A FFT-based approximation of tempered stable and tempered infinitely divisible distributions

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Abstract There is considerable empirical evidence that financial returns exhibit leptokurtosis and non-zero skewness. As a result, alternative distributions for modeling a time series of the financial returns have been proposed. A family of distributions that has shown considerable promise for modeling financial returns is the tempered stable and tempered infinitely divisible distributions. Two representative distributions are the classical tempered stable and the rapidly decreasing tempered stable. In this paper, we explain the practical implementation of these two distributions by (1) presenting how the density functions can be computed efficiently by applying the fast Fourier transform (FFT) and (2) how standardization helps to drive efficiency and effectiveness of maximum likelihood inference.

Keywords fast Fourier transform, stable Paretian distribution, tempered stable distribution, tempered infinitely divisible distribution, classical tempered stable distribution, rapidly decreasing tempered stable distribution

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1 Introduction

Portfolio construction and risk management depend on the modeling of the financial time series of asset returns. Prior to the pathbreaking works of Mandelbrot (1963) and Fama (1963), it was assumed that return distributions followed a normal distribution. Since the early 1960s, a considerable number of empirical studies have documented that the assumption that return distributions can be characterized by a normal distribution should be rejected.[1] The findings of these studies suggest that return distributions have heavier tails than the normal distribution (i.e., exhibit leptokurtosis) and have non-zero skewness (i.e., are asymmetric).

Given the overwhelming evidence rejecting the normal distribution, alternative distributions for dealing with the stylized facts observed for real-world financial returns were proposed. One such distribution proposed by both Mandelbrot and Fama is the stable Paretian distribution.[2] Despite being a significant improvement in modeling returns in comparison to the normal distribution, the stable Paretian distribution has two drawbacks: (1) it does not have a closed-form expression for the density function, leading to resource intensive applications and (2) the variance as well as higher moments are generally infinite. The first drawback has been overcome in recent years by using the fast Fourier transform (FFT).[3] It provides an efficient density approximation for the stable Paretian distribution and reduces computational requirements as discussed in Mittnik *et al.* (1999) and Menn and Rachev (2006).

Koponen (1995) proposed a model dealing with the second drawback-divergence of higher moments-by using tempered stable distributions. This model was enhanced by Boyarchenko and Levendorskii (2000) and Carr *et al.* (2002). Their works also mark the first application of such models to the financial market. With the seminal work of Rosiński (2007) and Bianchi *et al.* (2010) the family of tempered stable (TS) and the family of tempered infinitely divisible (TID) distributions were introduced. As discussed in Kim *et al.* (2010), the tempered stable (TS) and tempered infinitely divisible (TID) distributions also account for heavy tails and asymmetry in the empirical data. The improved modeling quality by using these distributions compared to the normal distribution has been supported by empirical studies investigating financial returns, e.g. Kim *et al.* (2008a) and Kim *et al.* (2008b).

[1] For a review of these studies, see Rachev *et al.* (2005).
[2] See Rachev and Mittnik (2000).
[3] See DuMouchel (1975).

From an implementation perspective, however, two key questions are still unanswered: (1) how to derive the density function of these distributions efficiently and (2) how to ensure the quality of the density approximation, especially for maximum likelihood estimation (MLE). In this paper, we address these two questions. First, we present a FFT-based approximation of the probability density and cumulative distribution function as well as the value-at-risk and average value-at-risk of TS and TID random variables. Second, we show how the FFT approach can be improved by standardizing the random variable. This standardization technique is exemplified by the classical tempered stable and the rapidly decreasing tempered stable distribution. In the following we introduce an approach to calibrate the FFT method in practice. Finally we apply our findings in the context of MLE.

2 Using FFT in the context of TS and TID distributions

A common drawback the TS and TID distributions share with the stable Paretian distributions is that in general there is no closed-form expression for their density function available. The probability law for these distributions is described by the Fourier transform of their density function which is called the characteristic function and denoted by ϕ. The general idea for an efficient algorithm computing the density function was suggested by DuMouchel (1975) for the stable Paretian distributions and thereafter used and refined in various studies. Based on this approach, we present here how the probability density function (PDF), the cumulative distribution function (CDF), and both the value-at-risk (VaR) and average value-at-risk (AVaR) can be computed efficiently using a FFT-based approximation method.

In our analysis, we assume the probability space $(\mathbb{R}, \wp(\mathbb{R}), P)$, where $\wp(\mathbb{R})$ denotes the Borel set on $\mathbb{R}$ and $X : \mathbb{R} \to \mathbb{R}$ as a $\mathbb{R}$-measurable function for which $P(X < x)$ is differentiable and invertible.

2.1 The probability density function

The PDF $f_X : \mathbb{R} \to [0, 1]$, $f_X(x) = \frac{dP(X<x)}{dx}$ of X can be computed from the characteristic function

$$\phi_X(u) := \mathrm{E}[e^{iuX}] \,, \tag{1}$$

using the inverse formula of the fourier transform

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \cdot \phi_X(u) du \,. \tag{2}$$

In the case of stable Paretian, TS, and TID distributions, there exists a closed-form expression for their characteristic functions $\phi_X(u)$. Given a stable Paretian random variable $X \sim S_\alpha(\alpha, \beta, c, m)$ with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $c \in \mathbb{R}_{>0}$, and $m \in \mathbb{R}$, $\phi_X(u)$ is defined as

$$\phi_X(u) = \begin{cases} \exp\left\{imu - c^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sign}(u) \tan(\frac{\pi\alpha}{2})\right]\right\} & : \quad \alpha \neq 1 \\ \exp\left\{imu - c|u| \left[1 + i\beta \operatorname{sign}(u) \frac{2}{\pi} \ln|u|\right]\right\} & : \quad \alpha = 1 . \end{cases}$$

As an example for the TS class, the classical tempered stable (CTS) distribution with parameters $C_+, C_-, \lambda_+, \lambda_- \in \mathbb{R}_{>0}$, $\alpha \in (0, 2)$, $m \in \mathbb{R}$ has the characteristic function

$$\begin{aligned} \phi(u; \alpha, C_+, C_-, \lambda_+, \lambda_-, m) = \exp\Big\{ & ium - iu\Gamma(1-\alpha)(C_+\lambda_+^{\alpha-1} - C_-\lambda_-^{\alpha-1}) \\ & + C_+\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) \\ & + C_-\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha)\Big\} . \end{aligned} \tag{3}$$

For a comprehensive defintion of the TS and TID class and some examples, we refer to Rosiński (2007) and Bianchi *et al.* (2010).

It is common knowledge that the Fourier transform can be numerically approximated by its discrete version: the discrete Fourier transform (DFT). The DFT is a specific mapping which assigns vector $X = (x_1, \ldots, x_N) \in \mathbb{R}^N$ to vector $Y = (y_1, \cdots, y_N) \in \mathbb{R}^N$ based on the equation

$$x_j = \sum_{k=1}^{N} y_k \cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}}, \ j = 1, \cdots, N . \tag{4}$$

The FFT is the generally acknowledged computationally efficient implementation of the DFT which exploits the periodicity of the unit roots. We denote the FFT mapping by $X = \text{FFT}[Y]$ where $X_j = \text{FFT}_j[Y]$. Furthermore $(y_k)_{k=1,\cdots,N}$ defines the vector Y by components. Given $a \in \mathbb{R}_{>0}$, $q \in \mathbb{N}_{>0}$, and the following set of definitions for all $j, k \in \{1, \cdots, N = 2^q\}$

$$\begin{aligned} u_k &:= -a + \frac{2a}{N}(k-1) \\ u_k^* &:= \frac{u_{k+1} + u_k}{2} \\ x_j &:= -\frac{N\pi}{2a} + \frac{\pi}{a}(j-1) \\ C_j &:= \frac{a}{N\pi}(-1)^{j-1} \cdot e^{i\frac{\pi(j-1)}{N}} , \end{aligned} \tag{5}$$

as well as the characteristic function $\phi(u)$ of a standardized random variable X, Menn and Rachev (2006) proved–in the context of stable Paretian distribution–that by applying FFT the PDF can be approximated at the points x_j

$$\begin{aligned} f^{MP}(x_j) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux_j}\cdot\phi(u)du \\ &\approx \frac{1}{2\pi}\int_{-a}^{a} e^{-iux_j}\cdot\phi(u)du \\ &\approx C_j\sum_{k=1}^{N}(-1)^{k-1}\cdot\phi(u_k^*)\cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}} \\ &= C_j\cdot \mathrm{FFT}_j\left[\left((-1)^{k-1}\cdot\phi(u_k^*)\right)_{k=1,\cdots,N}\right] . \end{aligned}$$

Parameter a determines the integration limits for the Fourier transform and q defines the number of integration steps $N = 2^q$.

The formula for $f^{MP}(x_j)$ makes use of the mid-points u_k^* (mid-point rule for integral approximation). In order to minimize approximation errors, Menn and Rachev (2006) resort to the Simpson rule which implies

$$f(x_j) \approx \frac{2}{3}\cdot f^{MP}(x_j) + \frac{1}{3}\cdot f^{LP}(x_j)\ , \tag{6}$$

The expression $f^{LP}(x_j)$ can be derived in the same manner as $f^{MP}(x_j)$ using the left-point rule

$$\begin{aligned} f^{LP}(x_j) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux_j}\cdot\phi(u)du \\ &\approx \sum_{k=1}^{N} e^{-iu_kx_j}\cdot\phi(u_k)\cdot\frac{2a}{N} . \end{aligned}$$

The equation $u_k\cdot x_j = \frac{N\pi}{2} - \pi(j-1) - \pi(k-1) + \frac{2\pi}{N}(k-1)(j-1)$ then yields

$$\begin{aligned} f^{LP}(x_j) &\approx D_j\sum_{k=1}^{N}(-1)^{k-1}\cdot\phi(u_k)\cdot e^{-i\frac{2\pi(j-1)(k-1)}{N}} \\ &= D_j\cdot \mathrm{FFT}_j\left[\left((-1)^{k-1}\cdot\phi(u_k)\right)_{k=1,\cdots,N}\right]\ , \end{aligned} \tag{7}$$

where $D_j := \frac{a}{N\pi}(-1)^{j-1}$.

Note that analyzing random variables with zero-mean and unit variance is sufficient if the influence of standardization on the distributional parameters is known. Then the method presented is applicable to any random variable from the given distributional family by means of standardization and rescaling. Furthermore this approximation technique works independently of the specific characteristic function, and that is why it can be directly applied to the case of TS and TID distributions. Note that the PDF at arbitrary points $x \in \mathbb{R}$ can be computed by any of the standard interpolation algorithms—preferably a cubic interpolation such as piecewise cubic Hermite interpolation.[4]

2.2 The cumulative distribution function

We know from Kim *et al.* (2009) that the CDF $F_X : \mathbb{R} \to [0,1],\ \ F_X(x) = P(X < x)$ of a TS or TID distributed random variable X can be calculated by

$$F_X(x) = \frac{e^{x\rho}}{\pi} \cdot \Re \left\{ \int_0^\infty e^{-ixu} \, \frac{\phi_X(u + i\rho)}{\rho - ui} \, du \right\} \ , \tag{8}$$

where $\rho > 0$ and $\phi_X(u)$ is the characteristic function of X with $|\phi_X(v)| < \infty$, for all complex numbers with $\Im(v) = \rho$. For simplicity, let us define

$$g(u) := \frac{\phi_X(u + i\rho)}{\rho - ui} \ . \tag{9}$$

Following Menn and Rachev (2006) we can approximate the integral using the mid-point rule for the DFT. Given $a \in \mathbb{R}_{>0}$, $q \in \mathbb{N}_{>0}$, $N = 2^q$, and for all $j, k \in \{1, \cdots, N\}$

$$\begin{aligned} u_k &:= \frac{2a}{N}\,(k-1) \\ u_k^* &:= \frac{u_k + u_{k+1}}{2} = \frac{a}{N}(2k-1) \\ x_j &:= -\frac{N\pi}{2a} + \frac{\pi}{a}\,(j-1) \end{aligned} \tag{10}$$

it holds that

$$\begin{aligned} \int_0^\infty e^{-ix_j\,u} \cdot \frac{\phi_X(u+i\rho)}{\rho - ui} du &\approx \int_0^a e^{-ix_j\,u} \cdot \frac{\phi_X(u+i\rho)}{\rho - ui}\, du \\ &\approx \sum_{k=1}^{N} e^{-ix_j u_k^*} \cdot g(u_k^*) \cdot \frac{2a}{N} \ . \end{aligned}$$

With the equations $x_j \cdot u_k^* = -\pi \frac{2k-1}{2} + \frac{2\pi}{N}\,\frac{2k-1}{2}(j-1)$ and $\frac{2k-1}{2} \cdot (j-1) =$

[4]See Fritsch and Carlson (1980).

$(k-1+\frac{1}{2})\cdot(j-1)=(k-1)(j-1)+\frac{1}{2}(j-1)$ it follows that

$$\begin{aligned}\sum_{k=1}^{N} e^{-ix_j u_k^*}\cdot g(u_k^*)\cdot\frac{2a}{N} &= \frac{2a}{N}\sum_{k=1}^{N} e^{-i\left[-\pi\frac{2k-1}{2}+\frac{2\pi}{N}\ \frac{2k-1}{2}(j-1)\right]}\cdot g(u_k^*)\\ &= \frac{2a}{N}\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot e^{-i\frac{2\pi}{N}\ \left[(k-1)(j-1)+\frac{1}{2}(j-1)\right]}\cdot g(u_k^*)\\ &= \frac{2a}{N}\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot e^{-i\frac{\pi}{N}(j-1)}\cdot g(u_k^*)\cdot e^{-i\frac{2\pi}{N}\ (k-1)(j-1)}\\ &= \frac{2a}{N} e^{-i\frac{\pi}{N}(j-1)}\cdot\sum_{k=1}^{N} e^{i\pi\frac{2k-1}{2}}\cdot g(u_k^*)\cdot e^{-i\frac{2\pi}{N}\ (k-1)(j-1)}\ .\end{aligned}$$

Hence

$$\begin{aligned}F_X(x_j) &\approx F_X^{MP}(x_j) \qquad (11)\\ &= \frac{e^{x_j\rho}}{\pi}\cdot\Re\left\{\frac{2a}{N}\,e^{-i\frac{\pi}{N}(j-1)}\cdot\text{FFT}_j\left[\left((-1)^{k-1}\cdot i\cdot g(u_k^*)\right)_{k=1,\cdots,N}\right]\right\}\ .\end{aligned}$$

The Simpson rule cannot be analogously applied for the CDF approximation. This is due to the fact that the integrand function in equation (8) is asymmetric which implies that three integrals: mid-point, left-point, and right-point have to be evaluated. This is, however, less efficient and therefore we suggest the use of the mid-point rule only.

2.3 The Value-at-Risk and Average Value-at-Risk

The VaR at confidence level $\delta\in[0,1]$ for a return modeled by a TS or TID random variable X is given by

$$\text{VaR}_\delta[X]=F_X^{-1}(\delta)\ , \qquad (12)$$

where $F_X^{-1}:[0,1]\to\mathbb{R}$ is the inverse CDF. From (8) the TS and TID CDF can be calculated and by (11) an efficient numerical procedure is given. Hence the $\text{VaR}_\delta[X]$ can also be computed efficiently.

Kim *et al.* (2009) provide a formula for AVaR at confidence level $\delta\in[0,1]$

$$\text{AVaR}_\delta[X]=\text{VaR}_\delta[X]-\frac{e^{\text{VaR}_\delta[X]\rho}}{\pi(1-\delta)}\Re\left\{\int_0^\infty e^{-iu\text{VaR}_\delta[X]}\ \frac{\phi_X(u+i\rho)}{(u+i\rho)^2}\ du\right\}\ . \qquad (13)$$

The $\text{VaR}_\delta[X]$ is thereby derived as described above. In practical applications, the integral in (13) is in most cases only evaluated for a restricted number of $\text{VaR}_\delta[X]$ (e.g. corresponding to the confidence levels $\delta=0.9,0.975,0.99$). Therefore there is no efficiency gain in applying the FFT approximation. The numerical integration in (13) can be dealt with using e.g. the standard trapezoid rule.

3 Standardizing TS and TID random variables

Before we present the standardization techniques for TS and TID distributions, we briefly motivate the link between FFT and standardization. For the effectiveness and efficiency of the FFT method, the choice of the parameters N and a is crucial. They both significantly influence the sampling grid $\{-a, -a + \frac{2a}{N}, \cdots, a - \frac{2a}{N}, a\}$ and the interpolation grid $\{-\frac{N\pi}{2a}, -\frac{N\pi}{2a} + \frac{\pi}{a}, \cdots, \frac{N\pi}{2a} - \frac{\pi}{a}, \frac{N\pi}{2a}\}$ which again determine the size of the approximation error. This error can be split into

1. the sampling error ϵ_1 which includes the loss of information due to (a) the restricted length of the sampling interval and (b) the discretization

2. the interpolation error ϵ_2 which comprises (a) the error between grid points and (b) error due to misspecifying the length of the interpolation interval.

Parameter a directly influences the size of the sampling interval $[-a, a]$ and the interpolation distances $\frac{\pi}{a}$. Increasing a can hence reduce errors 1(a) and 2(a). On the other hand, the ratio $\frac{a}{N}$ determines the sampling distance $\frac{2a}{N}$ and the size of the interpolation interval $[-\frac{N\pi}{2a}, \frac{N\pi}{2a}]$ which means that a low ratio decreases errors 1(b) and 2(b). Given that the size of N is upper-bounded due to computational efficiency, the optimal choice of a and N is a trade-off between minimizing the errors 1(a) and 2(a) on the one hand and the errors 1(b) and 2(b) on the other hand.

The standardization simplifies parameter selection. To demonstrate this, we focus on computing the PDF $f(0)$ of a CTS random variable X

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \cdot u \cdot 0} \cdot \phi(u) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u) du \, ,$$

which is up to a constant factor the integral over the characteristic function $\phi(u)$. The exact definition of a CTS random variable is provided later. We analyze three cases to highlight potential pitfalls in the FFT approximation

1. Standard: $X_1 \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$ with $\text{E}[X_1] = 0$ and $\text{V}[X_1] = 1$

2. Rescaled: $X_2 \sim \text{CTS}(\alpha, c \cdot C, \lambda_+, \lambda_-, m)$ with $c \in (0, 1)$

3. Shifted: $X_3 \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m + d)$ with $d \neq 0$

In order to avoid any confusion with value-at-risk, we denote the variance of a random variable X by $\text{V}[X]$.

Figure 1 illustrates the real part of the corresponding characteristic functions. The different curves imply different optimal selection for the FFT parameters. For the rescaled CTS, the sampling interval $[-a, a]$ must be widened to keep sampling error 1(a) low. Here we assume $c < 1$ implying $\text{V}[X_2] = c \cdot \text{V}[X_1] < 1$ because this is the relevant area of empirical return variances. The shifted CTS suggests,

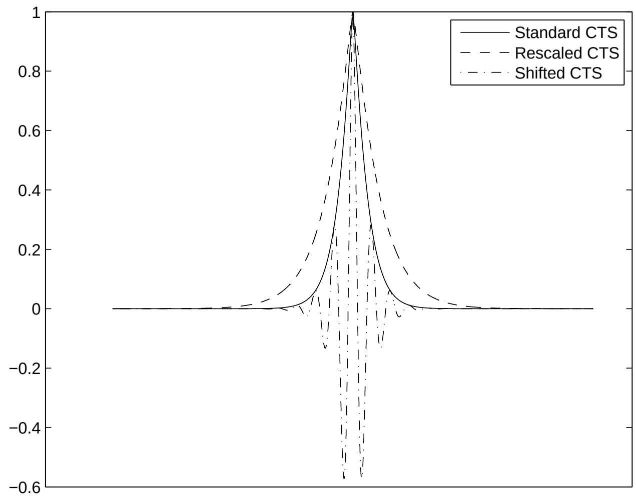

Figure 1: Real part of the CTS characteristic functions: standard $((\alpha, C, \lambda_+, \lambda_-, m) = (1.1, 0.0589, 0.1, 0.1, 0))$, rescaled ($c = 0.5$), and shifted ($d = 1$)

however, that the interpolation interval $[-\frac{N\pi}{2a}, \frac{N\pi}{2a}]$ is relocated by d or sufficiently increased in order to keep the interpolation error 2(b) small. The trade-off between errors (a) and (b) is hence intensified when looking at rescaled and shifted CTS distributions.

In conclusion, the use of standard CTS for the FFT eliminates the influence of the parameters C, and m on the choice of N and a and thereby helps practitioners to ensure the approximation quality of the PDF. The problem for general CTS distributions can be always reduced to the standard case by commonly known statistical means. This procedure is especially relevant in an inference setting, where the distributional parameters are a priori unknown. It is nevertheless important to mention that there is still a dependence between α, λ_+, and λ_- and the FFT parameters N and a.

3.1 Classical tempered stable distribution

Rosiński (2007) and Kim *et al.* (2008a) define the CTS distribution by the Lévy tupel (γ, σ^2, ν) with

$$\begin{aligned}\gamma &= m - \int_{|x|>1} x\nu(dx) \\ \sigma^2 &= 0 \\ \nu(dx) &= \left(C_+ e^{-\lambda_+ x}\mathbf{1}_{x>0} + C_- e^{-\lambda_- x}\mathbf{1}_{x<0}\right)\frac{dx}{|x|^{\alpha+1}}\ .\end{aligned}$$

where $C_+, C_-, \lambda_+, \lambda_- \in \mathbb{R}_{>0}$, $\alpha \in (0,2)$, $m \in \mathbb{R}$, and $\mathbf{1}_A$ denotes the indicator function. Using the Lévy-Khintchine representation in Sato (1999) yields

$$\begin{aligned}\phi(u;\alpha, C_+, C_-, \lambda_+, \lambda_-, m) = \exp\Big\{ & ium - iu\Gamma(1-\alpha)(C_+\lambda_+^{\alpha-1} - C_-\lambda_-^{\alpha-1}) \\ & + C_+\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) \\ & + C_-\Gamma(-\alpha)((\lambda_- + iu)^\alpha - \lambda_-^\alpha)\Big\}\ . \end{aligned} \tag{14}$$

The expected value and variance of a CTS random variable $X \sim \text{CTS}(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ are given by

$$\text{E}[X] = m \tag{15}$$

$$\text{V}[X] = \Gamma(2-\alpha)(C_+\lambda_+^{\alpha-2} + C_-\lambda_-^{\alpha-2}). \tag{16}$$

As discussed earlier, it is crucial to understand how X can be standardized in order to use the FFT efficiently. For our purpose, we consider the simplified CTS distribution with $C = C_+ = C_-$. Then we need to answer the question how Z defined by

$$Z := \frac{X - \text{E}[X]}{\sqrt{\text{V}[X]}} \tag{17}$$

is distributed if $X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$. Let us assume Z is standard CTS distributed $Z \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$.[5] Then the question reduces to whether or not there exists a combination of $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ such that equation (17) is true. The answer is given by the following proposition, which also defines suitable stdCTS parameters.

Proposition 1. *Let X be CTS distributed with $X \sim CTS(\alpha, C, \lambda_+, \lambda_-, m)$, then the standardized random variable*

$$Z := \frac{X - E[X]}{\sqrt{V[X]}}$$

[5] See Kim *et al.* (2008a) for definition of the stdCTS.

follows the distributional law

$$Z \sim stdCTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-), \tag{18}$$

where the standard deviation of X *is* $\sigma := \sqrt{\mathbb{V}[X]}$, $\tilde{\lambda}_+ := \lambda_+ \cdot \sigma$ *and* $\tilde{\lambda}_- := \lambda_- \cdot \sigma$. *The PDFs of* X *and* Z *are linked by*

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-m}{\sigma}\right) . \tag{19}$$

***Proof*.** The distribution of a CTS random variable X can be written as

$$P(X < x) = \int_{-\infty}^{x} f_X(t)dt .$$

By standardizing and substituting $s = \frac{t-m}{\sigma}$ we obtain

$$\begin{aligned} P(X < x) &= P\Big(\frac{X-m}{\sigma} < \frac{x-m}{\sigma}\Big) = P\Big(Z < \frac{x-m}{\sigma}\Big) \\ &= \int_{-\infty}^{\frac{x-m}{\sigma}} f_Z(s)ds = \int_{-\infty}^{x} f_Z\Big(\frac{t-m}{\sigma}\Big) \cdot \frac{1}{\sigma} \cdot dt . \end{aligned}$$

Consequently, a random variable Z is the standardization of X if and only if its PDF satisfies

$$f_Z\left(\frac{x-m}{\sigma}\right) = \sigma \cdot f_X(x) .$$

Let $\phi_Z(u)$ be the characteristic and $f_Z(z)$ the corresponding density function of the random variable $Z \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$. Using the definition of $\phi(u)$ we can write

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\cdot u\cdot z} \cdot \phi_Z(u)du .$$

Then $Z = \frac{X-m}{\sigma}$ yields

$$f_Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\cdot u\cdot \frac{x-m}{\sigma}} \cdot \phi_Z(u)du.$$

By substituting $u = s \cdot \sigma$ and $\frac{du}{ds} = \sigma$ we obtain

$$f_Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx+ism} \cdot \phi_Z(s \cdot \sigma)\sigma ds. \tag{20}$$

Since we know that $Z \sim \text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ we can write

$$\begin{aligned}
\phi_Z(s \cdot \sigma) =& \exp\Big\{ i\, s\, \sigma\, \tilde{C}\, \Gamma(1-\alpha)\, (\tilde{\lambda}_+^{\alpha-1} - \tilde{\lambda}_-^{\alpha-1}) \\
& + \tilde{C}\Gamma(-\alpha) \Big[(\tilde{\lambda}_+ - is\sigma)^\alpha - \tilde{\lambda}_+^\alpha + (\tilde{\lambda}_- + is\sigma)^\alpha - \tilde{\lambda}_-^\alpha \Big] \Big\} \\
=& \exp\Big\{ i\, s\, \sigma^\alpha\, \tilde{C}\, \Gamma(1-\alpha)\, (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) \\
& + \sigma^\alpha\, \tilde{C}\Gamma(-\alpha) \big[(\lambda_+ - is)^\alpha - \lambda_+^\alpha + (\lambda_- + is)^\alpha - \lambda_-^\alpha \big] \Big\} \\
=& \exp\Big\{ i\, s\, C\, \Gamma(1-\alpha)\, (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) \\
& + C\Gamma(-\alpha) \big[(\lambda_+ - is)^\alpha - \lambda_+^\alpha + (\lambda_- + is)^\alpha - \lambda_-^\alpha \big] \Big\} \\
=& e^{-i\, s\, m} \cdot \phi_X(s),
\end{aligned} \tag{21}$$

where $\phi_X(s)$ is the characteristic function of X. In the calculation above we made use of $\sigma^\alpha \cdot \tilde{C} = C$. This can be seen from the following equations

$$\begin{aligned}
\tilde{C} &= \Big[\Gamma(2-\alpha)\, (\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}) \Big]^{-1} \\
&= \big[\Gamma(2-\alpha)\, \sigma^{\alpha-2} (\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}) \big]^{-1} \text{ and}
\end{aligned} \tag{22}$$

$$\sigma^2 = C\, \Gamma(2-\alpha)\, (\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}) \tag{23}$$

Combining (23) and (22) yields

$$\sigma^\alpha \cdot \tilde{C} = \frac{\sigma^2}{\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})} = C$$

In conclusion, we obtain

$$\begin{aligned}
f_Z\left(\frac{x-m}{\sigma} \right) &= \sigma \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \cdot \phi_X(s) ds \\
&= \sigma \cdot f_X(x)\,,
\end{aligned}$$

which proves that the standardization Z of X is $\text{stdCTS}(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-)$ distributed. $\square$

3.2 Rapidly decreasing tempered stable distribution

Similar results can be derived for the RDTS distribution. As a representative of the family of TID distributions introduced in Bianchi *et al.* (2010), the RDTS is well-defined by the Lévy tupel $(\gamma, \sigma^2, \nu(dx))$ with

$$\begin{aligned} \gamma &= m - \int_{|x|>1} x\nu(dx) \\ \sigma^2 &= 0 \\ \nu(dx) &= \left(C_+ e^{-\lambda_+^2 \frac{x^2}{2}} \mathbf{1}_{x>0} + C_- e^{-\lambda_-^2 \frac{x^2}{2}} \mathbf{1}_{x<0} \right) \frac{dx}{|x|^{\alpha+1}} \,, \end{aligned} \tag{24}$$

where $C_+, C_-, \lambda_+, \lambda_- \in \mathbb{R}_{>0}$, $\alpha \in (0, 2)$, and $m \in \mathbb{R}$. For our purpose we consider the simplified RDTS distribution with $C = C_+ = C_-$. Using the Lévy-Khintchine representation, the characteristic function of a RDTS random variable $X \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$ takes the form

$$\begin{aligned} \phi(u) &= \exp\left\{ ium - iu \int_{|x|>1} x\nu(dx) + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1} \right) \nu(dx) \right\} \\ &= \exp\left\{ ium + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux \right) \nu(dx) \right\} \,. \end{aligned} \tag{25}$$

In Kim *et al.* (2010) this result was reformulated using the confluent hypergeometric function $M(a, b; z)$

$$\phi(u; \alpha, C, \lambda_+, \lambda_-, m) = \exp\Big\{ ium + C \cdot \left[G(iu; \alpha, \lambda_+) + G(-iu; \alpha, \lambda_-) \right] \Big\}, \tag{26}$$

where $G(x; \alpha, \lambda)$ is defined as

$$\begin{aligned} G(x; \alpha, \lambda) :=& 2^{-\frac{\alpha}{2}-1} \, \lambda^\alpha \, \Gamma\left(-\frac{\alpha}{2}\right) \, \left[M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1 \right] + \\ &+ 2^{-\frac{\alpha}{2}-\frac{1}{2}} \, \lambda^{\alpha-1} \, x \, \Gamma\left(\frac{1-\alpha}{2}\right) \, \left[M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1 \right]. \end{aligned}$$

The mean and variance of a RDTS distributed random variable $X \sim \text{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$ are given by

$$\text{E}[X] = m \tag{27}$$

$$\text{V}[X] = 2^{-\alpha/2} \, \Gamma\left(\frac{2-\alpha}{2}\right) \, C \, \left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2} \right) \,. \tag{28}$$

The following proposition provides the corresponding result to Proposition 1 for the RDTS distribution.

Proposition 2. *Let X be RDTS distributed with $X \sim$ RDTS$(\alpha, C, \lambda_+, \lambda_-, m)$, then the standardized random variable*

$$Z := \frac{X - E[X]}{\sqrt{V[X]}}$$

follows the distributional law

$$Z \sim stdRDTS(\alpha, \tilde{\lambda}_+, \tilde{\lambda}_-), \tag{29}$$

where the standard deviation of X is $\sigma := \sqrt{V[X]}$, $\tilde{\lambda}_+ := \lambda_+ \cdot \sigma$ and $\tilde{\lambda}_- := \lambda_- \cdot \sigma$. The PDFs of X and Z are linked by

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-m}{\sigma}\right) . \tag{30}$$

***Proof*.** The proposition can be proved in the same manner as Proposition 1. Hence it is left to show that the following equation holds

$$\phi_Z(s \cdot \sigma) = e^{ism} \cdot \phi_X(s)$$

Using the formulation of the characteristic function in (25) we can write

$$\phi_Z(s \cdot \sigma) = \exp\left\{ \int_{\mathbb{R}} \left(e^{is\sigma x} - 1 - is\sigma x\right) \nu(dx) \right\}.$$

Then the definition of the RDTS Lévy measure $\nu(dx)$ in (24) and the substitution $y = x \cdot \sigma$ yield

$$\phi_Z(s \cdot \sigma) = \exp\Bigg\{ \int_{\mathbb{R}} \left(e^{isy} - 1 - isy\right) \cdot \tilde{C} \cdot \left(e^{-\tilde{\lambda}_+^2 y^2/2\sigma^2} \mathbf{1}_{x>0} + e^{-\tilde{\lambda}_-^2 y^2/2\sigma^2} \mathbf{1}_{x<0}\right) \frac{dy}{\sigma \cdot |y|^{\alpha+1}} \cdot \sigma^{\alpha+1} \Bigg\}.$$

Using the defining expressions for $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ results in

$$\begin{aligned}
\phi_Z(s \cdot \sigma) &= \exp \left\{ \int_{\mathbb{R}} \left(e^{isy} - 1 - isy\right) \cdot C \cdot \left(e^{-\lambda_+^2 y^2/2} \mathbf{1}_{x>0} + e^{-\lambda_-^2 y^2/2} \mathbf{1}_{x<0}\right) \frac{dy}{|y|^{\alpha+1}} \right\} \\
&= \exp \left\{ -ism \right\} \cdot \exp\left\{ ism + \int_{\mathbb{R}} \left(e^{isy} - 1 - isy\right) \nu(dy) \right\} \\
&= \exp \left\{ -ism \right\} \cdot \phi_X(s) .
\end{aligned}$$

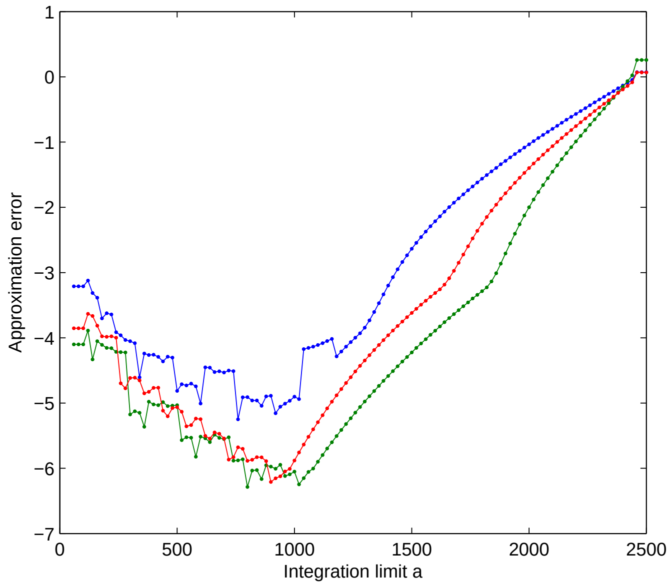

Figure 2: Logarithm of smoothed approximation error $\log_{10}(\epsilon_f(a, q))$ with $q = 13$ and $a \in \{60, 80, 100, ..., 2500\}$ for different standard CTS distributions

Again the equation $\tilde{C} \cdot \sigma^\alpha = C$ holds because

$$\tilde{C} = \left[2^{-\alpha/2}\, \Gamma\left(\frac{2-\alpha}{2}\right) \ \left(\tilde{\lambda}_+^{\alpha-2} + \tilde{\lambda}_-^{\alpha-2}\right)\right]^{-1}$$
$$= \left[2^{-\alpha/2}\, \Gamma\left(\frac{2-\alpha}{2}\right) \ \sigma^{\alpha-2} \ \left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right]^{-1} \quad \text{and}$$
$$\sigma^2 = 2^{-\alpha/2}\, \Gamma\left(\frac{2-\alpha}{2}\right) \ C \ \left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right) \ .$$

□

4 Minimizing the approximation error

For practical application of the FFT approach presented, the selection of the parameters q (and thereby N), a, and ρ is crucial because it determines the efficiency and the quality of the approximation. In the following, we focus on the standard CTS distribution and present a methodology for a potential calibration of the FFT-based PDF approximation.

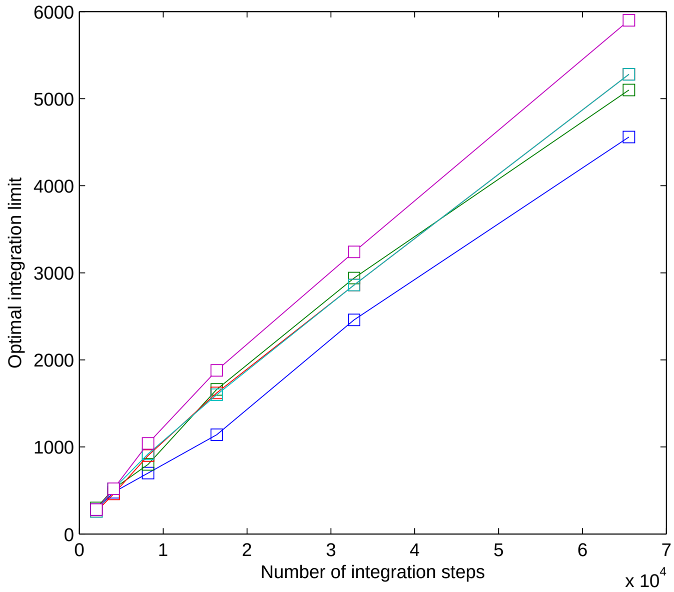

Figure 3: Optimal integration limits $a^*(q)$ for varying parameter q and different standard CTS distributions

In general the FFT parameters a and q should be chosen in a trade-off between computational complexity and required accuracy of PDF values. The complexity is determined by the number N of integration steps in the FFT. The accuracy can be measured by the approximation error $\epsilon_{f,X}(a,q)$ which is defined as

$$\epsilon_{f,X}(a,q) = \sup_{x\in\mathbb{R}} \left\{ f^{FFT}_{a,q}(x) - f^{Num}(x) \right\} \ , \tag{31}$$

where X is a stdCTS random variable, $f^{FFT}_{a,q}(x)$ and $f^{Num}(x)$ denote the PDF calculated by either the FFT method or numerical integration. We conclude that the quality of the approximation depends both on the FFT parameters and the distributional parameters. Figure 2 shows that the approximation error and the error-minimizing a^* for a given parameter q vary with the CTS parameters. In figure 3 the linear correlation between a^* and q is displayed. For MLE the exact CTS parameters are often unknown and that is why it is reasonable to estimate the approximation error based on the FFT parameters and the sample moments only.

In order to keep the optimization of the FFT parameters simple and robust, we suggest estimating q and a sequentially. q is determined first because it is closely related to the computational capacity and has a strong influence on the accuracy of the FFT method. That is why we need (1) a rough estimate $\hat{\epsilon}_f(q)$ of the dimension of the approximation error based on q and (2) the optimal parameter $a^*(q)$ given

α	λ_+	λ_-	$\mathrm{K}[X]$	q	a	$\log_{10}(\epsilon_{f,X}(a,q))$
0.5	1.5	0.8	7.6841	11	240	-3.8932
1.5	1.5	0.8	3.8180	11	300	-4.8747
0.8	1.5	1	5.0816	11	260	-4.5904
0.3	1	2	6.7798	11	260	-4.0448
1.6	1.1	1.05	5.0931	11	280	-5.3083
0.5	1.5	0.8	7.6841	12	500	-4.6696
1.5	1.5	0.8	3.8180	12	520	-5.5324
0.8	1.5	1	5.0816	12	460	-5.2018
0.3	1	2	6.7798	12	520	-4.7613
1.6	1.1	1.05	5.0931	12	520	-6.2735
0.5	1.5	0.8	7.6841	13	760	-5.2499
1.5	1.5	0.8	3.8180	13	800	-6.2868
0.8	1.5	1	5.0816	13	900	-6.2082
0.3	1	2	6.7798	13	920	-5.6326
1.6	1.1	1.05	5.0931	13	1040	-6.9148
0.5	1.5	0.8	7.6841	14	1600	-5.9663
1.5	1.5	0.8	3.8180	14	1660	-7.4091
0.8	1.5	1	5.0816	14	1620	-7.2649
0.3	1	2	6.7798	14	1600	-6.7678
1.6	1.1	1.05	5.0931	14	1880	-7.7314
0.5	1.5	0.8	7.6841	15	2840	-6.9696
1.5	1.5	0.8	3.8180	15	2940	-8.3406
0.8	1.5	1	5.0816	15	2860	-8.1683
0.3	1	2	6.7798	15	2860	-7.6024
1.6	1.1	1.05	5.0931	15	3240	-8.8701
0.5	1.5	0.8	7.6841	16	4980	-7.8939
1.5	1.5	0.8	3.8180	16	5100	-9.1084
0.8	1.5	1	5.0816	16	5280	-8.9881
0.3	1	2	6.7798	16	5280	-8.394
1.6	1.1	1.05	5.0931	16	5900	-9.4468

Table 1: Logarithm of approximation error $\epsilon_{f,X}(a,q)$ given parameter of FFT transform and standard CTS distribution

q. We use the following regression models for (1) and (2), where $\mathrm{K}[X]$ denotes the kurtosis of X:

$$\log_{10}(\hat{\epsilon}_f(q)) = a_0 + a_1 \cdot q + a_2 \cdot \mathrm{K}[X] \tag{32}$$
$$a^*(q) = b_0 + b_1 \cdot 2^q \; . \tag{33}$$

In order to reduce the risk of finding a local optimum, we smoothed the errors $\hat{\epsilon}_f(q)$ with window size 5.

For the data in table 1, the optimal parameters for model (32) are: $a_0 = 3.0558$, $a_1 = -0.8628$, and $a_2 = 0.3478$. The R^2 of 0.9525 indicates a strong correlation and adding information, e.g. the skewness $\mathrm{S}[X]$, does not result in a significant improvement of the explanatory quality ($R^2 = 0.9538$). For model (33) the optimal parameters are $b_0 = 229.1045$ and $b_1 = 0.0767$ with a R^2 of 0.9951.

Consequently the following two-step procedure provides a FFT parametrization (q^*, a^*) minimizing the approximation error.

1. Choose a q^* such that the dimension of $\hat{\epsilon}_f(q)$ is smaller or equal to the required accuracy. If q^* exceeds the computationally feasible limit q_{max} than use $q^* = q_{max}$.
2. Compute $a^*(q^*)$ using the regression formula (33).

Due to the use of the regression formula for $\hat{\epsilon}_f(q)$ and $a^*(q)$, the derived parameters may not represent the global optimum. The advantage of the procedure is nevertheless its robustness due to its independence of the CTS parameters and its simplicity. The convexity of the curves in figure 2 furthermore suggests that given q^*, we find a near-optimal solution.

The parameter selection problem for the CDF is more complex because of the additional parameter ρ and the slower convergence of the integral which should be approximated. Further research should be conducted to define parametrization rules for the CDF and VaR cases in the same manner as we outlined in this paper.

5 Empirical results for S&P 500 data

In this section, we present the MLE results for a simple CTS model based on S&P 500 return data. Therefore, we define four different data samples: (a) 4-year data from 26 June 2005 to 26 June 2009, (b) 6-year data from 26 June 2003 to 26 June 2009, (c) 8-year data from 26 June 2001 to 26 June 2009, and (d) 10-year data from 26 June 1999 to 26 June 2009. For each sample we perform 10 runs using randomized starting parameters and then we select the CTS parameters with the best fit. The log-likelihood function is approximated using two methods: method I (the FFT method with standardized CTS distribution presented in section 3) and

Data set	M		Method I	Method II
4 years	1008	$\text{logLH}_{\text{FFT}}$	2,973.560	2,975.087
		$\text{logLH}_{\text{Num}}$	2,973.798	2,972.374
		Relative error	0.008%	0.091%
6 years	1511	$\text{logLH}_{\text{FFT}}$	4,685.228	4,692.750
		$\text{logLH}_{\text{Num}}$	4,682.393	4,675.536
		Relative error	0.061%	0.368%
8 years	2011	$\text{logLH}_{\text{FFT}}$	6,040.171	6,036.312
		$\text{logLH}_{\text{Num}}$	6,038.343	6,038.048
		Relative error	0.030%	0.029%
10 years	2516	$\text{logLH}_{\text{FFT}}$	7,493.580	7,497.565
		$\text{logLH}_{\text{Num}}$	7,493.295	7,493.064
		Relative error	0.004%	0.060%

Table 2: Comparison of optimal logLH values from MLE for CTS distribution using four different S&P 500 data samples

method II (the FFT method without standardization). The goodness-of-fit for the estimated distributions is assessed with the help of the log-likelihood value calculated by numerical integration as well as by the corresponding FFT approximation, the p-value for the Kolmogorov-Smirnov statistic (KS),[6] and the Cramér-von Mises statistic (CvM).[7] The tail fit is measured by the squared Anderson-Darling statistic (AD^2).[8] According to the FFT parameter selection procedure presented, we choose $q^* = 13$ and $a^*(q^*) = 800$ for both FFT methods I and II.

A comparison of the optimal log-likelihood (logLH) values in table 2 reveals that the relative error ${}^{\text{logLH}_{\text{FFT}}-\text{logLH}_{\text{Num}}}/_{\text{logLH}_{\text{Num}}}$ for method I is less than 0.07%. For method II the maximum relative error observed is 0.368%, which is five times higher than for the MLE with standardization.

Table 3 shows that the goodness-of-fit results of method I are in general superior to method II: the logLH and p-values are higher and the AD^2 and CvM statistics are lower–both indicating a better fit. The exception for the 6-year data set, where the p-value of method I is worse, can be explained with the specifics of the KS statistic. It measures the maximum deviation which might be due to an outlier in the data set. The CvM statistic nevertheless coincides with the general results. The empirical findings support the theory that standardization leads to a better approximation quality because the optimal FFT parameters a^* and q^* do not depend on the sample mean and variance which reduces the risk of misspecification.

[6]See Kolmogorov (1933).
[7]See Cramér (1928).
[8]See Anderson and Darling (1952).

Data set	M		Method I	Method II	Δ_{I-II}
4 years	1008	$\text{logLH}_{\text{Num}}$	2,973.798	2,972.374	1.4244
		p-value	0.911	0.896	0.0154
		AD^2	0.267	0.370	-0.1032
		CvM	0.042	0.050	-0.0075
6 years	1511	$\text{logLH}_{\text{Num}}$	4,682.393	4,675.536	6.8572
		p-value	0.711	0.841	-0.1297
		AD^2	0.431	0.601	-0.1706
		CvM	0.056	0.061	-0.0049
8 years	2011	$\text{logLH}_{\text{Num}}$	6,038.343	6,038.048	0.2947
		p-value	0.834	0.751	0.0825
		AD^2	0.408	0.661	-0.2535
		CvM	0.047	0.110	-0.0622
10 years	2516	$\text{logLH}_{\text{Num}}$	7,493.295	7,493.064	0.2306
		p-value	0.831	0.424	0.4078
		AD^2	0.420	0.765	-0.3450
		CvM	0.057	0.118	-0.0613

Table 3: Goodness-of-fit results from MLE for CTS distribution using four different S&P 500 data samples

6 Conclusion

In this paper we outlined an efficient approximation for the PDF, CDF, VaR, and AVaR of TS and TID distributions. Based on knowledge of the characteristic function, the FFT method is used to compute the density and distribution functions. As Menn and Rachev (2006) argued for the stable Paretian case, this procedure is computationally efficient. We explained why standardization is important for the parameter choice of the FFT and presented how the standardized CTS and RDTS can be used to derive PDF values for any parameterization. For practical implementation purposes, this means reducing the risk of misspecifying the integration limit and the number of integration steps and thereby improving the effectiveness of the proposed FFT-based method. For the PDF of the CTS distribution, we proposed a two-step procedure for a near-optimal FFT parameter selection. In each step, a regression model is used to determine one optimal parameter. Empirical results using S&P 500 return data supported our two main theoretical results for the CTS case: applying the FFT method to the CTS distribution delivers good approximation quality and using standardization improves the effectiveness of MLE.

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