# A FFT-based approximation of tempered stable and tempered infinitely divisible distributions

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**Abstract** There is considerable empirical evidence that financial returns exhibit leptokurtosis and non-zero skewness. As a result, alternative distributions for modeling a time series of the financial returns have been proposed. A family of distributions that has shown considerable promise for modeling financial returns is the tempered stable and tempered infinitely divisible distributions. Two representative distributions are the classical tempered stable and the rapidly decreasing tempered stable. In this paper, we explain the practical implementation of these two distributions by (1) presenting how the density functions can be computed efficiently by applying the fast Fourier transform (FFT) and (2) how standardization helps to drive efficiency and effectiveness of maximum likelihood inference.

**Keywords** fast Fourier transform, stable Paretian distribution, tempered stable distribution, tempered infinitely divisible distribution, classical tempered stable distribution, rapidly decreasing tempered stable distribution

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## 1 Introduction

Portfolio construction and risk management depend on the modeling of the financial time series of asset returns. Prior to the pathbreaking works of Mandelbrot (1963) and Fama (1963), it was assumed that return distributions followed a normal distribution. Since the early 1960s, a considerable number of empirical studies have documented that the assumption that return distributions can be characterized by a normal distribution should be rejected.[1] The findings of these studies suggest that return distributions have heavier tails than the normal distribution (i.e., exhibit leptokurtosis) and have non-zero skewness (i.e., are asymmetric).

Given the overwhelming evidence rejecting the normal distribution, alternative distributions for dealing with the stylized facts observed for real-world financial returns were proposed. One such distribution proposed by both Mandelbrot and Fama is the stable Paretian distribution.[2] Despite being a significant improvement in modeling returns in comparison to the normal distribution, the stable Paretian distribution has two drawbacks: (1) it does not have a closed-form expression for the density function, leading to resource intensive applications and (2) the variance as well as higher moments are generally infinite. The first drawback has been overcome in recent years by using the fast Fourier transform (FFT) to provide an efficient density approximation for the stable Paretian distribution to reduce computational requirements as suggested by Menn and Rachev (2006).

Rosiński (2007) proposed dealing with the second drawback—divergence of higher moments—by introducing the family of tempered stable distributions. As discussed in Kim *et al.* (2009b), the tempered stable (TS) and tempered infinitely divisible (TID) distributions also account for heavy tails and asymmetry in the empirical data. At the same time, they possess finite moments of all orders and even finite exponential moments within a certain range. The improved modeling quality by using these distributions compared to the normal distribution has been supported by empirical studies investigating financial returns. The theoretical aspects of these distributions have been extensively studied by Carr *et al.* (2002), Rosiński (2007), Kim *et al.* (2008b), Bianchi *et al.* (2008).

From an implementation perspective, however, two key questions are still unanswered: (1) how to derive the density function of these distributions efficiently and (2) how to ensure the quality of the density approximation, especially for maximum likelihood estimation. In this paper, we address these two questions. First,

[1] For a review of these studies, see Rachev *et al.* (2005).
[2] See Rachev and Mittnik (2000).

we present a FFT-based approximation of the probability density and cumulative distribution function as well as the value-at-risk and average value-at-risk of TS and TID random variables. Second, we show how the FFT approach can be improved by standardizing the random variable. This standardization technique is exemplified by the classical tempered stable and the rapidly decreasing tempered stable distribution.

# 2 Using FFT in the context of TS and TID distributions

A common drawback the TS and TID distributions share with the stable Paretian distributions is that in general there is no closed-form expression for their density function. The probability law for these distributions is described by the Fourier transform of their density function which is called the characteristic function and denoted by $\phi$. The general idea for an efficient algorithm computing the density function was suggested by DuMouchel (1975) for the stable Paretian distributions and thereafter used and refined in various studies. Based on this approach, we present here how the probability density function (PDF), the cumulative distribution function (CDF), and both the value-at-risk (VaR) and average value-at-risk (AVaR) can be computed efficiently using a FFT-based approximation method.

In our analysis, we assume the probability space $(\mathbb{R}, \wp(\mathbb{R}), P)$, where $\wp(\mathbb{R})$ denotes the Borel set on $\mathbb{R}$ and $X : \mathbb{R} \to \mathbb{R}$ as a $\mathbb{R}$-measurable function for which $P(X < x)$ is differentiable and invertible.

## 2.1 The probability density function

The PDF $f_X : \mathbb{R} \to [0, 1]$, $f_X(x) = \frac{dP(X<x)}{dx}$ of $X$ can be computed from the characteristic function

$$\phi_X(u) := \mathrm{E}[e^{iuX}] \,. \tag{1}$$

As an example, Rachev and Mittnik (2000) demonstrate, that for a stable Paretian random variable $X \sim S_\alpha(\alpha, \beta, c, m)$ with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $c \in \mathbb{R}_{>0}$, $m \in \mathbb{R}$, and characteristic function

$$\phi_X(u) = \begin{cases} \exp\left\{imu - c^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sign}(u) \tan(\frac{\pi\alpha}{2})\right]\right\} & : \quad \alpha \neq 1 \\ \exp\left\{imu - c|u| \left[1 + i\beta \operatorname{sign}(u) \frac{2}{\pi} \ln|u|\right]\right\} & : \quad \alpha = 1 \end{cases}$$

the PDF is derived as follows

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \cdot \phi_X(u) du \,. \tag{2}$$

It is common knowledge that the Fourier transform can be numerically approximated by its discrete version: the discrete Fourier transform (DFT). The DFT is a specific mapping which assigns vector $X = (x_1, \ldots, x_N) \in \mathbb{R}^N$ to vector $Y = (y_1, \cdots, y_N) \in \mathbb{R}^N$ based on the equation

$$x_j = \sum_{k=1}^{N} y_k \cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}}, j = 1, \cdots, N . \tag{3}$$

The FFT is the generally acknowledged computationally efficient implementation of the DFT which exploits the periodicity of the unit roots. We denote the FFT mapping by $X = FFT[Y]$ where $X_j = FFT_j[Y]$. Furthermore $(y_k)_{k=1,\cdots,N}$ defines the vector $Y$ by components. Given $a \in \mathbb{R}_{>0}$, $q \in \mathbb{N}_{>0}$, and the following set of definitions for all $j, k \in \{1, \cdots, N = 2^q\}$

$$\begin{aligned} u_k &:= -a + \frac{2a}{N}(k-1) \\ u_k^* &:= \frac{u_{k+1} + u_k}{2} \\ x_j &:= -\frac{N\pi}{2a} + \frac{\pi}{a}(j-1) \\ C_j &:= \frac{a}{N\pi}(-1)^{j-1} \cdot e^{i\frac{\pi(j-1)}{N}} , \end{aligned} \tag{4}$$

as well as the characteristic function $\phi(u)$ of a standardized random variable $X$, Menn and Rachev (2006) proved that by applying FFT the PDF can be approximated at the points $x_j$

$$\begin{aligned} f^{MP}(x_j) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux_j} \cdot \phi(u) du \\ &\approx \frac{1}{2\pi}\int_{-a}^{a} e^{-iux_j} \cdot \phi(u) du \\ &\approx C_j \sum_{k=1}^{N} (-1)^{k-1} \cdot \phi(u_k^*) \cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}} \\ &= C_j \cdot FFT_j \left[ \left( (-1)^{k-1} \cdot \phi(u_k^*) \right)_{k=1,\cdots,N} \right] . \end{aligned}$$

This formulation for $f^{MP}(x_j)$ makes use of the mid-points $u_k^*$ (mid-point rule for integral approximation). In order to minimize approximation errors, Menn and Rachev (2006) resort to the Simpson rule which implies

$$f(x_j) \approx \frac{2}{3} \cdot f^{MP}(x_j) + \frac{1}{3} \cdot f^{LP}(x_j) , \tag{5}$$

The expression $f^{LP}(x_j)$ can be derived in the same manner as $f^{MP}(x_j)$ using the left-point rule

$$\begin{aligned} f^{LP}(x_j) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux_j} \cdot \phi(u) du \\ &\approx \sum_{k=1}^{N} e^{-iu_k x_j} \cdot \phi(u_k) \cdot \frac{2a}{N} . \end{aligned}$$

The equation $u_k \cdot x_j = \frac{N\pi}{2} - \pi(j-1) - \pi(k-1) + \frac{2\pi}{N}(k-1)(j-1)$ then yields

$$\begin{aligned} f^{LP}(x_j) &\approx D_j \sum_{k=1}^{N} (-1)^{k-1} \cdot \phi(u_k) \cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}} \\ &= D_j \cdot FFT_j \left[ \Big( (-1)^{k-1} \cdot \phi(u_k) \Big)_{k=1,\cdots,N} \right] , \end{aligned} \tag{6}$$

where $D_j := \frac{a}{N\pi}(-1)^{j-1}$.

Note that analyzing random variables with zero-mean and unit variance is sufficient if the parameter effects of the standardization are known. Then the method presented is applicable to any random variable from the given distributional family by means of standardization and rescaling. Furthermore this approximation technique works independently of the specific characteristic function, and that is why it can be directly applied to the case of TS and TID distributions. Note that the PDF at arbitrary points $x \in \mathbb{R}$ can be computed by any of the standard interpolation algorithms—preferably a cubic interpolation such as piecewise cubic Hermite interpolation.[3]

## 2.2 The cumulative distribution function

We know from Kim *et al.* (2009a) that the CDF $F_X : \mathbb{R} \to [0,1],\ F_X(x) = P(X < x)$ of a TS or TID distributed random variable $X$ can be calculated by

$$F_X(x) = \frac{e^{x\rho}}{\pi} \cdot \Re \left\{ \int_0^\infty e^{-ixu} \frac{\phi_X(u+i\rho)}{\rho - ui} \, du \right\} , \tag{7}$$

where $\rho > 0$ and $\phi_X(u)$ is the characteristic function of $X$ with $|\phi_X(v)| < \infty$, for all complex numbers with $\Im(v) = \rho$. For simplicity, let us define

$$g(u) := \frac{\phi_X(u+i\rho)}{\rho - ui} . \tag{8}$$

Following Menn and Rachev (2006) we can approximate the integral using the mid-point rule for the DFT. Given $a \in \mathbb{R}_{>0}$, $q \in \mathbb{N}_{>0}$, $N = 2^q$, and the definitions for all $j, k \in \{1, \cdots, N\}$

$$\begin{aligned} u_k &:= \frac{2a}{N}(k-1) \\ u_k^* &:= \frac{u_k + u_{k+1}}{2} = \frac{a}{N}(2k-1) \\ x_j &:= -\frac{N\pi}{2a} + \frac{\pi}{a}(j-1) \end{aligned} \tag{9}$$

[3]See Fritsch and Carlson (1980).

it holds that

$$\int_0^\infty e^{-ix_j\ u}\cdot\frac{\phi_X(u+i\rho)}{\rho-ui}du \approx \int_0^a e^{-ix_j\ u}\cdot\frac{\phi_X(u+i\rho)}{\rho-ui}\,du$$
$$\approx \sum_{k=1}^{N} e^{-ix_j u_k^*}\cdot g(u_k^*)\cdot\frac{2a}{N}\,.$$

With the equations $x_j\cdot u_k^* = -\pi\frac{2k-1}{2}+\frac{2\pi}{N}\,\frac{2k-1}{2}(j-1)$ and $\frac{2k-1}{2}\cdot(j-1) = (k-1+\frac{1}{2})\cdot(j-1) = (k-1)(j-1)+\frac{1}{2}(j-1)$ it follows that

$$\sum_{k=1}^{N} e^{-ix_j u_k^*}\cdot g(u_k^*)\cdot\frac{2a}{N} = \frac{2a}{N}\sum_{k=1}^{N} e^{-i\left[-\pi\frac{2k-1}{2}+\frac{2\pi}{N}\,\frac{2k-1}{2}(j-1)\right]}\cdot g(u_k^*)$$
$$= \frac{2a}{N}\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot e^{-i\frac{2\pi}{N}\,\left[(k-1)(j-1)+\frac{1}{2}(j-1)\right]}\cdot g(u_k^*)$$
$$= \frac{2a}{N}\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot e^{-i\frac{\pi}{N}(j-1)}\cdot g(u_k^*)\cdot e^{-i\frac{2\pi}{N}\,(k-1)(j-1)}$$
$$= \frac{2a}{N}\,e^{-i\frac{\pi}{N}(j-1)}\cdot\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot g(u_k^*)\cdot e^{-i\frac{2\pi}{N}\,(k-1)(j-1)}$$

Hence

$$F_X(x_j) \approx F_X^{MP}(x_j) \tag{10}$$
$$= \frac{e^{x_j\rho}}{\pi}\cdot\Re\left\{\frac{2a}{N}\,e^{-i\frac{\pi}{N}(j-1)}\cdot FFT_j\left[\left((-1)^{k-1}\cdot i\cdot g(u_k^*)\right)_{k=1,\cdots,N}\right]\right\}\,.$$

The Simpson rule cannot be applied for the CDF approximation in the same manner as it was presented for the PDF. This is due to the fact that the integrand function in equation (7) is not symmetric which implies that three terms: mid-point, left-point, and right-point have to be calculated. This is, however, less efficient and therefore we suggest the use of the mid-point rule only.

## 2.3 The Value-at-Risk and Average Value-at-Risk

The VaR at confidence level $\delta\in[0,1]$ for a return modeled by a TS or TID random variable $X$ is given by

$$VaR_\delta[X] = F_X^{-1}(\delta)\,, \tag{11}$$

where $F_X^{-1}:[0,1]\to\mathbb{R}$ is the inverse CDF. From (7) the TS and TID CDF can be calculated and by (10) an efficient numerical procedure is given. Hence the $VaR_\delta[X]$ can also be computed efficiently.

The formula for AVaR at confidence level $\delta \in [0,1]$ is given by Kim *et al.* (2009a)

$$AVaR_\delta[X] = VaR_\delta[X] - \frac{e^{VaR_\delta[X]\rho}}{\pi(1-\delta)}\Re\left\{\int_0^\infty e^{-iuVaR_\delta[X]}\,\frac{\phi_X(u+i\rho)}{(u+i\rho)^2}\,du\right\}. \tag{12}$$

The $VaR_\delta[X]$ is thereby derived as described above. In practical applications, the integral in (12) is in most cases only evaluated for a restricted number of $VaR_\delta[X]$ (e.g. corresponding to the confidence levels $\delta = 0.9, 0.975, 0.99$). Therefore there is no efficiency gain in applying the FFT approximation. The numerical integration in (12) can be dealt with using e.g. the standard trapezoid rule.

# 3 Standardizing TS and TID random variables

Before we present the standardization techniques for TS and TID distributions, we briefly motivate the link between FFT and standardization. For the effectiveness and efficiency of the FFT method, the choice of the parameters $N$ and $a$ is crucial. They both influence the sampling grid $\{-a, -a+\frac{2a}{N}, \cdots, a-\frac{2a}{N}, a\}$ and the interpolation grid $\{-\frac{N\pi}{2a}, -\frac{N\pi}{2a}+\frac{\pi}{a}, \cdots, \frac{N\pi}{2a}-\frac{\pi}{a}, \frac{N\pi}{2a}\}$ significantly which determine the size of the approximation error. The latter can be split into

1. the sampling error $\epsilon_1$ which includes the loss of information due to (a) the restricted length of the sampling interval and (b) the discretization
2. the interpolation error $\epsilon_2$ which comprises (a) the approximation errors between grid points and (b) errors due to misspecifying the length of the interpolation interval.

Parameter $a$ directly influences the size of the sampling interval $[-a, a]$ and the interpolation distances $\frac{\pi}{a}$. Increasing $a$ can hence reduce errors 1(a) and 2(a). On the other hand, the ratio $\frac{a}{N}$ determines the sampling distance $\frac{2a}{N}$ and the size of the interpolation interval $[-\frac{N\pi}{2a}, \frac{N\pi}{2a}]$ which means that a low ratio decreases errors 1(b) and 2(b). Given that the size of $N$ is upper-bounded due to computational efficiency, the optimal choice of $a$ and $N$ is a trade-off between minimizing the errors 1(a) and 2(a) on the one hand and the errors 1(b) and 2(b) on the other hand.

The standardization simplifies parameter selection. To demonstrate this, we focus on computing the PDF $f(0)$ of a classical tempered stable (CTS) random variable $X$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\cdot u\cdot 0}\cdot\phi(u)du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(u)du\,,$$

which is up to a constant factor the integral over the characteristic function $\phi(u)$. The exact definition of a CTS random variable is provided later. We analyze three cases to highlight potential pitfalls in the FFT approximation

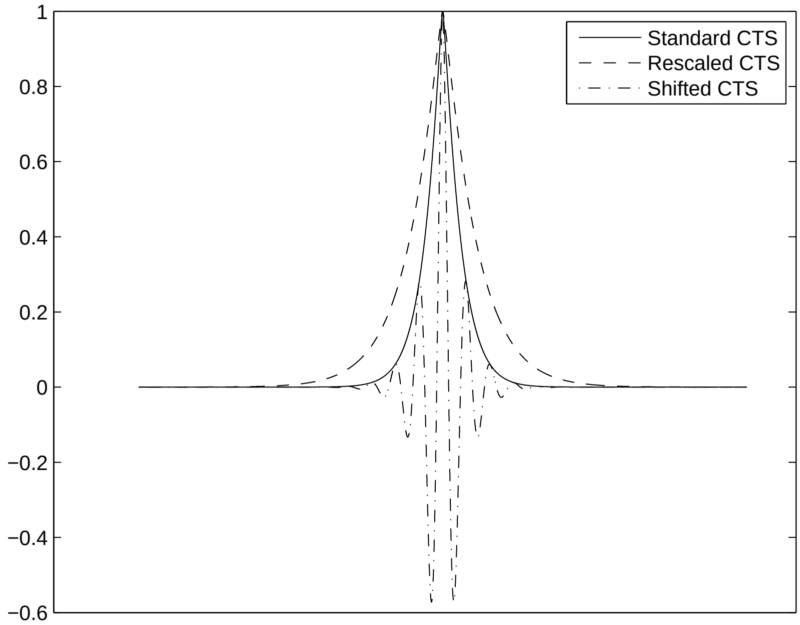


Figure 1: Real part of the CTS characteristic functions: standard $((\alpha, C, \lambda_+, \lambda_-, m) = (1.1, 0.0589, 0.1, 0.1, 0))$, rescaled ($c = 0.5$), and shifted ($d = 1$)

1. Standard: $X_1 \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$ with $\text{E}[X_1] = 0$ and $\text{V}[X_1] = 1$
2. Rescaled: $X_2 \sim \text{CTS}(\alpha, c \cdot C, \lambda_+, \lambda_-, m)$ with $c \in (0, 1)$
3. Shifted: $X_3 \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m + d)$ with $d \neq 0$

In order to avoid any confusion with value-at-risk, we denote the variance of a random variable $X$ by $V[X]$.

Figure 1 illustrates the real part of the corresponding characteristic functions. The different curves imply different optimal selection for the FFT parameters. For the rescaled CTS, the sampling interval $[-a, a]$ must be widened to keep sampling error 1(a) low. Here we assume $c < 1$ implying $\text{V}[X_2] = c \cdot \text{V}[X_1] < 1$ because this is the relevant area of empirical return variances. The shifted CTS suggests, however, that the interpolation interval $[-\frac{N\pi}{2a}, \frac{N\pi}{2a}]$ is relocated by $d$ or sufficiently increased in order to keep the interpolation error 2(b) small. The trade-off between errors (a) and (b) is hence intensified when looking at rescaled and shifted CTS distributions.

In conclusion, the use of standard CTS for the FFT eliminates the influence of the parameters $C$, and $m$ on the choice of $N$ and $a$ and thereby helps practitioners to ensure the approximation quality of the PDF. The problem for general CTS distributions can be always reduced to the standard case by commonly known statistical means. This procedure is especially relevant in an inference setting, where the distributional parameters are a priori unknown. It is nevertheless important to mention that there is still a dependence between $\alpha$, $\lambda_+$, and $\lambda_-$ and the FFT parameters $N$ and $a$.

### 3.1 Classical tempered stable distribution

Rosiński (2007) and Kim *et al.* (2008a) define the CTS distribution by the Lévy tupel $(\gamma, \sigma^2, \nu)$ with

$$\begin{aligned}\gamma &= m - \int_{|x|>1} x\nu(dx)\\ \sigma^2 &= 0\\ \nu(dx) &= \left(C_+ e^{-\lambda_+ x}\mathbf{1}_{x>0} + C_- e^{-\lambda_- x}\mathbf{1}_{x<0}\right)\frac{dx}{|x|^{\alpha+1}}\,.\end{aligned}$$

where $C_+, C_-, \lambda_+, \lambda_- \in \mathbb{R}_{>0}$, $\alpha \in (0,2)$, $m \in \mathbb{R}$, and $\mathbf{1}_A$ denotes the indicator function. Using the Lévy-Khintchine representation in Sato (1999) yields

$$\begin{aligned}\phi(u;\alpha,C_+,C_-,\lambda_+,\lambda_-,m) = \exp\Big\{ & ium - iu\Gamma(1-\alpha)(C_+\lambda_+^{\alpha-1} - C_-\lambda_-^{\alpha-1})\\ & + C_+\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha)\\ & + C_-\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha)\Big\}\,. \end{aligned} \tag{13}$$

The expected value and variance of a CTS random variable $X \sim \text{CTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ are given by

$$\text{E}[X] = m \tag{14}$$

$$\text{V}[X] = \Gamma(2-\alpha)(C_+\lambda_+^{\alpha-2} + C_-\lambda_-^{\alpha-2}). \tag{15}$$

As discussed earlier, it is crucial to understand how $X$ can be standardized in order to use the FFT efficiently. For our purpose, we consider the simplified CTS distribution with $C = C_+ = C_-$. Then we need to answer the question how $Z$ defined by

$$Z := \frac{X - \text{E}[X]}{\sqrt{\text{V}[X]}} \tag{16}$$

is distributed if $X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$. Let us assume $Z$ is standard CTS distributed $Z \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$.[4] Then the question reduces to whether or

[4] See Kim *et al.* (2008a) for definition of the stdCTS.

not there exists a combination of $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ such that equation (16) is true. The answer is given by the following proposition, which also defines suitable stdCTS parameters.

**Proposition 1.** *Let $X$ be CTS distributed with $X \sim$ CTS$(\alpha, C, \lambda_+, \lambda_-, m)$, then the standardized random variable*

$$Z := \frac{X - E[X]}{\sqrt{V[X]}}$$

*follows the distributional law*

$$Z \sim \textit{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-), \tag{17}$$

*where the standard deviation of $X$ is $\sigma := \sqrt{V[X]}$, $\tilde{\lambda}_+ := \lambda_+ \cdot \sigma$ and $\tilde{\lambda}_- := \lambda_- \cdot \sigma$. The PDFs of $X$ and $Z$ are linked by*

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-m}{\sigma}\right) . \tag{18}$$

***Proof.*** The distribution of a CTS random variable $X$ can be written as

$$P(X < x) = \int_{-\infty}^{x} f_X(t)dt \, .$$

By standardizing and substituting $s = \frac{t-m}{\sigma}$ we obtain

$$\begin{aligned} P(X < x) &= P\Big(\frac{X-m}{\sigma} < \frac{x-m}{\sigma}\Big) = P\Big(Z < \frac{x-m}{\sigma}\Big) \\ &= \int_{-\infty}^{\frac{x-m}{\sigma}} f_Z(s)ds = \int_{-\infty}^{x} f_Z\Big(\frac{t-m}{\sigma}\Big) \cdot \frac{1}{\sigma} \cdot dt \, . \end{aligned}$$

Consequently, a random variable $Z$ is the standardization of $X$ if and only if its PDF satisfies

$$f_Z\left(\frac{x-m}{\sigma}\right) = \sigma \cdot f_X(x) \, .$$

Let $\phi_Z(u)$ be the characteristic and $f_Z(z)$ the corresponding density function of the random variable $Z \sim$ stdCTS$(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$. Using the definition of $\phi(u)$ we can write

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \cdot u \cdot z} \cdot \phi_Z(u)du \, .$$

Then $Z = \frac{X-m}{\sigma}$ yields

$$f_Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \cdot u \cdot \frac{x-m}{\sigma}} \cdot \phi_Z(u)du.$$

By substituting $u = s \cdot \sigma$ and $\frac{du}{ds} = \sigma$ we obtain

$$f_Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx+ism} \cdot \phi_Z(s\cdot\sigma)\sigma ds. \tag{19}$$

Since we know that $Z \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ we can write

$$\begin{aligned}
\phi_Z(s\cdot\sigma) =& \exp\Big\{i\, s\, \sigma\, \tilde{C}\, \Gamma(1-\alpha)\, (\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}) \qquad (20)\\
&+ \tilde{C}\Gamma(-\alpha)\left[(\tilde{\lambda}_+ - is\sigma)^\alpha - \tilde{\lambda}_+^\alpha + (\tilde{\lambda}_- + s\sigma)^\alpha - \tilde{\lambda}_-^\alpha\right]\Big\}\\
=& \exp\Big\{i\, s\, \sigma^\alpha\, \tilde{C}\, \Gamma(1-\alpha)\, (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})\\
&+ \sigma^\alpha\, \tilde{C}\Gamma(-\alpha)\left[(\lambda_+ - is)^\alpha - \lambda_+^\alpha + (\lambda_- + s)^\alpha - \lambda_-^\alpha\right]\Big\}\\
=& \exp\Big\{i\, s\, C\, \Gamma(1-\alpha)\, (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})\\
&+ C\Gamma(-\alpha)\left[(\lambda_+ - is)^\alpha - \lambda_+^\alpha + (\lambda_- + s)^\alpha - \lambda_-^\alpha\right]\Big\}\\
=& e^{-i\,s\,m} \cdot \phi_X(s),
\end{aligned}$$

where $\phi_X(s)$ is the characteristic function of $X$. In the calculation above we made use of $\sigma^\alpha \cdot \tilde{C} = C$. This can be seen from the following equations

$$\begin{aligned}
\tilde{C} &= \left[\Gamma(2-\alpha)\, (\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2})\right]^{-1} \qquad (21)\\
&= \left[\Gamma(2-\alpha)\, \sigma^{\alpha-2}(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})\right]^{-1} \text{ and}\\
\sigma^2 &= C\, \Gamma(2-\alpha)\, (\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}) \qquad (22)
\end{aligned}$$

Combining (22) and (21) yields

$$\sigma^\alpha \cdot \tilde{C} = \frac{\sigma^2}{\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} = C$$

In conclusion, we obtain

$$\begin{aligned}
f_Z\left(\frac{x-m}{\sigma}\right) &= \sigma \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \cdot \phi_X(s) ds\\
&= \sigma \cdot f_X(x)\,,
\end{aligned}$$

which proves that the standardization $Z$ of $X$ is stdCTS-distributed with parameters $(\alpha, \tilde{\lambda_+}, \tilde{\lambda_-})$.

□

## 3.2 Rapidly decreasing tempered stable distribution

Similar results can be derived for the rapidly decreasing tempered stable distribution (RDTS). As a representative of the family of TID distributions introduced in Bianchi *et al.* (2008), the RDTS is well-defined by the Lévy tupel $(\gamma, \sigma^2, \nu(dx))$ with

$$\begin{aligned}\gamma &= m - \int_{|x|>1} x\nu(dx) \\ \sigma^2 &= 0 \\ \nu(dx) &= \left( C_+ e^{-\lambda_+^2 \frac{x^2}{2}} \mathbf{1}_{x>0} + C_- e^{-\lambda_-^2 \frac{x^2}{2}} \mathbf{1}_{x<0} \right) \frac{dx}{|x|^{\alpha+1}} \,, \end{aligned} \tag{23}$$

where $C_+, C_-, \lambda_+, \lambda_- \in \mathbb{R}_{>0}$, $\alpha \in (0,2)$, and $m \in \mathbb{R}$. For our purpose we consider the simplified RDTS distribution with $C = C_+ = C_-$. Using the Lévy-Khintchine representation, the characteristic function of a RDTS random variable $X \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$ takes the form

$$\begin{aligned}\phi(u) &= \exp\left\{ ium - iu \int_{|x|>1} x\nu(dx) + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}\right) \nu(dx) \right\} \\ &= \exp\left\{ ium + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \nu(dx) \right\} \,. \end{aligned} \tag{24}$$

In Kim *et al.* (2009b) this result was reformulated using the confluent hypergeometric function $M(a, b; z)$

$$\phi(u; \alpha, C, \lambda_+, \lambda_-, m) = \exp\left\{ ium + C \cdot [G(iu; \alpha, \lambda_+) + G(-iu; \alpha, \lambda_-)] \right\}, \tag{25}$$

where $G(x; \alpha, \lambda)$ is defined as

$$\begin{aligned}G(x; \alpha, \lambda) :=& 2^{-\frac{\alpha}{2}-1} \lambda^{\alpha} \, \Gamma\left(-\frac{\alpha}{2}\right) \left[ M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1 \right] + \\ &+ 2^{-\frac{\alpha}{2}-\frac{1}{2}} \lambda^{\alpha-1} \, x \, \Gamma\left(\frac{1-\alpha}{2}\right) \left[ M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1 \right]. \end{aligned}$$

The mean and variance of a RDTS distributed random variable $X \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$ are given by

$$\text{E}[X] = m \tag{26}$$

$$\text{V}[X] = 2^{-\alpha/2} \, \Gamma\left(\frac{2-\alpha}{2}\right) \, C \, \left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right) \,. \tag{27}$$

The following proposition provides the corresponding result to Proposition 1 for the RDTS distribution.

**Proposition 2.** *Let $X$ be RDTS distributed with $X \sim$ RDTS$(\alpha, C, \lambda_+, \lambda_-, m)$, then the standardized random variable*

$$Z := \frac{X - E[X]}{\sqrt{V[X]}}$$

*follows the distributional law*

$$Z \sim stdRDTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-), \tag{28}$$

*where the standard deviation of $X$ is $\sigma := \sqrt{V[X]}$, $\tilde{\lambda}_+ := \lambda_+ \cdot \sigma$ and $\tilde{\lambda}_- := \lambda_- \cdot \sigma$. The PDFs of $X$ and $Z$ are linked by*

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x - m}{\sigma}\right). \tag{29}$$

***Proof.*** The proposition can be proved in the same manner as Proposition 1. Hence it is left to show that the following equation holds

$$\phi_Z(s \cdot \sigma) = e^{ism} \cdot \phi_X(s)$$

Using the formulation of the characteristic function in (24) we can write

$$\phi_Z(s \cdot \sigma) = \exp\left\{ \int_{\mathbb{R}} \left(e^{is\sigma x} - 1 - is\sigma x\right) \nu(dx) \right\}.$$

Then the definition of the RDTS Lévy measure $\nu(dx)$ in (23) and the substitution $y = x \cdot \sigma$ yield

$$\begin{aligned} \phi_Z(s \cdot \sigma) = \exp\Bigg\{ \int_{\mathbb{R}} &\left(e^{isy} - 1 - isy\right) \cdot \tilde{C} \cdot \\ &\left(e^{-\tilde{\lambda}_+^2 y^2/2\sigma^2} \mathbf{1}_{x>0} + e^{-\tilde{\lambda}_-^2 y^2/2\sigma^2} \mathbf{1}_{x<0}\right) \frac{dy}{\sigma \cdot |y|^{\alpha+1}} \cdot \sigma^{\alpha+1} \Bigg\}. \end{aligned}$$

Using the defining expressions for $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ results in

$$\begin{aligned} \phi_Z(s \cdot \sigma) &= \exp \Bigg\{ \int_{\mathbb{R}} \left(e^{isy} - 1 - isy\right) \cdot C \cdot \\ &\qquad \left(e^{-\lambda_+^2 y^2/2} \mathbf{1}_{x>0} + e^{-\lambda_-^2 y^2/2} \mathbf{1}_{x<0}\right) \frac{dy}{|y|^{\alpha+1}} \Bigg\} \\ &= \exp \Big\{ - ism \Big\} \cdot \exp\left\{ ism + \int_{\mathbb{R}} \left(e^{isy} - 1 - isy\right) \nu(dy) \right\} \\ &= \exp \Big\{ - ism \Big\} \cdot \phi_X(s). \end{aligned}$$

Again the equation $\tilde{C} \cdot \sigma^\alpha = C$ holds because

$$\begin{aligned}\tilde{C} &= \left[2^{-\alpha/2}\,\Gamma\left(\frac{2-\alpha}{2}\right)\;\left(\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}\right)\right]^{-1} \\ &= \left[2^{-\alpha/2}\,\Gamma\left(\frac{2-\alpha}{2}\right)\;\sigma^{\alpha-2}\;\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right]^{-1} \quad \text{and} \\ \sigma^2 &= 2^{-\alpha/2}\,\Gamma\left(\frac{2-\alpha}{2}\right)\;C\;\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\;.\end{aligned}$$

□

# 4 An empirical example

For practical application of the FFT approach presented, the selection of the parameters $q$ (and thereby $N$), $a$, and $\rho$ is crucial because it determines the efficiency and the quality of the approximation. In the following, we focus on the CTS distribution and present possible parameter settings for the FFT-based approximation. In order to assess the quality of the FFT method, we calculate the approximation errors for the PDF and the CDF of a CTS random variable $X$

$$\epsilon_f(a,q) = \sup_{x\in\mathbb{R}} \left\{ f_{a,q}^{FFT}(x) - f^{Num}(x) \right\} \tag{30}$$

$$\epsilon_F(a,q,\rho) = \sup_{x\in\mathbb{R}} \left\{ F_{a,q,\rho}^{FFT}(x) - F^{Num}(x) \right\}\;, \tag{31}$$

where $f_{a,q}^{FFT}(x)$ denotes the PDF derived from our FFT method and $f^{Num}(x)$ denotes the PDF computed by numerical integration. The terms $F_{a,q,\rho}^{FFT}(x)$ and $F^{Num}(x)$ are analogously defined for the CDF. These definitions were chosen for their practicability. Since the benchmark value is derived by numerical integration, there exists, however, an imprecision in the derived error values. Furthermore the error still depends slightly on the parameters of the CTS distribution. Therefore we provide upper bounds in our empirical results.

For the PDF, the parameters $q = 15$ ($N = 32,768$) and $a = 2,000$ yield errors $\epsilon_f(a,q) < 10^{-7}$, if the FFT is applied to the standardized distribution. A further analysis shows that the sampling error is in this case below $10^{-10}$. The sampling error can be determined by restricting the search space from $\mathbb{R}$ to a set of grid points $x_j$ from the sampling grid. This way there is no interpolation necessary. These results show clearly that reducing the interpolation error could leverage the quality of the PDF approximation substantially.

Setting the parameters $q = 17$ ($N = 131,072$), $a = 50$, and $\rho = 0.001$ for the CDF leads to approximation errors $\epsilon_F(a,q,\rho) < 10^{-3}$. Hence the computations for the CDF are not only less efficient, but also deliver lower quality when compared to the PDF case. This is caused by the fact that the Laplace integral for the CDF converges slower than the Fourier integral for the PDF approximation. Therefore, the improvement of the CDF approximation should be a focus for further research on this topic. Since the VaR is directly computed from the inverse CDF, the approximation quality equals the CDF case. The quality and efficiency of the AVaR approximation based on the formula presented are, however, expected to be lower due to the additional integration step.

## 5 Conclusion

In this paper we outlined an efficient approximation for the PDF, CDF, VaR and AVaR of TS and TID distributions. Based on knowledge of the characteristic function, the FFT method is used to compute the density and distribution functions. As Menn and Rachev (2006) argued for the stable Paretian case, this procedure is computationally efficient. We explained why standardization is important for the parameter choice of the FFT and show how the standardized CTS and RDTS can be used to derive PDF values for any parameterization. For practical implementation purposes, this means reducing the risk of misspecifying $a$ and $N$ and thereby improving the effectiveness of the proposed FFT-based method. As an example, we present possible parameterizations to achieve reasonably good approximations for the PDF and CDF of the CTS distribution.

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