $\epsilon = 0.5$ implies that the CVaR averages all losses below the median of the distribution. In effect, a significant part of the body of the distribution has an impact on $CVaR_{0.5}(X)$ and, therefore, we would expect a relatively smaller impact of skewness and kurtosis. The choices 0.05 and 0.01 are standard. $CVaR_{0.05}(X)$ averages the losses beyond the 5% quantile of the distribution. Finally, we consider the CVaR at $\epsilon = 0.01$. At this tail probability, CVaR averages the losses beyond the 1% quantile.

Table 3 contains the results from the one-dimensional regressions which include the slope coefficients, the corresponding 95% confidence intervals, and the R^2 statistics. We draw three conclusions from the regression results. First, the standard deviation is the characteristic most affected by the MCTR-based rebalancing on the three tail probability levels. Its significance tends to decrease with the decrease in tail probability but stays relatively high for tail probabilities as low as 0.01. Second, the relative importance of kurtosis tends to increase a lot as tail probability decreases. It becomes more important than the mean at about $\epsilon = 0.05$. Third, the mean is the second most important parameter for $\epsilon = 0.5$. Its significance, however, decreases as tail probability decreases.

Table 4 contains the results from the multivariate regressions. The overall explanatory power of the multivariate regression does not change much and stays quite high as the tail probability changes from 0.5 to 0.05. It deteriorates as ϵ decreases further to 0.01 but remains at the level of 0.65.

3.2 Minimum-variance initial portfolios

There are situations in which standard deviation cannot be reduced further. In such a case, it cannot be the main characteristic influenced by the MCTR-based rebalancing. Such a situation arises if the starting portfolio is the minimum-variance portfolio. Starting from the minimum-variance portfolio, there is certainly room for improvement of CVaR because the minimum-variance portfolio may not coincide with the minimum CVaR portfolio.

We repeat the three experiments from Section 3.1. Table 5 includes the results from the one-dimensional regressions and Table 6 includes the results from the multivariate regression. We draw the following three conclusions. First, the importance of the mean relative to the skewness and kurtosis is approximately the same as in the inefficient portfolios case. Second, unlike the inefficient initial portfolio case, MCTR-based rebalancing of CVaR at small tail probabilities affects portfolio skewness most and not portfolio kurtosis.

Third, the explanatory power in all cases is significantly smaller than in the inefficient initial portfolios case.

4 Conclusion and implications

In this paper, we considered two important questions concerning reducing portfolio VaR and CVaR through marginal rebalancing. Two fat-tailed classes of distributions, Student's t and stable distributions, suggest that the portfolio scale parameter is most efficient in reducing portfolio VaR and CVaR for tail probabilities as low as 0.01. Increasing portfolio mean is, generally, the second most effective way to reduce portfolio VaR and CVaR. Reducing tail thickness may be more effective than portfolio mean only for CVaR with tail probabilities below 0.01 and relatively fat-tailed portfolio return distributions. It is not surprising that portfolio scale and mean are most effective in reducing portfolio VaR. It is, however, quite surprising that almost the same conclusions hold for CVaR. All conclusions are based on the assumption that the corresponding portfolio return characteristics can be changed by marginal rebalancing.

In the non-parametric case, we considered the approach of MCTR rebalancing based on CVaR being theoretically the most effective way of reducing portfolio CVaR. We found that starting from arbitrary portfolios constructed from the constituents of the DAX30, CAC40, and SMI, the standard deviation is the characteristic most affected by the rebalancing. Portfolio kurtosis is the second most affected characteristic only if the tail probability is beyond $\epsilon = 0.01$. These conclusions are relatively in line with the theoretical results based on Student's t and stable distributions.

Starting from the corresponding minimum-variance portfolio seems to reveal a slightly different picture. The portfolio mean is the most affected parameter but only for relatively high tail probabilities. If $\epsilon \leq 0.05$, portfolio skewness is the most affected parameter.

The fact that the scale, or standard deviation in the non-parametric case, appears to be the parameter to which both VaR and CVaR are very sensitive implies the following two key points. First, portfolio managers have to be very careful about properly estimating the portfolio scale parameter. As a result, not properly taking into account the clustering of volatility effect can have an very adverse effect on both portfolio VaR and CVaR. The clustering of volatility effect turns out be more important than tail thickness in the sense that a small error there can have a bigger impact than a small error in tail modeling. Second, a marginal reduction in portfolio VaR and CVaR can be very effectively implemented through the traditional approach of selling

proportionately parts of the holdings in the risky stocks and buying risk-free debt obligations.

The high relative importance of the mean has the following two implications. First, high quality mean forecasts are notoriously hard to obtain. Mean estimation may involve expert opinion and specific statistical models. As a result, there is a significant risk that a modeling error or an incorrect expert opinion can impact portfolio VaR and CVaR. Second, it is counterintuitive that the mean can be more significant than tail thickness and skewness for VaR at very low tail probabilities and even for CVaR which is a tail risk measure. An increase in the mean forecasts of 1% leads to a reduction of 1% in portfolio VaR and CVaR. Since means can be subjectively specified, there is room for manipulation of portfolio risk numbers.

Finally, the fact that the mean and standard deviation are significant factors in marginal rebalancing does not imply that VaR and CVaR basically reduce to the classical mean-variance portfolio framework formulated by Markowitz. Optimal portfolios based on CVaR minimization can have better return characteristics than optimal portfolios based on variance minimization. This, however, concerns global optimization and not the marginal improvement of portfolio CVaR.

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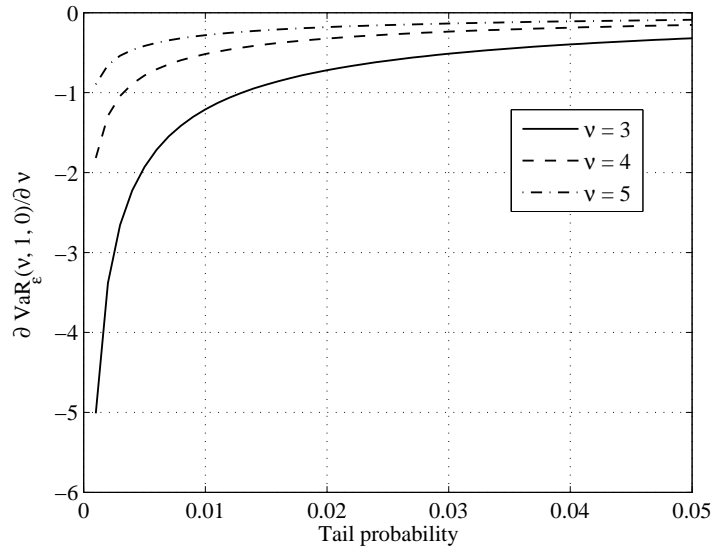
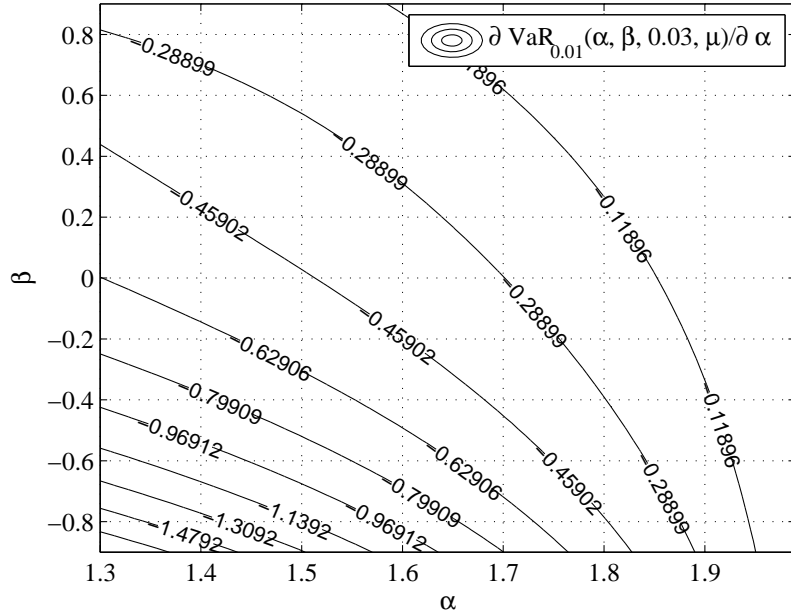
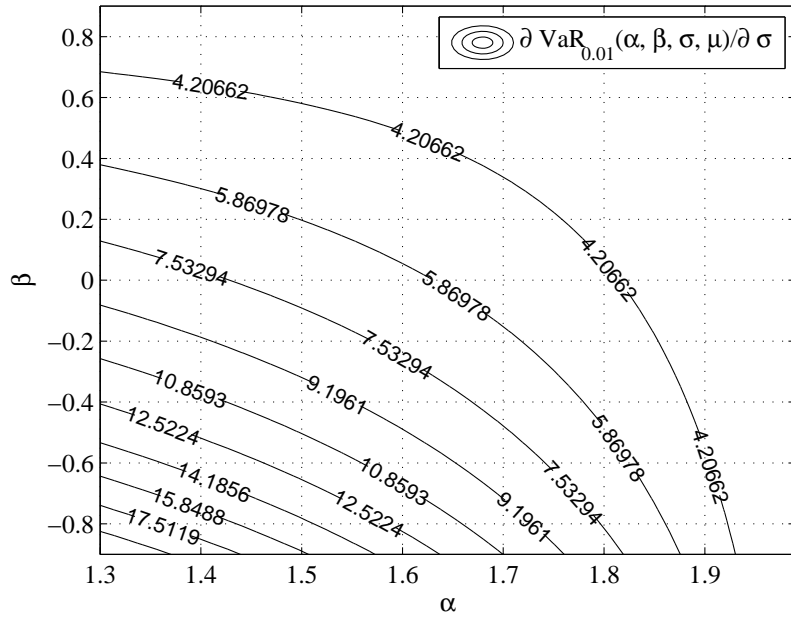


Figure 1: The derivative with respect to ν as a function of the tail probability, ϵ , and ν for $\sigma = 1$.



(a)



(b)

Figure 2: Plot (a) shows the contour lines of the derivative of stable VaR with respect to α for $\epsilon = 0.01$ and $\sigma = 0.03$; Plot (b) shows the contour lines of the derivative with respect to σ for $\epsilon = 0.01$.

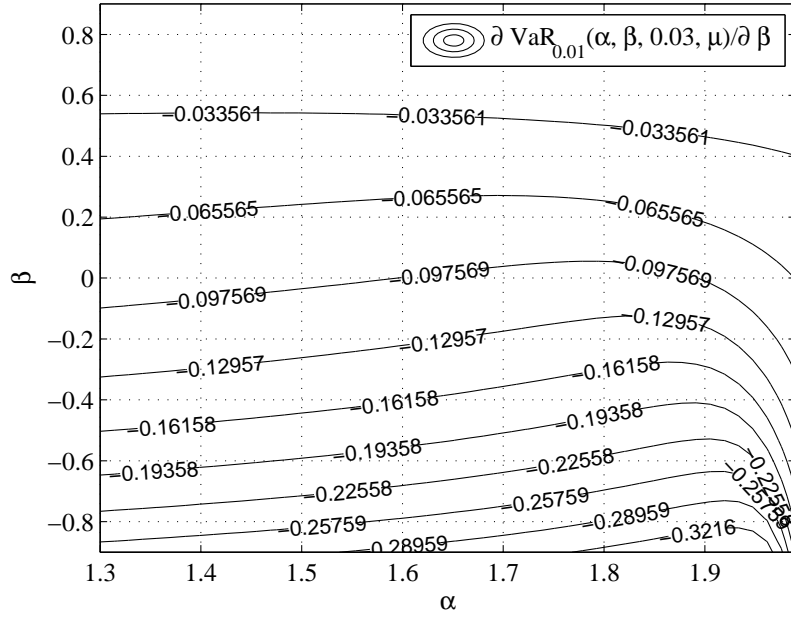


Figure 3: The contour lines of the derivative of stable VaR with respect to β for $\epsilon = 0.01$ and $\sigma = 0.03$.

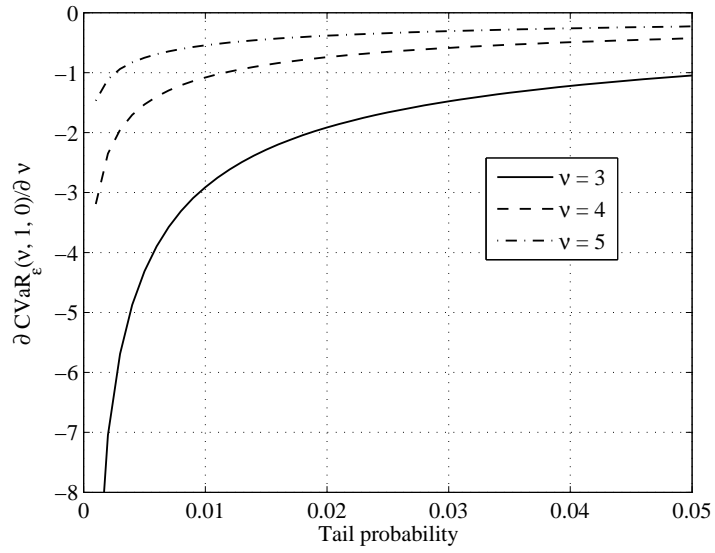
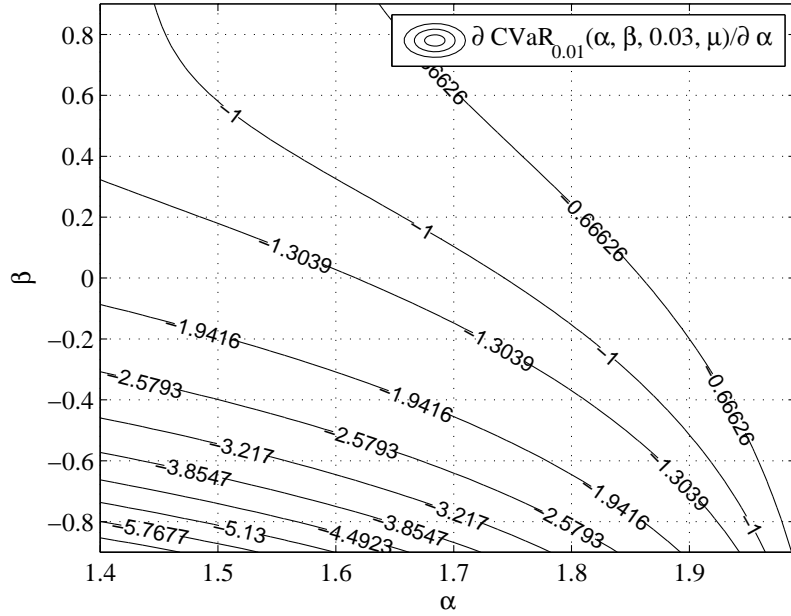
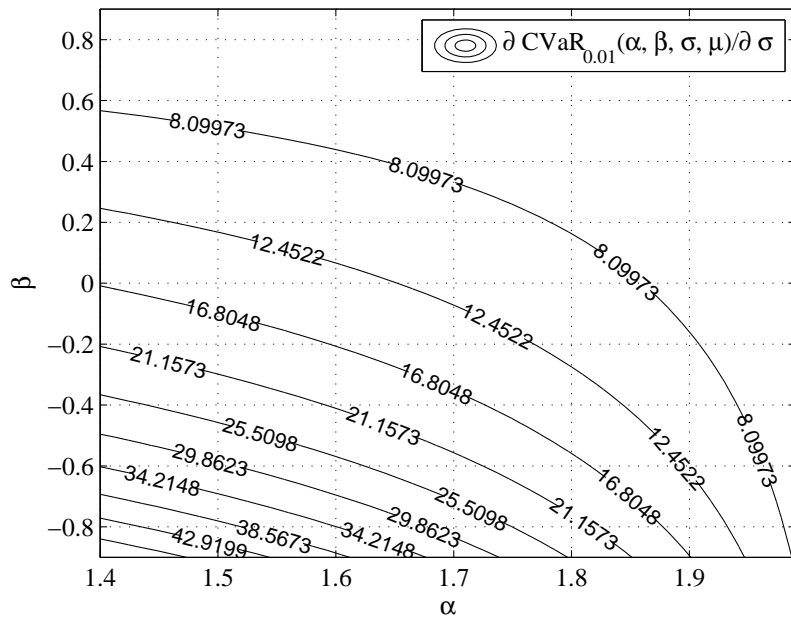


Figure 4: The derivative of Student's t CVaR with respect to ν as a function of the tail probability, ϵ , and ν for $\sigma = 1$.



(a)



(b)

Figure 5: Plot (a) shows the contour lines of the derivative of stable CVaR with respect to α for $\epsilon = 0.01$ and $\sigma = 0.03$; Plot (b) shows the contour lines of the derivative with respect to σ for $\epsilon = 0.01$.

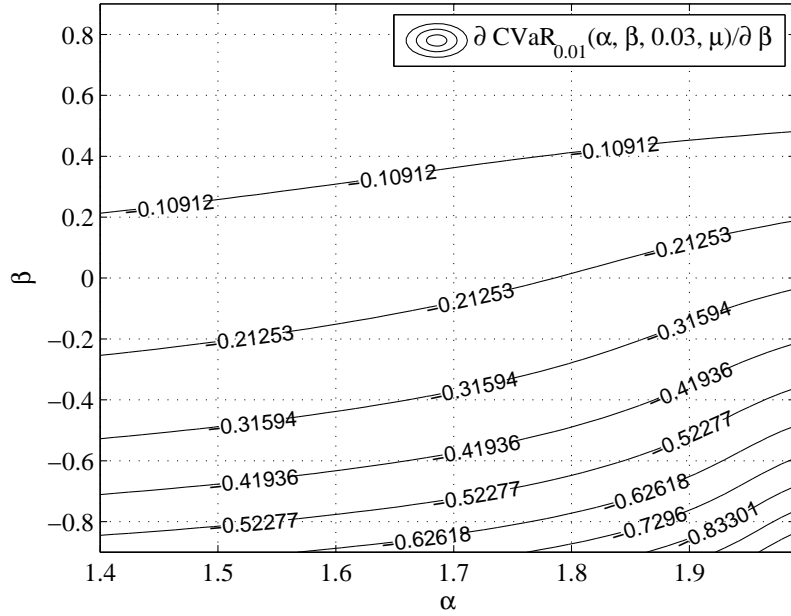


Figure 6: The derivative of stable CVaR with respect to β for $\epsilon = 0.01$ and $\sigma = 0.03$.

Parameter	95% confidence interval
$\hat{\alpha}$	[1.5, 1.93]
$\hat{\beta}$	[-0.9, 0.7]
$\hat{\sigma}$	[0.0089, 0.0265]

Table 1: The 95% confidence interval of stable distribution parameters fitted on the daily returns of the constituents of DAX30, CAC40, and SMI.

Characteristic	2.5% quantile	median	97.5% quantile
Skewness	-0.712	-0.114	0.312
Excess kurtosis	0.156	1.42	3.565

Table 2: The distribution of sample skewness and excess kurtosis of the 80 stocks comprising the DAX30, CAC40, and SMI.

	One dimensional model	Slope	95% CI	R^2
$\epsilon = 0.5$	$\Delta CVaR = a + b\Delta\mu + \delta$	-1.94	(-2.264, -1.6161)	0.22
	$\Delta CVaR = a + b\Delta\sigma + \delta$	0.56733	(0.54042, 0.59424)	0.78
	$\Delta CVaR = a + b\Delta s + \delta$	-0.00092	(-0.001, -0.0004)	0.03
	$\Delta CVaR = a + b\Delta k + \delta$	0.0003	(0.0001, 0.0004)	0.05
$\epsilon = 0.05$	$\Delta CVaR = a + b\Delta\mu + \delta$	-7.0572	(-8.6017, -5.5127)	0.14
	$\Delta CVaR = a + b\Delta\sigma + \delta$	2.3326	(2.2028, 2.4623)	0.71
	$\Delta CVaR = a + b\Delta s + \delta$	-0.0043	(-0.0059, -0.0026)	0.05
	$\Delta CVaR = a + b\Delta k + \delta$	0.0023	(0.0019, 0.0028)	0.18
$\epsilon = 0.01$	$\Delta CVaR = a + b\Delta\mu + \delta$	-10.3055	(-13.252, -7.3591)	0.09
	$\Delta CVaR = a + b\Delta\sigma + \delta$	3.0236	(2.7101, 3.3372)	0.42
	$\Delta CVaR = a + b\Delta s + \delta$	-0.005	(-0.009, -0.002)	0.02
	$\Delta CVaR = a + b\Delta k + \delta$	0.006	(0.005, 0.007)	0.25

Table 3: The slope coefficients, their 95% confidence intervals, and the R^2 statistic from one-dimensional regressions with inefficient initial portfolios and different choices for the tail probability. The magnitude of all intercept coefficients is below 2×10^{-4} .

	Variable	Slope	95% CI
$\epsilon = 0.5$	Mean	-1.2946	(-1.4267, -1.1625)
	St. dev.	0.6269	(0.60754, 0.64627)
	Skewness	0.0005	(0.0003, 0.0006)
	Kurtosis	-0.0003	(-0.0004, -0.0003)
	$R^2 = 0.93$		
$\epsilon = 0.05$	Mean	-1.3914	(-2.0396, -0.74315)
	St. dev.	2.2618	(2.1662, 2.3575)
	Skewness	-0.006	(-0.0076, -0.0063)
	Kurtosis	0.001	(0.00093, 0.0013)
	$R^2 = 0.89$		
$\epsilon = 0.01$	Mean	-3.8448	(-6.0896, -1.6)
	St. dev.	2.685	(2.3955, 2.9745)
	Skewness	-0.01	(-0.013, -0.008)
	Kurtosis	0.005	(0.004, 0.005)
	$R^2 = 0.65$		

Table 4: The slope coefficients, their 95% confidence intervals, and the R^2 statistic from the multivariate regressions with inefficient initial portfolios and different choices for the tail probability. The magnitude of the intercept is below 2×10^{-4} .

	One dimensional model	Slope	95% CI	R^2
$\epsilon = 0.5$	$\Delta CVaR = a + b\Delta\mu + \delta$	-0.59231	(-0.67781, -0.50681)	0.27
	$\Delta CVaR = a + b\Delta s + \delta$	-0.00011	(-0.00018, -4.7e-005)	0.02
	$\Delta CVaR = a + b\Delta k + \delta$	-8.2e-005	(-0.0001, -5.7e-005)	0.08
$\epsilon = 0.05$	$\Delta CVaR = a + b\Delta\mu + \delta$	-2.3796	(-2.9271, -1.8322)	0.13
	$\Delta CVaR = a + b\Delta s + \delta$	-0.0028	(-0.0032, -0.0024)	0.27
	$\Delta CVaR = a + b\Delta k + \delta$	0.00013	(-2.84e-005, 0.00029)	0.01
$\epsilon = 0.01$	$\Delta CVaR = a + b\Delta\mu + \delta$	-3.273	(-4.9396, -1.6064)	0.03
	$\Delta CVaR = a + b\Delta s + \delta$	-0.0071	(-0.0083, -0.0059)	0.22
	$\Delta CVaR = a + b\Delta k + \delta$	-0.00055	(-0.00097, -0.00012)	0.01

Table 5: The slope coefficients, their 95% confidence intervals, and the R^2 statistic from one-dimensional regressions with minimum-variance initial portfolios and different choices for the tail probability. The magnitude of all intercept coefficients is below 2×10^{-4} .

	Variable	Slope	95% CI
$\epsilon = 0.5$	Mean	-0.66722	(-0.74849, -0.58595)
	Skewness	7.3e-005	(1.44e-005, 0.00013)
	Kurtosis	-0.00011	(-0.00013, -9e-005)
	$R^2 = 0.4$		
$\epsilon = 0.05$	Mean	-1.7955	(-2.2426, -1.3483)
	Skewness	-0.0032	(-0.0036, -0.0028)
	Kurtosis	0.00055	(0.00042, 0.00069)
	$R^2 = 0.43$		
$\epsilon = 0.01$	Mean	-2.6838	(-4.1586, -1.209)
	Skewness	-0.0076	(-0.0089, -0.0063)
	Kurtosis	0.00047	(6.2e-005, 0.00088)
	$R^2 = 0.25$		

Table 6: The slope coefficients, their 95% confidence intervals, and the R^2 statistic from the multivariate regressions with mean-variance initial portfolios and different choices for the tail probability. The magnitude of the intercept is below 2×10^{-4} .