Computing VaR and AVaR In Infinitely Divisible Distributions

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Acknowledgment: Rachev gratefully acknowledges research support by grants from the Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst. The authors are grateful to Hyung Keun Koo for his helpful comments and suggestions.

Abstract

In this paper we derive closed-form solutions for the cumulative density function and the average value-at-risk for five subclasses of the infinitely divisible distributions: classical tempered stable distribution, Kim-Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. We present empirical evidence using the daily performance of the S&P 500 for the period January 2, 1997 through December 29, 2006.

Key words: tempered stable distribution, infinitely divisible distribution, value-at-risk, conditional value-at-risk, average value-at-risk.

2000 AMS Mathematics Subject Classifications: 60E07, 91B28

JEL Classifications: G11 G21

1 Introduction

In finance, numerous studies of return and price distributions of different asset classes and national financial markets reject the notion that the distributions are normal. The most popular alternative to the normal distribution is the class α-stable and tempered stable distributions. Although the α-stable distribution does not have finite moments, generally, tempered stable distributions have finite moments for all orders and finite exponential moments. Moreover, tempered stable distributions include non-Gaussian α-stable distributions as the limiting case. For this reason, tempered stable distributions have been preferred to the normal and used as extension of α-stable distributions for modeling the distribution of asset returns.

There is ample empirical evidence that daily asset returns are skewed and leptokurtic. These well-documented findings reported for asset returns are not mere academic conclusions that hold little interest for practitioners. Rather, they have important implications for asset managers and risk managers. Not properly accounting for these stylized facts can result in models that result in inferior investment performance by asset managers and disastrous financial consequences for financial institutions that rely upon them for risk management. More specifically, a thorough understanding of the tail loss distribution for a portfolio or trading position is critical for the design of stress tests. The failure of stress tests in identifying potential losses has been identified by several researchers as the cause of the failure of risk management systems to identify the losses suffered by the major dealers in the subprime mortgage market in 2007−2008. Although the risk management systems of these financial entities were structured such that they were compatible with what was thought to be the historical performance of subprime mortgage returns, they proved to be inadequate because of their failure to focus on the distribution in the tails. Better modeling of asset return distributions is an essential component of stress testing and should be considered in bank stress tests that are currently being formulated by bank regulators.

It is important to mention that random variables with tempered heavy tails, which are still infinitely divisible distributed, retain many of the properties of random variables with the usual heavy tails such as α-stable random variable (see Grabchak and Samorodnitsky, 2009, Klebanov *et al.*, 2006, and Rachev and Mittnik, 2000).

In particular, risk calculations with tempered heavy tails will have much in common with risk calculations with the usual heavy tails, although they are not identical. In particular, the density functions of a tempered stable random variable and a α-stable are comparable in the center, even if the tail behavior is slightly different. Furthermore, at level of processes, if the time scale increases, a tempered stable process converges to a Brownian motion, while if the time scale decreases it converges to a α-stable one. This property seems to be common for financial asset return processes. For this reason, in the class of infinitely divisible distributions we select distribution that belong to the tempered stable family.

The value-at-risk (VaR) measure has been adopted as a standard risk measure in the financial industry. Nevertheless, it has a number of well-known limitations as a risk measure. For example it does not satisfy the subadditivity property, and hence VaR is not a coherent risk measure.[1]

The average value-at-risk (AVaR) is the average of VaRs larger than the VaR for a given tail probability.[2] AVaR is a superior alternative to VaR because it satisfies all axioms of coherient risk measures and it is consistent with preference relations of risk-averse investors (see Rachev *et al.*, 2007).[3] Moreover, AVaR is still a coherent measure, while ETL is not. Consequently, in dealing with risk management and portfolio optimization problems, it is important to compute AVaR accurately for non-normal distributions. The closed-form solution for AVaR for

[1]The notion of a coherent risk measure was introduced by Artzner et al. (1999).

[2]AVaR is also known as conditional value-at-risk (CVaR). See Pflug (2000) and Rockafellar and Uryasev (2000, 2002)

[3]AVaR and another popular risk measure, expected tail loss (ETL), coincide if the loss distribution is continuous at the corresponding VaR level. However, if there is discontinuity, then AVaR and ETL differ

the α-stable distribution and on the skewed-t distribution have been presented by Stoyanov *et al.* (2006) and Dokov *et al.* (2008), respectively. Explicit formulas for VaR and AVaR are of great importance in operational risk assessment because of the need to calculate these risk measures at the extreme tail when the use of Monte Carlo methods is impractical.

In this paper, we develop a closed-form solution for the calculation of the VaR and AVaR on some subclass of infinitely divisible distributions. We apply this formula to five classes of tempered stable distributions, which are a parametric subclass of infinitely divisible distributions. The remainder of this paper is organized as follows. The integral representation of the cumulative density function and the AVaR are presented in Section 2. Section 3 discusses the computational issues. Section 4 reviews the five classes of tempered stable distributions and applies the formula for VaR and AVaR to each class. The empirical results are reported in Section 5. Section 6 summarizes the principal conclusions of the paper.

2 VaR and AVaR on infinitely divisible distributions

In this section, the random variable X represents the loss of a portfolio, and $F_X(x) = P(X \leq x)$, $\bar{F}_X(x) = P(X \geq x)$, $f_X(x) = \frac{d}{dx}F_X(x)$, $\phi_X(u) = E[e^{iuX}]$ stand for the cumulative density function (cdf), the complementary cumulative density function (ccdf), the probability density function (pdf), and the characteristic function (ch.f) of X, respectively. For convenience, in this paper we denote $(x)^+ = \max(x, 0)$, and $\Re(z)$ and $\Im(z)$ represent the real part and imaginary part of a complex number z, respectively.

We first investigate an integral representation of F_X.

Proposition 1. *Suppose a random variable X is infinitely divisible.*
(i) if there is $\rho > 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then

$$F_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_X(u+i\rho)}{\rho - ui}du\right), \quad \text{for } x \in \mathbb{R}. \tag{1}$$

(ii) if there is $\rho < 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then

$$\bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_X(u+i\rho)}{ui - \rho}du\right), \quad \text{for } x \in \mathbb{R}. \tag{2}$$

Proof. (i) By the definition of the cumulative density function, we have

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t)dt.$$

The probability density function $f_X(t)$ can be obtained from the characteristic function ϕ_X by the complex inverse formula (see Doetsch, 1970); that

$$f_X(t) = \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-itz} \phi_X(z) dz,$$

and we have

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-itz} \phi_X(z) dz dt \\ &= \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \int_{-\infty}^{x} e^{-itz} dt \, \phi_X(z) dz. \end{aligned}$$

Note that if $\rho > 0$, then

$$\lim_{t\to-\infty} \left| e^{-it(a+i\rho)} \right| = \lim_{t\to\infty} \left| e^{it(a+i\rho)} \right| = \lim_{t\to\infty} e^{-\rho t} = 0, \quad a \in \mathbb{R},$$

and hence

$$\int_{-\infty}^{x} e^{-itz} dt = -\frac{1}{iz} \left[e^{-itz} \right]_{-\infty}^{x} = -\frac{1}{iz} e^{-ixz}$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Thus, we have

$$\begin{aligned} F_X(x) &= -\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{1}{iz} e^{-ixz} \phi_X(z) dz \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i(u+i\rho)} e^{-ix(u+i\rho)} \phi_X(u+i\rho) du \\ &= \frac{e^{x\rho}}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} \frac{\phi_X(u+i\rho)}{\rho - iu} du. \end{aligned}$$

Let

$$g_\rho(u) = \frac{\phi_X(u+i\rho)}{\rho - iu},$$

then we can show that $g_\rho(-u) = \overline{g_\rho(u)}$ with $u \in \mathbb{R}$, and hence we have

$$\int_{-\infty}^{\infty} e^{-ixu} g_\rho(u) du = 2\Re \left(\int_{0}^{\infty} e^{-ixu} g_\rho(u) du \right).$$

Therefore we obtain (1).

(ii) By the definition of the complementary cumulative density function and the complex inverse formula, we have

$$\begin{aligned} \bar{F}_X(x) &= \int_{x}^{\infty} f_X(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \int_{x}^{\infty} e^{-itz} dt \, \phi_X(z) dz. \end{aligned}$$

Note that if $\rho < 0$, then

$$\lim_{t\to\infty} \left| e^{-it(a+i\rho)} \right| = \lim_{t\to\infty} e^{\rho t} = 0, \quad a \in \mathbb{R},$$

and hence

$$\int_x^\infty e^{-itz} dt = -\frac{1}{iz} \left[e^{-itz} \right]_x^\infty = \frac{1}{iz} e^{-ixz}$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Using similar arguments as in the proof of (i), we can prove (ii). $\square$

The VaR of X at tail probability ε is defined as

$$\begin{aligned} \text{VaR}_\varepsilon(X) &= \inf\{y \in \mathbb{R} : P(X \geq y) \leq (1-\varepsilon)\} \\ &= \inf\{y \in \mathbb{R} : F_X(y) \geq \varepsilon\}. \end{aligned}$$

The AVaR at tail probability ε is defined as the average of the VaRs which are larger than $\text{VaR}_\varepsilon(X)$, that is

$$\text{AVaR}_\varepsilon(X) = \frac{1}{1-\varepsilon} \int_\varepsilon^1 \text{VaR}_t(X) dt. \tag{3}$$

If the cumulative density function $F_X(x)$ is continuous, then we have

$$\text{VaR}_\varepsilon(X) = F_X^{-1}(\varepsilon) = \bar{F}_X^{-1}(1-\varepsilon) \tag{4}$$

and

$$\int_\varepsilon^1 \text{VaR}_t(X) dt = \int_\varepsilon^1 F_X^{-1}(t) dt = \int_{F_X^{-1}(\varepsilon)}^\infty s\, dF_X(s) = E\left[X 1_{\{X \geq F_X^{-1}(\varepsilon)\}} \right].$$

By (3), we obtain

$$\begin{aligned} \text{AVaR}_\varepsilon(X) &= \frac{1}{1-\varepsilon} E\left[X 1_{\{X \geq \text{VaR}_\varepsilon(X)\}} \right] \\ &= \frac{1}{1-\varepsilon} E[\text{VaR}_\varepsilon(X) 1_{\{X \geq \text{VaR}_\varepsilon(X)\}} + (X - \text{VaR}_\varepsilon(X))^+] \\ &= \text{VaR}_\varepsilon(X) + \frac{1}{1-\varepsilon} E[(X - \text{VaR}_\varepsilon(X))^+]. \end{aligned} \tag{5}$$

In order to obtain the closed-form solution of $\text{AVaR}_\varepsilon(X)$ for the infinitely divisible random variable X, we need the following lemma.

Lemma 1. *Let $K \in \mathbb{R}$. If X is infinitely divisible and there is $\rho < 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$E[(X-K)^+] = -\frac{e^{K\rho}}{\pi} \Re\left(\int_0^\infty e^{-iuK} \frac{\phi_X(u+i\rho)}{(u+i\rho)^2} du \right). \tag{6}$$

Proof. Since the characteristic function $\phi_X(z)$ of X is defined for all complex z with $\Im(z) = \rho$, the probability density function $f_X(x)$ of X is equal to

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz} \phi_X(z) dz$$

by the complex inversion formula. Thus we have,

$$\begin{aligned} E[(X-K)^+] &= \int_K^\infty (x-K) f_X(x) dx \\ &= \int_K^\infty (x-K) \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz} \phi_X(z) dz dx \\ &= \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \left(\int_K^\infty (x-K) e^{-ixz} dx \right) \phi_X(z) dz. \end{aligned}$$

Note that if $\rho < 0$, then

$$\lim_{x \to \infty} \left| \frac{1 + ix(a+i\rho)}{(a+i\rho)^2} e^{-ix(a+i\rho)} - \frac{iK}{(a+i\rho)} e^{-ix(a+i\rho)} \right| = 0,$$

for $a \in \mathbb{R}$. We have

$$\int_K^\infty (x-K) e^{-ixz} dx = \left[\frac{1+ixz}{z^2} e^{-ixz} - \frac{iK}{z} e^{-ixz} \right]_K^\infty = -\frac{e^{-iKz}}{z^2}$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Therefore,

$$E[(X-K)^+] = -\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{e^{-iKz}}{z^2} \phi_X(z) dz.$$

By using $u + i\rho$ instead of z, we obtain

$$E[(X-K)^+] = -\frac{e^{K\rho}}{2\pi} \int_{-\infty}^{\infty} e^{-iKu} \frac{\phi_X(u+i\rho)}{(u+i\rho)^2} du.$$

Let

$$h_\rho(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)^2},$$

then we can show that $h_\rho(-u) = \overline{h_\rho(u)}$ with $u \in \mathbb{R}$, and hence

$$\int_{-\infty}^{\infty} e^{-iuK} h_\rho(u) du = 2\Re \left(\int_0^\infty e^{-iuK} h_\rho(u) du \right),$$

which completes the proof. □

By Lemma 1, we obtain the closed-form solution of AVaR for continuous and infinitely divisible random variable as follows:

Proposition 2. *Suppose X is infinitely divisible and $F_X(x)$ is continuous. If there is $\rho < 0$ such that $|\phi_X(u)| < \infty$ for all u with $\Im(u) = \rho$, then*

$$\begin{aligned}&\text{AVaR}_\varepsilon(X) \\ &= \text{VaR}_\varepsilon(X) - \frac{e^{\text{VaR}_\varepsilon(X)\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_\varepsilon(X)}\frac{\phi_X(u+i\rho)}{(u+i\rho)^2}du\right).\end{aligned} \tag{7}$$

Proof. Equation (5) leads to equation (7) by substituting $K = \text{VaR}_\varepsilon(X)$ into equation (6). □

In operational risk management, the loss is always positive, and its distribution is right skewed and has a heavy right tail.[4] For this reason, the log-normal and log-α-stable random variable have been often used to model operational loss. We will derive the closed-form solution of the AVaR for a more general class of distributions, including the log-normal distribution.

Consider a random variable Y such that $\log Y$ is infinitely divisible. Then the random variable Y is referred to as the *log infinitely divisible random variable*. Since a normal random variable is infinitely divisible, the log-normal random variable is also log infinitely divisible. Using Proposition 1, we obtain the following corollary.

Corollary 1. *Assume random variable Y is log infinitely divisible and $\phi_{\log Y}$ is the characteristic function of $\log Y$.*
(i) If there is $\rho > 0$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then

$$F_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\log Y}(u+i\rho)}{\rho-ui}du\right), \quad y > 0. \tag{8}$$

(ii) If there is $\rho < 0$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then

$$\bar{F}_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\log Y}(u+i\rho)}{ui-\rho}du\right), \quad y > 0. \tag{9}$$

Proof. Since

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\log Y \le \log y) = F_{\log Y}(\log y), \quad y > 0,$$

where $F_{\log Y}$ is the cumulative density function of $\log Y$, we can prove (i) and (ii) by substituting $x = \log y$ into (i) and (ii) of Proposition 1, respectively. □

[4] See Chernobai *et al.* (2007).

If the cumulative density function $F_Y(y)$ of a log infinitely divisible random variable Y is continuous, then we have

$$\mathrm{VaR}_\varepsilon(Y) = F_Y^{-1}(\varepsilon) = \bar{F}_Y^{-1}(1-\varepsilon). \tag{10}$$

In order to obtain a closed-form solution of $\mathrm{AVaR}_\varepsilon(Y)$ for the log infinitely divisible random variable Y, we need the following lemma.

Lemma 2. *Assume random variable Y is the log infinitely divisible and $\phi_{\log Y}$ is the characteristic function of $\log Y$. If there is $\rho < -1$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$E[(Y-K)^+] = -\frac{K^{1+\rho}}{\pi}\Re\left(\int_0^\infty \frac{K^{-iu}\phi_{\log Y}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right), \tag{11}$$

for $K > 0$.

Proof. Let $X = \log Y$. Since the characteristic function $\phi_X(z)$ of X is defined for all complex z with $\Im(z) = \rho$, the probability density function $f_X(x)$ of X is equal to

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz}\phi_X(z)dz$$

by the complex inversion formula. Thus we have,

$$\begin{aligned}
E[(Y-K)^+] &= E[(e^X-K)^+] \\
&= \int_{\log K}^\infty (e^x-K)f_X(x)dx \\
&= \int_{\log K}^\infty (e^x-K)\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz}\phi_X(z)dzdx \\
&= \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho}\left(\int_{\log K}^\infty (e^x-K)e^{-ixz}dx\right)\phi_X(z)dz.
\end{aligned}$$

Note that if $\rho < -1$, then

$$\lim_{x\to\infty}\left|\frac{e^{(1-i(a+i\rho))x}}{1-i(a+i\rho)} + \frac{Ke^{-i(a+i\rho)x}}{i(a+i\rho)}\right| = 0,$$

for $a \in \mathbb{R}$. We have

$$\int_{\log K}^\infty (e^x-K)e^{-ixz}dx = \left[\frac{e^{(1-iz)x}}{1-iz} + \frac{Ke^{-izx}}{iz}\right]_{\log K}^\infty = -\frac{K^{1-iz}}{z(i+z)},$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Therefore,

$$E[(Y-K)^+] = -\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{K^{1-iz}}{z(i+z)}\phi_X(z)dz.$$

By using $u + i\rho$ instead of z, we obtain

$$\begin{aligned} E[(Y-K)^+] &= -\frac{K}{2\pi}\int_{-\infty}^{\infty} \frac{K^{-i(u+i\rho)}\phi_X(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du \\ &= -\frac{K^{1+\rho}}{2\pi}\int_{-\infty}^{\infty} \frac{K^{-iu}\phi_X(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du. \end{aligned}$$

Let

$$h_\rho(u) = \frac{K^{-iu}\phi_X(u+i\rho)}{(u+i\rho)(u+i(1+\rho))},$$

then we can show that $h_\rho(-u) = \overline{h_\rho(u)}$ with $u \in \mathbb{R}$, and hence

$$\int_{-\infty}^{\infty} h_\rho(u)du = 2\Re\left(\int_0^{\infty} h_\rho(u)du\right),$$

which completes the proof. □

Proposition 3. *Let Y be a log infinitely divisible random variable, and F_Y and $\phi_{\log Y}$ be the cumulative density function of Y and the characteristic function of $\log Y$, respectively. If $F_Y(x)$ is continuous for $x > 0$ and there is $\rho < -1$ such that $|\phi_{\log Y}(u)| < \infty$ for all u with $\Im(u) = \rho$, then*

$$\begin{aligned} &\text{AVaR}_\varepsilon(Y) \qquad (12) \\ &= \text{VaR}_\varepsilon(Y) - \frac{(\text{VaR}_\varepsilon(Y))^{1+\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^{\infty} \frac{(\text{VaR}_\varepsilon(Y))^{-iu}\phi_{\log Y}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right). \end{aligned}$$

Proof. Equation (5) leads to equation (12) by substituting $K = \text{VaR}_\varepsilon(Y)$ into equation (11). □

3 Computational issues

According to Proposition 1, the cumulative density function and the complementary cumulative density function of an infinitely divisible random variable X are equal to

$$\begin{aligned} F_X(x) &= \frac{e^{x\rho}}{\pi}\Re\left(\int_0^{\infty} e^{-ixu}g_1(u)du\right), \\ \bar{F}_X(x) &= \frac{e^{x\rho}}{\pi}\Re\left(\int_0^{\infty} e^{-ixu}g_2(u)du\right), \end{aligned}$$

where

$$g_1(u) = \frac{\phi_X(u+i\rho)}{\rho - ui} \text{ and } g_2(u) = \frac{\phi_X(u+i\rho)}{ui-\rho}.$$

By Proposition 2, AVaR of X is also obtained by

$$\text{AVaR}_\varepsilon(X) = x - \frac{e^{x\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iux} g_3(u)du\right),$$

where $x = \text{VaR}_\varepsilon(X)$ and

$$g_3(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)^2}.$$

By Proposition 3, AVaR of a log infinitely divisible random variable Y is also obtained by

$$\text{AVaR}_\varepsilon(Y) = e^x - \frac{e^{x(1-\rho)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iux} g_4(u)du\right),$$

where $x = \log \text{VaR}_\varepsilon(Y)$ and

$$g_4(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}.$$

Therefore, we can obtain the cumulative density function, the complementary cumulative density function, and AVaR, if we can compute the integral

$$\hat{g}(x) := \int_0^\infty e^{-ixu} g(u)du.$$

If $x_j = \frac{2\pi j}{N\varDelta u}$, $j = 0, 1, 2, \cdots, N-1$ for sufficiently large positive integer N and sufficiently small positive real value $\varDelta u$, then $\hat{g}(x_j)$ can be approximated by

$$\hat{g}(x_j) \approx \sum_{n=0}^{N-1} e^{-ix_j(n\varDelta u)} g(n\varDelta u)\varDelta u = \sum_{n=0}^{N-1} w^{nj} g_n,$$

where $w = e^{-2\pi i/N}$ and $g_n = g(n\varDelta u)\varDelta u$. If $x \in (x_j, x_{j+1})$, then $\hat{g}(x)$ is obtained by the interpolation between $\hat{g}(x_j)$ and $\hat{g}(x_{j+1})$. The approximation $\sum_{n=0}^{N-1} w^{nj} g_n$ is calculated efficiently using the fast Fourier transform (FFT) method. The FFT method is implemented by many numerical software packages.

The value $\text{Var}_\varepsilon(X) = \bar{F}_X^{-1}(1-\varepsilon)$ is a solution to the following equation:

$$\bar{F}_X(x) + \varepsilon - 1 = 0.$$

We can find the solution by various numerical methods such as the Newton-Raphson method. Using Newton-Raphson method, we iterate the following

$$x_{i+1} = x_i + \frac{\bar{F}_X(x_i) + \varepsilon - 1}{\bar{F}'_X(x_i)},$$

until the relative error between x_j and x_{j+1} becomes sufficiently small. In this case, $\bar{F}'_X(x_i) = -f_X(x_i)$ is the pdf of X, and it can be obtained numerically. The Newton-Raphson method is also implemented by many numerical software packages.

4 Tempered Stable Distributions

In this section, we present five subclasses of infinitely divisible distributions for modeling a portfolio loss distribution: classical tempered stable distribution, Kim-Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. In the literature, these distributions have been referred to as tempered stable distributions. In general, these distributions do not have closed-form solution for the probability density function. Instead, they are defined by their characteristic functions.

Below we will let a random variable X denote a tempered stable distributed random variable. Consider a random variable Y such that $\log Y$ is a tempered stable distribution. Then the random variable Y is referred to as the *log tempered stable random variable*.

4.1 Classical Tempered Stable Distribution

Let $\alpha \in (0,2)$, $C, \lambda_+, \lambda_- > 0$, and $m \in \mathbb{R}$. X is said to follow the classical tempered stable (CTS) distribution [5] if the characteristic function of X is given by

$$\begin{aligned}\phi_{\text{CTS}}(u) &:= \phi_X(u)\\ &= \exp(ium - iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})\\ &\qquad + C\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha)),\end{aligned}$$

and we denote $X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$. The mean of X is m, and cumulants $c_n(X) = \frac{d^n}{du^n} \log \phi_X(u)|_{u=0}$ of X are

$$c_n(X) = C\Gamma(n-\alpha)(\lambda_+^{\alpha-n} + (-1)^n \lambda_-^{\alpha-n}),$$

for $n = 2, 3, \cdots$.

[5] See Koponen (1995), Boyarchenko and Levendorskiǐ (2000), and Carr *et al.* (2002).

By analytic continuation in complex analysis, the function $\phi_{\text{CTS}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : \Im(z) \in (-\lambda_+, \lambda_-)\}$, that is $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) \in (-\lambda_+, \lambda_-)$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 1, equation (4), and Proposition 2 as follows:

$$
\begin{aligned}
&\bar{F}_{\text{CTS}}(x) := \bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{CTS}}(u+i\rho)}{ui-\rho}du\right),\\
&\text{VaR}_{\text{CTS}}(\varepsilon) := \text{VaR}_\varepsilon(X) = \bar{F}_{\text{CTS}}^{-1}(1-\varepsilon),\\
&\text{AVaR}_{\text{CTS}}(\varepsilon) := \text{AVaR}_\varepsilon(X)\\
&= \text{VaR}_{\text{CTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{CTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{CTS}}(\varepsilon)}\frac{\phi_{\text{CTS}}(u+i\rho)}{(u+i\rho)^2}du\right),
\end{aligned}
$$

for $-\lambda_+ < \rho < 0$.

If a random variable Y is a log infinitely divisible distribution such that

$$\log Y \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-CTS random variable*. If $\lambda_+ > 1$, then there exists $\rho < -1$ such that $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 1, equation (10), and Proposition 3 as follows:

$$
\begin{aligned}
&\bar{F}_{\text{logCTS}}(y) := \bar{F}_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\text{CTS}}(u+i\rho)}{ui-\rho}du\right), \quad y > 0,\\
&\text{VaR}_{\text{logCTS}}(\varepsilon) := \text{VaR}_\varepsilon(Y) = \bar{F}_{\text{logCTS}}^{-1}(1-\varepsilon),
\end{aligned}
$$

for $-\lambda_+ < \rho < 0$, and

$$
\begin{aligned}
&\text{AVaR}_{\text{logCTS}}(\varepsilon) := \text{AVaR}_\varepsilon(Y)\\
&= \text{VaR}_{\text{logCTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logCTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \frac{(\text{VaR}_{\text{logCTS}}(\varepsilon))^{iu}\phi_{\text{CTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right),
\end{aligned}
$$

for $-\lambda_+ < \rho < -1$.

4.2 Kim-Rachev Distribution

Let $\alpha \in (0,2) \setminus \{1\}$, , $k_+, k_-, r_+, r_- > 0$, $p_+, p_- \in \{p > -\alpha \mid p \neq -1, p \neq 0\}$, and $m \in \mathbb{R}$. X is said to follow the Kim-Rachev (KR) distribution[6] if the

[6]See Kim *et al.* (2008a,b).

characteristic function of X is given by

$$\begin{aligned}\phi_{\text{KR}}(u) &:= \phi_X(u)\\ &= \exp\Big(ium - iu\Gamma(1-\alpha)\left(\frac{k_+r_+}{p_++1} - \frac{k_-r_-}{p_-+1}\right)\\ &\qquad + k_+H(iu;\alpha,r_+,p_+) + k_-H(-iu;\alpha,r_-,p_-)\Big)\end{aligned}$$

where

$$H(x;\alpha,r,p) = \frac{\Gamma(-\alpha)}{p}\left({}_2F_1(p,-\alpha;1+p;rx) - 1\right)$$

where ${}_2F_1$ is the hypergeometric function,[7] and we denote $X \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$. The mean of X is m, and cumulants of X are

$$c_n(X) = \Gamma(n-\alpha)\left(\frac{k_+r_+^n}{p_++n} + (-1)^n\frac{k_-r_-^n}{p_-+n}\right),$$

for $n = 2, 3, \cdots$. If p_+ and p_- approach to the infinite, then KR distribution converges to the CTS distribution.

The function $\phi_{\text{KR}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : \Im(z) \in (-r_+^{-1}, r_-^{-1})\}$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{KR}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 1, equation (4), and Proposition 2 as follows:

$$\begin{aligned}&\bar{F}_{\text{KR}}(x) := \bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{KR}}(u+i\rho)}{ui-\rho}du\right),\\ &\text{VaR}_{\text{KR}}(\varepsilon) := \text{VaR}_\varepsilon(X) = \bar{F}_{\text{KR}}^{-1}(1-\varepsilon),\\ &\text{AVaR}_{\text{KR}}(\varepsilon) := \text{AVaR}_\varepsilon(X)\\ &= \text{VaR}_{\text{KR}}(\varepsilon) - \frac{e^{\rho\text{VaR}_{\text{KR}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{KR}}(\varepsilon)}\frac{\phi_{\text{KR}}(u+i\rho)}{(u+i\rho)^2}du\right),\end{aligned}$$

for $-r_+^{-1} < \rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$$

then Y is referred to as a *log-KR random variable*. If $1/r_+ > 1$, then there exists $\rho < -1$, such that $|\phi_{\text{KR}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence,

[7] See Andrews (1998).

$\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 1, equation (10), and Proposition 3 as follows:

$$\bar{F}_{\text{logKR}}(y) := \bar{F}_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\text{KR}}(u+i\rho)}{ui-\rho}du\right), \quad y>0,$$
$$\text{VaR}_{\text{logKR}}(\varepsilon) := \text{VaR}_\varepsilon(Y) = \bar{F}^{-1}_{\text{logKR}}(1-\varepsilon),$$

for $-r_+^{-1} < \rho < 0$, and

$$\begin{aligned}&\text{AVaR}_{\text{logKR}}(\varepsilon) := \text{AVaR}_\varepsilon(Y)\\ &= \text{VaR}_{\text{logKR}}(\varepsilon) - \frac{(\text{VaR}_{\text{logKR}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \frac{(\text{VaR}_{\text{logKR}}(\varepsilon))^{iu}\phi_{\text{KR}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right),\end{aligned}$$

for $-r_+^{-1} < \rho < -1$.

4.3 Modified Tempered Stable Distribution

Let $\alpha \in (0,2)\setminus\{1\}$, $C, \lambda_+, \lambda_- > 0$, and $m \in \mathbb{R}$. X is said to follow the modified tempered stable (MTS) distribution[8] if the characteristic function of X is given by

$$\begin{aligned}\phi_{MTS}(u) &:= \phi_X(u)\\ &= \exp(ium + C(G_R(u;\alpha,C,\lambda_+) + G_R(u;\alpha,C,\lambda_-))\\ &\quad + iuC(G_I(u;\alpha,\lambda_+) - G_I(u;\alpha,\lambda_-))),\end{aligned}$$

where for $u \in \mathbb{R}$,

$$G_R(x;\alpha,\lambda) = 2^{-\frac{\alpha+3}{2}}\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\left((\lambda^2+x^2)^{\frac{\alpha}{2}} - \lambda^\alpha\right)$$

and

$$G_I(x;\alpha,\lambda) = 2^{-\frac{\alpha+1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\lambda^{\alpha-1}\left[{}_2F_1\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{x^2}{\lambda^2}\right) - 1\right],$$

and we denote $X \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$. The mean of X is m, and cumulants of X are equal to

$$c_n(X) = 2^{n-\frac{\alpha+3}{2}}C\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)(\lambda_+^{\alpha-n} + (-1)^n\lambda_-^{\alpha-n}),$$

for $n = 2, 3, \cdots$.

[8] See Kim *et al.* (2008c).

The function $\phi_{\text{MTS}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : |\Im(z)| < \min\{\lambda_+, \lambda_-\}\}$, that is $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $|\Im(z)| < \min\{\lambda_+, \lambda_-\}$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, VaR$_\varepsilon(X)$, and AVaR$_\varepsilon(X)$ are obtained by Proposition 1, equation (4), and Proposition 2 as follows:

$$\begin{aligned}
&\bar{F}_{\text{MTS}}(x) := \bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{MTS}}(u+i\rho)}{ui-\rho}du\right),\\
&\text{VaR}_{\text{MTS}}(\varepsilon) := \text{VaR}_\varepsilon(X) = \bar{F}^{-1}_{\text{MTS}}(1-\varepsilon),\\
&\text{AVaR}_{\text{MTS}}(\varepsilon) := \text{AVaR}_\varepsilon(X)\\
&= \text{VaR}_{\text{MTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{MTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{MTS}}(\varepsilon)}\frac{\phi_{\text{MTS}}(u+i\rho)}{(u+i\rho)^2}du\right),
\end{aligned}$$

for $-\min\{\lambda_+, \lambda_-\} < \rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-MTS random variable*. If $\lambda_+ > 1$ and $\lambda_- > 1$, then there exists $\rho < -1$ such that $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_Y(x)$, VaR$_\varepsilon(Y)$, and AVaR$_\varepsilon(Y)$ are obtained by Corollary 1, equation (10), and Proposition 3 as follows:

$$\begin{aligned}
&\bar{F}_{\text{logMTS}}(y) := \bar{F}_Y(y) = \frac{y^{\rho}}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\text{MTS}}(u+i\rho)}{ui-\rho}du\right), \quad y > 0,\\
&\text{VaR}_{\text{logMTS}}(\varepsilon) := \text{VaR}_\varepsilon(Y) = \bar{F}^{-1}_{\text{logMTS}}(1-\varepsilon),
\end{aligned}$$

for $-\min\{\lambda_+, \lambda_-\} < \rho < 0$, and

$$\begin{aligned}
&\text{AVaR}_{\text{logMTS}}(\varepsilon) := \text{AVaR}_\varepsilon(Y)\\
&= \text{VaR}_{\text{logMTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logMTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \frac{(\text{VaR}_{\text{logMTS}}(\varepsilon))^{iu}\phi_{\text{MTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right),
\end{aligned}$$

for $-\min\{\lambda_+, \lambda_-\} < \rho < -1$.

4.4 Normal Tempered Stable Distribution

Let $\alpha \in (0,2)$, $C, \lambda > 0$, $|\beta| < \lambda$, and $m \in \mathbb{R}$. X is said to follow the normal tempered stable (NTS) distribution[9] if the characteristic function of X is given by

$$\begin{aligned}\phi_{NTS}(u) &:= \phi_X(u)\\ &= \exp\Big(ium + iu2^{-\frac{\alpha+1}{2}} C\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\alpha\beta(\lambda^2-\beta^2)^{\frac{\alpha}{2}-1}\\ &\qquad + 2^{-\frac{\alpha+1}{2}} C\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\left((\lambda^2-(\beta+iu)^2)^{\frac{\alpha}{2}} - (\lambda^2-\beta^2)^{\frac{\alpha}{2}}\right)\Big),\end{aligned}$$

and we denote $X \sim \text{NTS}(\alpha, C, \lambda, \beta, m)$. The mean of X is m. The general expressions for cumulants of X are omitted since they are rather complicated. Instead of the general form, we present three cumulants that

$$\begin{aligned}c_2(X) &= \kappa\alpha(\lambda^2-\beta^2)^{\frac{\alpha}{2}-2}(\alpha\beta^2-\lambda^2-\beta^2),\\ c_3(X) &= -\kappa\alpha\beta(\lambda^2-\beta^2)^{\frac{\alpha}{2}-3}(\alpha^2\beta^2-3\alpha\lambda^2-3\alpha\beta^2+6\lambda^2+2\beta^2),\\ c_4(X) &= \kappa\alpha(\alpha-2)(\lambda^2-\beta^2)^{\frac{\alpha}{2}-4}\\ &\qquad\times(\alpha^2\beta^4-6\alpha\lambda^2\beta^2-4\alpha\beta^4+3\beta^4+18\lambda^2\beta^2+3\lambda^4),\end{aligned}$$

where $\kappa = 2^{-\frac{\alpha+1}{2}} C\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)$.

The function $\phi_{NTS}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : \Im(z) \in (-\lambda+\beta, \lambda+\beta)\}$, that is $|\phi_{\text{NTS}}(z)| < \infty$ for all complex z with $\Im(z) \in (-\lambda+\beta, \lambda+\beta)$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{NTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 1, equation (4), and Proposition 2 as follows:

$$\begin{aligned}&\bar{F}_{\text{NTS}}(x) := \bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{NTS}}(u+i\rho)}{ui-\rho}du\right),\\ &\text{VaR}_{\text{NTS}}(\varepsilon) := \text{VaR}_\varepsilon(X) = \bar{F}_{\text{NTS}}^{-1}(1-\varepsilon),\\ &\text{AVaR}_{\text{NTS}}(\varepsilon) := \text{AVaR}_\varepsilon(X)\\ &\ = \text{VaR}_{\text{NTS}}(\varepsilon) - \frac{e^{\rho\text{VaR}_{\text{NTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{NTS}}(\varepsilon)}\frac{\phi_{\text{NTS}}(u+i\rho)}{(u+i\rho)^2}du\right),\end{aligned}$$

for $-\lambda+\beta < \rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{NTS}(\alpha, C, \lambda, \beta, m),$$

then Y is referred to as a *log-NTS random variable*. If $\lambda - \beta > 1$, then there exists $\rho < -1$, such that $|\phi_{\text{NTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence,

[9]See Barndorff-Nielsen and Levendorskii (2001) and Kim *et al.* (2008d).

$\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 1, equation (10), and Proposition 3 as follows:

$$\bar{F}_{\text{logNTS}}(y) := \bar{F}_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\text{NTS}}(u+i\rho)}{ui-\rho}du\right), \quad y>0,$$
$$\text{VaR}_{\text{logNTS}}(\varepsilon) := \text{VaR}_\varepsilon(Y) = \bar{F}^{-1}_{\text{logNTS}}(1-\varepsilon),$$

for $-\lambda+\beta<\rho<0$, and

$$\begin{aligned}&\text{AVaR}_{\text{logNTS}}(\varepsilon) := \text{AVaR}_\varepsilon(Y)\\ &= \text{VaR}_{\text{logNTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logNTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \frac{(\text{VaR}_{\text{logNTS}}(\varepsilon))^{iu}\phi_{\text{NTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right),\end{aligned}$$

for $-\lambda+\beta<\rho<-1$.

4.5 Rapidly Decreasing Tempered Stable Distribution

Let $\alpha\in(0,2)\backslash\{1\}$, $C,\lambda_+,\lambda_->0$, and $m\in\mathbb{R}$. X is said to follow the rapidly decreasing tempered stable (RDTS) distribution[10] if the characteristic function of X is given by

$$\begin{aligned}\phi_{RDTS}(u) &= \phi_X(u)\\ &= \exp\left(ium + C(G(iu;\alpha,\lambda_+)+G(-iu;\alpha,\lambda_-))\right),\end{aligned}$$

where

$$\begin{aligned}G(x;\alpha,\lambda) &= 2^{-\frac{\alpha}{2}-1}\lambda^\alpha\Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2},\frac{1}{2};\frac{x^2}{2\lambda^2}\right)-1\right)\\ &\quad + 2^{-\frac{\alpha}{2}-\frac{1}{2}}\lambda^{\alpha-1}x\Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2},\frac{3}{2};\frac{x^2}{2\lambda^2}\right)-1\right),\end{aligned}$$

and M is the confluent hypergeometric function Andrews (1998), and we denote $X\sim\text{RDTS}(\alpha,C,\lambda_+,\lambda_-,m)$. The mean of X is m, and cumulants of X are

$$c_n(X) = 2^{\frac{n-\alpha-2}{2}}C\Gamma\left(\frac{n-\alpha}{2}\right)\left(\lambda_+^{\alpha-n}+(-1)^n\lambda_-^{\alpha-n}\right),$$

for $n=2,3,\cdots$.

The function $\phi_{RDTS}(u)$ is expandable to an entire function on $\mathbb{C}$. Hence, $\text{AVaR}_\varepsilon(X)$ is obtained by equation (7) if $\rho<0$, that is $|\phi_{\text{RDTS}}(z)|<\infty$ for all

[10] See Bianchi *et al.* (2008) and Kim *et al.* (2009).

complex z. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{RDTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 1, equation (4), and Proposition 2 as follows:

$$
\begin{aligned}
&\bar{F}_{\text{RDTS}}(x) := \bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{RDTS}}(u+i\rho)}{ui-\rho}du\right),\\
&\text{VaR}_{\text{RDTS}}(\varepsilon) := \text{VaR}_\varepsilon(X) = \bar{F}^{-1}_{\text{RDTS}}(1-\varepsilon),\\
&\text{AVaR}_{\text{RDTS}}(\varepsilon) := \text{AVaR}_\varepsilon(X)\\
&= \text{VaR}_{\text{RDTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{RDTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{RDTS}}(\varepsilon)}\frac{\phi_{\text{RDTS}}(u+i\rho)}{(u+i\rho)^2}du\right),
\end{aligned}
$$

for $\rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-RDTS random variable*. Since $|\phi_{\text{RDTS}}(z)| < \infty$ for all complex z, we have $|\phi_{\text{RDTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho < -1$. Hence, $\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 1, equation (10), and Proposition 3 as follows:

$$
\begin{aligned}
&\bar{F}_{\text{logRDTS}}(y) := \bar{F}_Y(y) = \frac{y^{\rho}}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\text{RDTS}}(u+i\rho)}{ui-\rho}du\right), \quad y > 0,\\
&\text{VaR}_{\text{logRDTS}}(\varepsilon) := \text{VaR}_\varepsilon(Y) = \bar{F}^{-1}_{\text{logRDTS}}(1-\varepsilon),
\end{aligned}
$$

for $\rho < 0$, and

$$
\begin{aligned}
&\text{AVaR}_{\text{logRDTS}}(\varepsilon) := \text{AVaR}_\varepsilon(Y)\\
&= \text{VaR}_{\text{logRDTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logRDTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \frac{(\text{VaR}_{\text{logRDTS}}(\varepsilon))^{iu}\phi_{\text{RDTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))}du\right),
\end{aligned}
$$

for $\rho < -1$.

Characteristic functions, complementary cumulative density functions, VaRs, and AVaRs of tempered stable and log tempered stable random variables are presented in Table 1 and Table 2.

5 Empirical Example

In this section, we estimate parameters for the five tempered stable distributions and the normal distribution ($N(\mu, \sigma^2)$), and then calculate the AVaR for each using those estimated parameters. We use daily closing prices of the S&P 500 index from January 2, 1997 through December 29, 2006. Data were obtained from Yahoo! Finance. Daily losses are observed by taking the minus sign for daily log-returns of the S&P 500 index. The parameters are estimated using the maximum likelihood estimation (MLE). In this empirical study, we do not focus the operational risk. Hence, the closed-form solution of AVaR for log infinitely divisible random variables will not be concerned in this section.

We report the estimated parameters in Table 3. For the assessment of the goodness-of-fit, we utilize the Kolmogrov-Smirnov (KS) test. We also calculate the Anderson-Darling (AD) statistic to better evaluate the tail fit.[11] Table 3 gives the KS and AD statistics, and the p-value of the KS statistic. The normal distribution is rejected for our sample, but the other five tempered stable distributions are not rejected based on the p-value of KS statistic. The AD statistic of the normal fit is dramatically larger than the other classes; that is, the normal distribution does not capture the tail property of the empirical distribution.

In Table 4, the variance, skewness, and excess kurtosis are provided. As can be seen, the variance of the normal distribution is similar to the variance of the empirical distribution, but the normal distribution cannot describe the nonzero skewness and nonzero excess kurtosis of the empirical distribution. The five tempered stable distributions have positive skewness values and positive excess kurtosis values. From Table 4 it can be seen that the CTS and the RDTS distributions have the closest excess kurtosis and the closest skewness to the empirical values, respectively. Therefore, the tempered stable distributions are more realistic distributions in describing the data than the normal distribution for the historical data investigated.

Comparing the performance of AVaRs, we use *empirical AVaR* provided in Rachev *et al.* (2007) as a benchmark value. Denote (1) the observed portfolio (or asset) losses by $x_1,\ x_2,\ \cdots,\ x_n$ at time instant $t_1,\ t_2,\ \cdots,\ t_n$ and (2) the sorted sample by $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$. The empirical AVaR of the loss at tail

[11] The KS statistic is defined as

$$KS = \sup_{x_i} |F(x_i) - \hat{F}(x_i)|,$$

and the AD statistic is defined as

$$AD = \sup_{x_i} \frac{|F(x_i) - \hat{F}(x_i)|}{\sqrt{F(x_i)(1 - F(x_i))}},$$

where F is the cumulative distribution function with estimated parameters and $\hat{F}$ is the empirical cumulative distribution function for a given observation $\{x_i\}$.

probability ε is estimated by

$$\widehat{\text{AVaR}}_\varepsilon = \frac{1}{1-\varepsilon}\left(\frac{1}{n}\sum_{\lceil n\varepsilon\rceil+1}^{n} x_{(k)} + \left(\frac{\lceil n\varepsilon\rceil}{n} - \varepsilon\right) x_{(\lceil n\varepsilon\rceil)}\right),$$

where the notation $\lceil x\rceil$ stands for the smallest integer larger than x. In addition, *empirical VaR* is defined by

$$\widehat{\text{VaR}}_\varepsilon = x_{(\lceil n\varepsilon\rceil)}.$$

The VaR and the AVaR values for confidence levels $\{90\%, 91\%, \cdots, 99\%\}$ are provided in Table 5 and Table 6, respectively, and the values are also plotted in Figure 1 and Figure 2, respectively. The VaR and the AVaR values for confidence levels for extreme events, $\{99.1\%, 99.\%, \cdots, 99.9\%\}$, are provided in Table 9 and Table 10, respectively, and the values are also plotted in Figure 3 and Figure 4, respectively. Since the VaR and the AVaR values of the CTS, the MTS, the NTS, and the KR distributions are very similar, and the values of the RDTS distribution is more or less different from the KR case, we plot only the values of the KR and the RDTS distributions in the figures.

According to Table 5 and Figure 1, the normal VaR is larger than the empirical VaR, and the tempered stable VaRs are smaller, if the confidence level is less than or equal to 95%. If the confidence level is larger than 96%, the tempered stable VaRs are larger than the empirical VaR, and the normal VaR is smaller. If one uses the normal VaR with 99% confidence level for measuring the risk, the measured risk is less than the real risk in this empirical study. Moreover, according to Table 6 and Figure 2, the AVaRs of the tempered stable distributions are relatively similar to the empirical AVaR compared to the normal distribution.

According to Table 9 and Figure 3, normal VaR values are always smaller than empirical VaR values, and tempered stable VaR values are larger than empirical VaR values if confidence levels are less than 99.9%. According to Table 10 and Figure 4, the AVaRs of the tempered stable distributions are relatively similar to the empirical AVaR compared to the normal distribution considering extreme events.

Table 7 and Table 11 report relative errors between the empirical VaR and normal and tempered stable VaR, and Table 8 and Table 12 between the empirical AVaR and normal and tempered stable AVaR. The average of relative errors are also presented in the bottom line of both tables. The average errors of the normal VaR and normal AVaR have the largest value in those tables. The average errors of the KR VaR and the MTS VaR are the smallest in Table 7 and Table 11, respectively. The average errors of the RDTS AVaR and the KR AVaR are the smallest in Table 8 and Table 12, respectively.

6 Conclusion

In this paper, we derive closed-form solution of the AVaR for five subclasses of the infinitely divisible distribution. If a loss distribution is infinitely divisible and the characteristic function of the loss distribution is defined on the complex subset $\{z \in \mathbb{C} : \Im(z) = \rho\}$ for some $\rho < 0$, then we can obtain the closed-form solution of the AVaR. If a loss distribution is log infinitely divisible and its characteristic function is defined on the complex subset $\{z \in \mathbb{C} : \Im(z) = \rho\}$ for some $\rho < -1$, then we can also obtain closed-form solutions of the cumulative density function and the AVaR. In order to apply the closed-form solution we derived, we considered five tempered stable distributions: classical tempered stable distribution, Kim-Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. We estimated the parameters of those distributions for the S&P 500 index, and obtained VaR and AVaR values using closed-form solutions with the estimated parameters. In our investigation, the tempered stable VaR and AVaR are more realistic than the normal VaR and the normal AVaR.

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Table 1: The ch.f, ccdf, VaR, and AVaR for the tempered stable distribution

CTS	$X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ch.f	$\phi_{\text{CTS}}(u) = \exp[ium - iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})$ $+C\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha)]$
ccdf	$\bar{F}_{\text{CTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{CTS}}(u+i\rho)}{ui-\rho}du\right)$
VaR	$\text{VaR}_{\text{CTS}}(\varepsilon) = \bar{F}_{\text{CTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{CTS}}(\varepsilon) = \text{VaR}_{\text{CTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{CTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{CTS}}(\varepsilon)}\frac{\phi_{\text{CTS}}(u+i\rho)}{(u+i\rho)^2}du\right)$
Range of ρ	$-\lambda_+ < \rho < 0$
MTS	$X \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ch.f	$\phi_{\text{MTS}}(u) = \exp[ium + C(G_R(u;\alpha,C,\lambda_+) + G_R(u;\alpha,C,\lambda_-))$ $+iuC(G_I(u;\alpha,\lambda_+) - G_I(u;\alpha,\lambda_-))]$
	where $G_R(x;\alpha,\lambda) = 2^{-\frac{\alpha+3}{2}}\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\left((\lambda^2+x^2)^{\frac{\alpha}{2}} - \lambda^\alpha\right)$
	and $G_I(x;\alpha,\lambda) = 2^{-\frac{\alpha+1}{2}}\Gamma\left(\frac{1-\alpha}{2}\right)\lambda^{\alpha-1}\left[{}_2F_1\left(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{x^2}{\lambda^2}\right) - 1\right]$
ccdf	$\bar{F}_{\text{MTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{MTS}}(u+i\rho)}{ui-\rho}du\right)$
VaR	$\text{VaR}_{\text{MTS}}(\varepsilon) = \bar{F}_{\text{MTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{MTS}}(\varepsilon) = \text{VaR}_{\text{MTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{MTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{MTS}}(\varepsilon)}\frac{\phi_{\text{MTS}}(u+i\rho)}{(u+i\rho)^2}du\right)$
Range of ρ	$-\min\{\lambda_+, \lambda_-\} < \rho < 0$
NTS	$X \sim \text{NTS}(\alpha, C, \lambda, \beta, m)$
ch.f	$\phi_{\text{NTS}}(u) = \exp[ium + iu\kappa\beta(\lambda^2-\beta^2)^{\frac{\alpha}{2}-1} + \kappa((\lambda^2-(\beta+iu)^2)^{\frac{\alpha}{2}} - (\lambda^2-\beta^2)^{\frac{\alpha}{2}})]$
	where $\kappa = 2^{-\frac{\alpha+1}{2}}C\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)$
ccdf	$\bar{F}_{\text{NTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{NTS}}(u+i\rho)}{ui-\rho}du\right)$
VaR	$\text{VaR}_{\text{NTS}}(\varepsilon) = \bar{F}_{\text{NTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{NTS}}(\varepsilon) = \text{VaR}_{\text{NTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{NTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{NTS}}(\varepsilon)}\frac{\phi_{\text{NTS}}(u+i\rho)}{(u+i\rho)^2}du\right)$
Range of ρ	$-\lambda + \beta < \rho < 0$

Table 1: (continue) The ch.f, ccdf, VaR, and AVaR for the tempered stable distribution

KR	$X \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$
ch.f	$\phi_{\text{KR}}(u) = \exp\Big[ium - iu\Gamma(1-\alpha)\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1}\right)$
	$+k_+ H(iu; \alpha, r_+, p_+) + k_- H(-iu; \alpha, r_-, p_-)\Big]$
	where $H(x; \alpha, r, p) = \frac{\Gamma(-\alpha)}{p}\left({}_2F_1(p, -\alpha; 1+p; rx) - 1\right)$
ccdf	$\bar{F}_{\text{KR}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{KR}}(u+i\rho)}{ui-\rho}du\right)$
VaR	$\text{VaR}_{\text{KR}}(\varepsilon) = \bar{F}_{\text{KR}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{KR}}(\varepsilon) = \text{VaR}_{\text{KR}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{KR}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{KR}}(\varepsilon)}\frac{\phi_{\text{KR}}(u+i\rho)}{(u+i\rho)^2}du\right)$
Range of ρ	$-1/r_+ < \rho < 0$
RDTS	$X \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ch.f	$\phi_{\text{RDTS}}(u) = \exp[iumt + tC(G(iu; \alpha, \lambda_+) + G(-iu; \alpha, \lambda_-))$
	where $G(x; \alpha, \lambda) = 2^{-\frac{\alpha}{2}-1}\lambda^\alpha\Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)$
	$+2^{-\frac{\alpha}{2}-\frac{1}{2}}\lambda^{\alpha-1}x\Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)$
ccdf	$\bar{F}_{\text{RDTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\text{RDTS}}(u+i\rho)}{ui-\rho}du\right)$
VaR	$\text{VaR}_{\text{RDTS}}(\varepsilon) = \bar{F}_{\text{RDTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{RDTS}}(\varepsilon) = \text{VaR}_{\text{RDTS}}(\varepsilon) - \frac{e^{\rho \text{VaR}_{\text{RDTS}}(\varepsilon)}}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty e^{-iu\text{VaR}_{\text{RDTS}}(\varepsilon)}\frac{\phi_{\text{RDTS}}(u+i\rho)}{(u+i\rho)^2}du\right)$
Range of ρ	$\rho < 0$

Table 2: The ccdf, VaR, and AVaR for the log tempered stable distribution

log-CTS	$\log Y \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ccdf	$\bar{F}_{\text{logCTS}}(y) = \frac{y^\rho}{\pi} \Re \left(\int_0^\infty y^{-iu} \frac{\phi_{\text{CTS}}(u+i\rho)}{ui-\rho} du \right)$
VaR	$\text{VaR}_{\text{logCTS}}(\varepsilon) = \bar{F}^{-1}_{\text{logCTS}}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{logCTS}}(\varepsilon) = \text{VaR}_{\text{logCTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logCTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)} \Re \left(\int_0^\infty \frac{(\text{VaR}_{\text{logCTS}}(\varepsilon))^{iu} \phi_{\text{CTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))} du \right)$
Range of ρ	$-\lambda_+ < \rho < -1$
log-MTS	$\log Y \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ccdf	$\bar{F}_{\text{logMTS}}(y) = \frac{y^\rho}{\pi} \Re \left(\int_0^\infty y^{-iu} \frac{\phi_{\text{MTS}}(u+i\rho)}{ui-\rho} du \right)$
VaR	$\text{VaR}_{\text{logMTS}}(\varepsilon) = \bar{F}^{-1}_{\text{logMTS}}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{logMTS}}(\varepsilon) = \text{VaR}_{\text{logMTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logMTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)} \Re \left(\int_0^\infty \frac{(\text{VaR}_{\text{logMTS}}(\varepsilon))^{iu} \phi_{\text{MTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))} du \right)$
Range of ρ	$-\min\{\lambda_+, \lambda_-\} < \rho < -1$
log-NTS	$\log Y \sim \text{NTS}(\alpha, C, \lambda, \beta, m)$
ccdf	$\bar{F}_{\text{logNTS}}(y) = \frac{y^\rho}{\pi} \Re \left(\int_0^\infty y^{-iu} \frac{\phi_{\text{NTS}}(u+i\rho)}{ui-\rho} du \right)$
VaR	$\text{VaR}_{\text{logNTS}}(\varepsilon) = \bar{F}^{-1}_{\text{logNTS}}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{logNTS}}(\varepsilon) = \text{VaR}_{\text{logNTS}}(\varepsilon) - \frac{(\text{VaR}_{\text{logNTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)} \Re \left(\int_0^\infty \frac{(\text{VaR}_{\text{logNTS}}(\varepsilon))^{iu} \phi_{\text{NTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))} du \right)$
Range of ρ	$-\lambda + \beta < \rho < -1$

Table 2: (continue) The ccdf, VaR, and AVaR for the log tempered stable distribution

log-KR	$\log Y \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$
ccdf	$\bar{F}_{\mathrm{logKR}}(y) = \frac{y^\rho}{\pi} \Re \left(\int_0^\infty y^{-iu} \frac{\phi_{\mathrm{KR}}(u+i\rho)}{ui-\rho} du \right)$
VaR	$\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logKR}}(1-\varepsilon)$
AVaR	$\mathrm{AVaR}_{\mathrm{logKR}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logKR}}(\varepsilon) - \frac{(\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)} \Re \left(\int_0^\infty \frac{(\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon))^{iu} \phi_{\mathrm{KR}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))} du \right)$
Range of ρ	$-1/r_+ < \rho < -1$
log-RDTS	$\log Y \sim \mathrm{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$
ccdf	$\bar{F}_{\mathrm{logRDTS}}(y) = \frac{y^\rho}{\pi} \Re \left(\int_0^\infty y^{-iu} \frac{\phi_{\mathrm{RDTS}}(u+i\rho)}{ui-\rho} du \right)$
VaR	$\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logRDTS}}(1-\varepsilon)$
AVaR	$\mathrm{AVaR}_{\mathrm{logRDTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon) - \frac{(\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon))^{1-\rho}}{\pi(1-\varepsilon)} \Re \left(\int_0^\infty \frac{(\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon))^{iu} \phi_{\mathrm{RDTS}}(u+i\rho)}{(u+i\rho)(u+i(1+\rho))} du \right)$
Range of ρ	$\rho < -1$

Table 3: Maximum likelihood estimation and statistic of goodness of fit tests.

Model	**Model Parameters**					KS	p-value	AD
	μ	σ						
Normal	$-2.5734E-4$	$1.1467E-2$				0.0485	$1.5045E-5$	19.5700
	m	α	C	λ_+	λ_-			
CTS	$-2.5934E-4$	0.8444	$1.1247E-2$	77.8805	82.4561	0.0151	0.6191	0.0368
	m	α	C	λ_+	λ_-			
MTS	$-2.7172E-4$	0.9983	$3.7780E-3$	86.7298	91.4724	0.0150	0.6240	0.0380
	m	α	C	λ	β			
NTS	$-2.5921E-4$	0.9986	$3.7708E-3$	89.0979	2.2992	0.0152	0.6129	0.0367
	m	α	k_+	r_+				
KR	$-2.5841E-4$	0.8300	2.9496	0.0143		0.0151	0.6157	0.0369
		$p_+ = p_-$	k_-	r_-				
		6.1491	3.0929	0.0135				
	m	α	C	λ_+	λ_-			
RDTS	$-3.0201E-4$	1.3033	$5.6806E-4$	30.5918	31.7721	0.0155	0.5807	0.0430

Table 4: Standard deviation, skewness, and excess kurtosis

Model	standard deviation	skewness	excess kurtosis
Empirical	0.0115	0.0546	3.0856
Normal	0.0115	0	0
CTS	0.0115	0.0772	2.9438
MTS	0.0115	0.0663	2.8741
NTS	0.0115	0.0757	2.8749
KR	0.0115	0.0778	2.9721
RDTS	0.0130	0.0489	4.2597

Table 5: One-day VaR (90%−99% confidence level)

Confidence Level (ε)	Empirical	Normal	CTS	MTS	NTS	KR	RDTS
0.90	0.0135	0.0145	0.0129	0.0129	0.0129	0.0129	0.0128
0.91	0.0142	0.0152	0.0137	0.0137	0.0137	0.0137	0.0136
0.92	0.0151	0.0159	0.0146	0.0146	0.0146	0.0146	0.0145
0.93	0.0160	0.0167	0.0156	0.0156	0.0156	0.0156	0.0155
0.94	0.0170	0.0176	0.0168	0.0168	0.0168	0.0168	0.0166
0.95	0.0183	0.0187	0.0182	0.0182	0.0182	0.0182	0.0180
0.96	0.0195	0.0199	0.0199	0.0199	0.0199	0.0199	0.0198
0.97	0.0219	0.0214	0.0222	0.0221	0.0222	0.0222	0.0221
0.98	0.0247	0.0234	0.0254	0.0254	0.0254	0.0254	0.0254
0.99	0.0290	0.0265	0.0311	0.0311	0.0311	0.0311	0.0312

Table 6: One-day AVaR (90%−99% confidence level)

Confidence Level (ε)	Empirical	Normal	CTS	MTS	NTS	KR	RDTS
0.90	0.0207	0.0153	0.0208	0.0207	0.0208	0.0208	0.0205
0.91	0.0214	0.0161	0.0216	0.0216	0.0216	0.0216	0.0213
0.92	0.0223	0.0170	0.0225	0.0225	0.0225	0.0225	0.0222
0.93	0.0232	0.0179	0.0236	0.0235	0.0236	0.0236	0.0233
0.94	0.0243	0.0190	0.0248	0.0248	0.0248	0.0248	0.0245
0.95	0.0257	0.0203	0.0263	0.0262	0.0263	0.0263	0.0260
0.96	0.0274	0.0215	0.0281	0.0281	0.0281	0.0281	0.0278
0.97	0.0296	0.0239	0.0305	0.0304	0.0305	0.0305	0.0301
0.98	0.0328	0.0272	0.0339	0.0338	0.0339	0.0339	0.0335
0.99	0.0390	0.0337	0.0399	0.0397	0.0398	0.0399	0.0393

Table 7: Relative errors between the empirical VaR and the parametric VaR (90%−99% confidence level)

Confidence Level (ε)	Relative error= $\frac{\lvert\widehat{\text{VaR}}_\varepsilon - \text{VaR}_\varepsilon(X)\rvert}{\widehat{\text{VaR}}_\varepsilon}$					
	Normal	CTS	MTS	NTS	KR	RDTS
0.90	0.0719	0.0433	0.0451	0.0433	0.0433	0.0528
0.91	0.0666	0.0349	0.0368	0.0348	0.0351	0.0448
0.92	0.0562	0.0298	0.0316	0.0296	0.0300	0.0398
0.93	0.0464	0.0224	0.0241	0.0220	0.0227	0.0322
0.94	0.0372	0.0120	0.0136	0.0115	0.0124	0.0213
0.95	0.0207	0.0054	0.0068	0.0046	0.0059	0.0136
0.96	0.0220	0.0235	0.0221	0.0244	0.0228	0.0166
0.97	0.0229	0.0128	0.0116	0.0140	0.0121	0.0086
0.98	0.0532	0.0287	0.0275	0.0299	0.0278	0.0277
0.99	0.0864	0.0713	0.0698	0.0721	0.0705	0.0738
Average	0.0483	0.0284	0.0289	0.0286	0.0283	0.0331

Table 8: Relative errors between the empirical AVaR and the parametric AVaR (90%−99% confidence level)

Confidence Level (ε)	Relative error= $\frac{\lvert\widehat{\text{AVaR}}_\varepsilon - \text{AVaR}_\varepsilon(X)\rvert}{\widehat{\text{AVaR}}_\varepsilon}$					
	Normal	CTS	MTS	NTS	KR	RDTS
0.90	0.2587	0.0050	0.0030	0.0052	0.0046	0.0096
0.91	0.2492	0.0085	0.0066	0.0087	0.0081	0.0058
0.92	0.2382	0.0124	0.0104	0.0127	0.0120	0.0015
0.93	0.2305	0.0162	0.0142	0.0164	0.0158	0.0026
0.94	0.2194	0.0202	0.0182	0.0204	0.0197	0.0071
0.95	0.2104	0.0248	0.0227	0.0249	0.0243	0.0122
0.96	0.2142	0.0270	0.0248	0.0270	0.0266	0.0150
0.97	0.1906	0.0311	0.0287	0.0309	0.0308	0.0196
0.98	0.1713	0.0343	0.0314	0.0336	0.0341	0.0226
0.99	0.1373	0.0222	0.0182	0.0203	0.0224	0.0076
Average	0.2120	0.0202	0.0178	0.0200	0.0198	0.0104

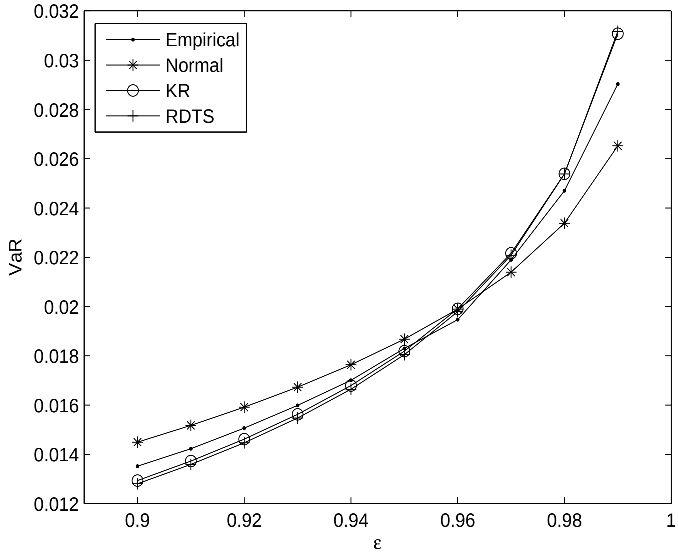

Figure 1: One-day VaR (90%−99% confidence level)

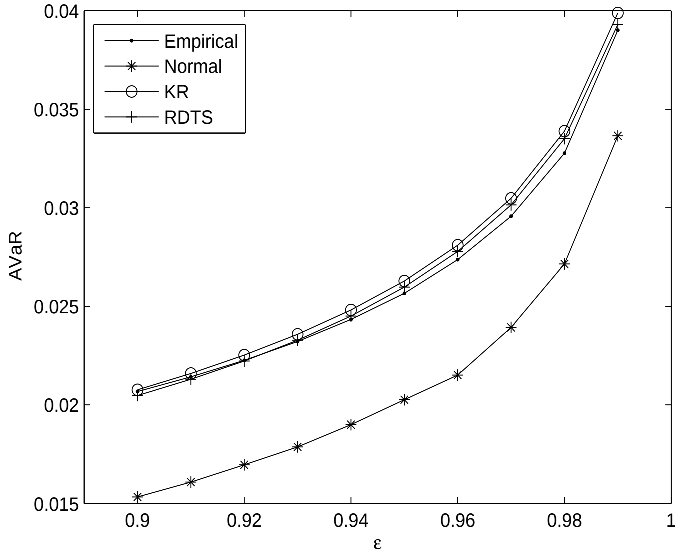

Figure 2: One-day AVaR (90%−99% confidence level)

Table 9: One-day VaR (99.1%−99.9% confidence level)

Confidence Level (ε)	Empirical	Normal	CTS	MTS	NTS	KR	RDTS
0.991	0.0301	0.0270	0.0320	0.0319	0.0320	0.0320	0.0321
0.992	0.0306	0.0275	0.0330	0.0329	0.0330	0.0330	0.0331
0.993	0.0316	0.0280	0.0341	0.0340	0.0341	0.0341	0.0342
0.994	0.0328	0.0287	0.0354	0.0354	0.0354	0.0354	0.0355
0.995	0.0349	0.0294	0.0370	0.0369	0.0370	0.0370	0.0370
0.996	0.0359	0.0303	0.0389	0.0388	0.0389	0.0389	0.0389
0.997	0.0391	0.0314	0.0415	0.0413	0.0414	0.0415	0.0414
0.998	0.0424	0.0329	0.0451	0.0449	0.0450	0.0451	0.0448
0.999	0.0600	0.0353	0.0514	0.0511	0.0512	0.0515	0.0506

Table 10: One-day AVaR (99.1%−99.9% confidence level)

Confidence Level (ε)	Empirical	Normal	CTS	MTS	NTS	KR	RDTS
0.991	0.0400	0.0348	0.0408	0.0406	0.0407	0.0408	0.0402
0.992	0.0412	0.0356	0.0418	0.0417	0.0417	0.0419	0.0412
0.993	0.0427	0.0367	0.0430	0.0428	0.0429	0.0430	0.0423
0.994	0.0444	0.0400	0.0444	0.0442	0.0443	0.0444	0.0436
0.995	0.0465	0.0431	0.0460	0.0458	0.0459	0.0461	0.0451
0.996	0.0493	0.0471	0.0481	0.0478	0.0479	0.0481	0.0470
0.997	0.0530	0.0548	0.0507	0.0504	0.0505	0.0508	0.0493
0.998	0.0592	0.0666	0.0545	0.0541	0.0542	0.0546	0.0527
0.999	0.0686	0.1010	0.0611	0.0605	0.0606	0.0612	0.0583

Table 11: Relative errors between the empirical VaR and the parametric VaR (99.1%−99.9% confidence level)

Confidence Level (ε)	Relative error= $\frac{\|\widehat{\text{VaR}}_\varepsilon - \text{VaR}_\varepsilon(X)\|}{\widehat{\text{VaR}}_\varepsilon}$					
	Normal	CTS	MTS	NTS	KR	RDTS
0.991	0.1042	0.0622	0.0605	0.0628	0.0614	0.0648
0.992	0.1013	0.0788	0.0769	0.0793	0.0780	0.0814
0.993	0.1116	0.0811	0.0791	0.0814	0.0804	0.0836
0.994	0.1259	0.0805	0.0782	0.0805	0.0799	0.0826
0.995	0.1577	0.0603	0.0578	0.0600	0.0598	0.0616
0.996	0.1560	0.0859	0.0829	0.0851	0.0856	0.0857
0.997	0.1976	0.0610	0.0574	0.0595	0.0609	0.0583
0.998	0.2250	0.0634	0.0588	0.0608	0.0637	0.0561
0.999	0.4118	0.1438	0.1490	0.1475	0.1429	0.1579
Average	0.1768	0.0797	0.0778	0.0797	0.0792	0.0813

Table 12: Relative errors between the empirical AVaR and the parametric AVaR (99.1%−99.9% confidence level)

Confidence Level (ε)	Relative error= $\frac{\|\widehat{\text{AVaR}}_\varepsilon - \text{AVaR}_\varepsilon(X)\|}{\widehat{\text{AVaR}}_\varepsilon}$					
	Normal	CTS	MTS	NTS	KR	RDTS
0.991	0.1300	0.0194	0.0153	0.0173	0.0197	0.0041
0.992	0.1358	0.0151	0.0107	0.0127	0.0155	0.0011
0.993	0.1401	0.0083	0.0036	0.0056	0.0088	0.0090
0.994	0.0994	0.0005	0.0055	0.0036	0.0001	0.0192
0.995	0.0720	0.0092	0.0146	0.0127	0.0084	0.0297
0.996	0.0439	0.0243	0.0300	0.0282	0.0232	0.0470
0.997	0.0351	0.0424	0.0488	0.0471	0.0411	0.0683
0.998	0.1248	0.0799	0.0870	0.0855	0.0781	0.1105
0.999	0.4723	0.1102	0.1190	0.1176	0.1076	0.1502
Average	0.1393	0.0344	0.0372	0.0367	0.0336	0.0488

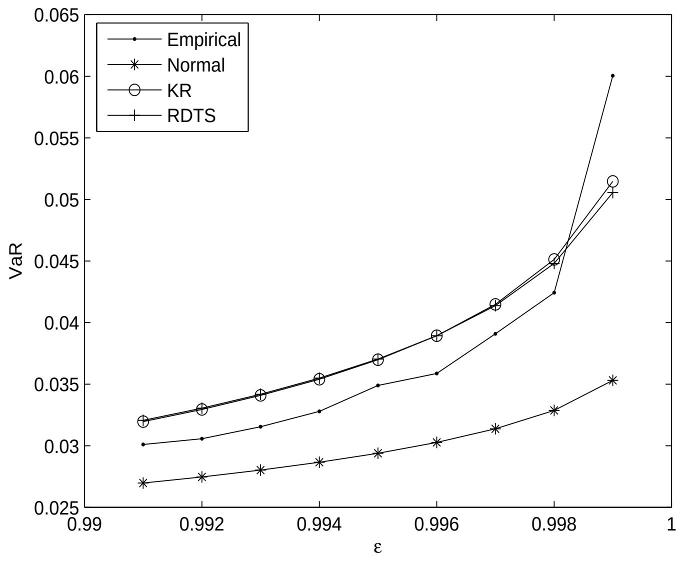

Figure 3: One-day VaR (99.1%−99.9% confidence level)

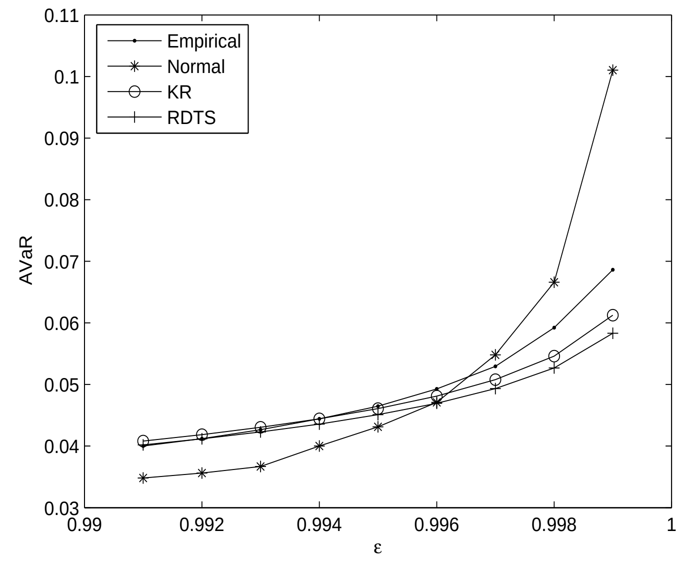

Figure 4: One-day AVaR (99.1%−99.9% confidence level)