A comparison of the Lee-Carter model and AR-ARCH model for forecasting mortality rates

Rosella Giacometti[a], Marida Bertocchi[b], Svetlozar T. Rachev[c], Frank J. Fabozzi[d,e]

[a] Rosella Giacometti Department of Mathematics, Statistics,Computer Science and Applications, Bergamo, University, Via dei Caniana, 2, Bergamo 24127, Italy (rosella.giacometti@unibg.it)

[b] Marida Bertocchi, Department of Mathematics, Statistics,Computer Science and Applications, Bergamo, University, Via dei Caniana, 2, Bergamo 24127, Italy (marida.bertocchi@unibg.it)

[c]Svetlozar T. Rachev, School of Economics and Business Engineering, University of Karlsruhe, Postfach 6980, 76128 Karlsruhe, Germany, Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106-3110, USA , and Chief-Scientist, FinAnalytica Inc. (rachev@kit.edu)

[d] Frank J. Fabozzi, Yale School of Management, 135 Prospect Street, Box 208200, New Haven, Connecticut 06520-8200, USA (frank.fabozzi@yale.edu and fabozzi321@aol.com)

[e]**CONTACT AUTHOR:** Frank J. Fabozzi, Yale School of Management, 135 Prospect Street, Box 208200, New Haven, Connecticut 06520-8200, USA (frank.fabozzi@yale.edu and fabozzi321@aol.com)

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Abstract:
With the decline in the mortality level of populations, national social security systems and insurance companies of most developed countries are reconsidering their mortality tables taking into account longevity risk. The Lee and Carter model is the first model to consider the increased life expectancy trends in mortality rates and is still broadly used today. In this paper, we propose an alternative to the Lee-Carter model: an AR(1)-ARCH(1) model. More specifically, we compare performance of these two models with respect to forecasting age-specific mortality in Italy. We fit the two models, with Gaussian and *t*-student innovations, for the matrix of Italian death rates from 1960 to 2003. We compare the forecast ability of the two approaches in out-of-sample analysis for the period 2004-2006 and find that the AR(1)-ARCH(1) model with *t*-student innovations provides the best fit among the models studied in this paper.

Key words: mortality rates, Lee-Carter model, autoregression-autoregressive conditional heteroskedasticity model, AR(1)-ARCH(1) model
JEL Classification: C51, C52, C53, C59, G22

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1. Introduction

Mortality risk is the risk of having a higher percentage of deaths than expected, which implies a higher probability of death. Factors that impact mortality are wars, epidemics or pandemics, leading to a higher mortality rate (i.e., people living less than expected). Longevity risk is the risk of people surviving longer than expected or observed death rates being lower than expected. Advances in medical science, technological improvements and lifestyle changes tend to reduce the number of deaths.

From an economic perspective, the decline of mortality has a significant adverse impact on pension plans and annuity insurers (i.e., entities who provide old age benefits). A significant overestimate of mortality the rate implies a high risk profile for pension funds and annuity insurers. Longevity risk can be hedged with reinsurance contracts and with longevity derivatives. A typical contract is a longevity bond which pays a coupon that is proportional to the number of survivors in a selected birth cohort. The pricing of such products requires handling the uncertainty of life expectancy going forward.

The prevailing literature deals with different scenarios of mortality risk and/or stochastic death distributions.[1] The model presented by Lee and Carter (1992) appears to be the first model that considers the increased life expectancy trends in mortality rates. Although originally applied to U.S. mortality data, it is now applied to all-cause and cause-specific mortality data from many countries.[2] Moreover, many of the recent approaches that have been proposed in the literature are consistent with Lee-Carter model (see Lifemetrics (2007) and the reference therein) and the consensus in the literature in the last decade appears to consider the Lee-Carter model the leading statistical model for forecasting mortality.[3]

The Lee-Carter method combines a demographic model with a statistical model of time series to forecast mortality rates. As pointed out by Girosi and King (2007), the Lee-Carter model can be viewed as a special type of a multivariate process in which the covariance matrix depends on the drift vector and the innovations are intertemporally correlated.

In this paper, we present an alternative econometric model to the Lee-Carter model for

[1] See Milevsky and Promislow (2001), Ballotta and Haberman (2006), and Giacometti et al. (2009, 2010).
[2] See, for example, Tuljapurkar et al. (2000).
[3] See, among others, Lee and Miller (2000), Lee (2000), Deaton and Paxson (2004), and Denuit (2009).

forecasting mortality and then empirically compare the forecasting properties of the two models. The model we propose is the autoregression (AR)-autoregressive conditional heteroskedasticity (ARCH) model. More specifically, we propose the AR(1)-ARCH(1) model. The paper is organized as follows. In the next section, we review the main characteristics of the Lee-Carter model. Our econometric approach is described in Section 3, followed by a description of our database and the results of estimates of the models in the Section 4. In Section 5, we compare the results of all models.

2. The Lee-Carter model

Let $m_{x,t}$ be the death rate for age x in year t. Lee and Carter (1992) suggested a log-bilinear form for the force of mortality $\mu_{x,t}$, that is

$$\mu_{x,t} = \ln\left(m_{x,t}\right) = \alpha_x + \beta_x k_t + \varepsilon_{x,t}$$

$$x=1,\ldots, A;\ t=1,\ldots,T, \qquad (1)$$

where α_x, β_x are age-specific parameters, k_t is a time-varying parameter representing a common factor risk, and $\varepsilon_{x,t}$ is a zero mean Gaussian error $N(0,\sigma^2)$. The random term $\varepsilon_{x,t}$ reflects a particular age-specific historical influence. The coefficients α_x are age-specific constants that describe the general shape of the age-mortality profile. The index k_t serves to capture the main temporal level of mortality. Since the parameterization in (1) is invariant with respect to the transformations:

$$(\beta_x, k_t) \rightarrow (c\beta_x, k_t/c) \text{ for } (\alpha_x, k_t) \rightarrow (\alpha_x - c\beta_x, k_t + c) \text{ for some } c \in \Re \backslash \{0\}, \qquad (2)$$

then the parameters β_x, k_t should satisfy the constraints:

$$\sum_{x=1}^{A} \beta_x = 1;\ \sum_{t=1}^{T} k_t = 0 \ (3),$$

in order to ensure the identifiability of the model. The constraint $\sum_{t=1}^{T} k_t = 0$ implies that the estimates of parameters α_x, are given by the averages of the force of mortality over the time period, that is, $\hat{\alpha}_x = \frac{1}{T}\sum_{t=1}^{T} \mu_{x,t}$.

Considering that $\mu_{x,t} - \hat{\alpha}_x = \beta_x k_t + \varepsilon_{x,t} \approx N\left(\beta_x k_t, \sigma^2\right)$ are Gaussian distributed with mean $\beta_x \cdot k_t$ and variance σ^2, then the parameters β_x and k_t can be estimated via maximum likelihood. In particular, as remarked by Lee and Carter (1992), the optimal solution can be found using the Singular Value Decomposition (SVD) of the matrix of the centered age profiles $z_{x,t} = \mu_{x,t} - \hat{\alpha}_x$.

Given the matrix $\mathbf{Z} = [z_{x,t}]_{x=1,\ldots,A,\ t=1,\ldots,T}$, we can compute the normalized eigenvector

$\boldsymbol{u_1}=[u_{1,1},\dots,u_{1,T}]'$ (respectively $v_1=[v_{1,1},\dots,v_{1,A}]'$) of the matrix $\mathbf{Z}'\mathbf{Z}$ (respectively $\mathbf{Z}\mathbf{Z}'$) corresponding to the largest eigenvalue λ_1. Then the optimal estimates satisfying the constraints (3) imposed on the parameters are given by the vectors:

$$\hat{\boldsymbol{\beta}} = [\hat{\beta}_1,\dots,\hat{\beta}_A]' = \frac{\mathbf{v_1}}{\sum_{j=1}^{A} v_{1,j}} \quad \text{and} \quad \hat{\mathbf{k}} = [\hat{k}_1,\dots,\hat{k}_T]' = \lambda_1 \left(\sum_{j=1}^{A} v_{1,j}\right)\mathbf{u_1} \quad .$$

Typically for low-mortality populations, the approximation $\mathbf{Z}\approx\lambda_1\mathbf{v_1}\mathbf{u}'_\mathbf{1}$ accounts for more than 90% of the variance of $\ln(m_{x,t})$. A further re-estimation step for the parameters k_t is required because with the above procedure the number of fitted deaths does not equal the number of observed deaths. The parameters $\hat{k}_t$ are adjusted (taking estimates $\hat{\alpha}_x$, $\hat{\beta}_x$ as given) such that the new estimates $\overline{\overline{k_t}}$ solve the equations

$$D_t = \sum_{x=1}^{A} N_{x,t} \exp\left(\hat{\alpha}_x + \hat{\beta}_x \overline{\overline{k_t}}\right) \qquad t=1,\dots,T,$$

where D_t and $N_{x,t}$ are, respectively, the total number of deaths in year t and the total population with age x in year t.

In order to forecast future mortality rates, Lee and Carter assume that α_x and β_x remain constant over time and the time factor k_t is intrinsically viewed as a stochastic process. They suggest using the following random walk with drift model for k_t:

$$\hat{k}_t = \hat{k}_{t-1} + \theta + \xi_t , \qquad (4)$$

where $\xi_t \approx N\left(0,\sigma_{rw}^2\right)$ are independent and identically distributed (i.i.d.) Gaussian distributed with null mean and variance σ_{rw}^2. The maximum likelihood estimate of the drift parameter θ is given by $\hat{\theta} = \left(\hat{k}_T - \hat{k}_1\right)/(T-1)$ and the variance estimate is $\hat{\sigma}_{rw}^2 = \frac{1}{T-1}\sum_{t=1}^{T-1}\left(\hat{k}_{t+1} - \hat{k}_t - \hat{\theta}\right)^2$. To estimate $\hat{k}_{T+\Delta t}$ at time $T+\Delta t$ we get $\hat{k}_{T+\Delta t} = \hat{k}_T + \left(\Delta t\right)\hat{\theta} + \sqrt{\Delta t}\,\tilde{\xi}$ where $\tilde{\xi} \approx N\left(0,\sigma_{rw}^2\right)$ and the expected log-mortality can be approximated as follows:

$$\hat{\mu}_{x,T+\Delta t} = \hat{\alpha}_x + \hat{\beta}_x\left(\hat{k}_T + \left(\Delta t\right)\hat{\theta}\right) = \hat{\alpha}_x + \hat{\beta}_x\left(\hat{k}_T + \Delta t\,\frac{\left(\hat{k}_T - \hat{k}_1\right)}{(T-1)}\right). \qquad (5)$$

3. The AR-ARCH model

In this section, we propose as an alternative to the Lee-Carter model estimating an AR(1)-ARCH(1) model for forecasting the force of mortality $\mu_{x,t}$. We analyze separately columns and rows of the mortality table. As a first step, we analyze the data by rows; that is, we fix a specific age *x* and consider a time series process $\{\mu_x\}_t$ with t = 1,…,*T*. Assuming the time series presents a polynomial trend of degree *n*, we estimate the following univariate AR(1)-ARCH(1) on the residuals, for each age *x,* with *x=1,..,A*:

$$\begin{aligned} \mu_{x,t} &= p_x(t) + \alpha_1 \mu_{x,t-1} + \varepsilon_{x,t} \\ \sigma^2_{x,t} &= \beta_0 + \beta_1 \varepsilon^2_{x,t-1} \end{aligned} \qquad (6)$$

where $\varepsilon_{x,t}$ is the innovation of the time series process $\{\mu_x\}_t$, with $\varepsilon_{x,t} = z_t \sigma_{x,t}$, z_t an i.i.d process with zero mean and constant variance, and $p_x(t)$ is polynomial of degree *n.*

As a second step, we analyze the data by columns: that is, we fix a specific year *t* and consider the age process $\{\mu_t\}_x$ with *x*=1,…,A. We assume a polynomial trend and we estimate for each year *t,* with *t=1,…T*

$$\begin{aligned} \mu_{x,t} &= p_t(x) + \phi_1 \mu_{x-1,t} + \eta_{x,t} \\ \sigma^2_{x,t} &= \varphi_0 + \varphi_1 \eta^2_{x-1,t} \end{aligned} \qquad (7)$$

where $\eta_{x,t}$ is the innovation of the age process $\{\mu_t\}_x$ for a fixed *t* , with $\eta_{x,t} = z_x \sigma_{x,t}$, z_x is an i.i.d. process with zero mean and constant variance, and $p_t(x)$ is a polynomial of degree *n*.

4. An empirical analysis based on the Italian mortality rate

In this section, we report the results of fitting the Lee-Carter model and the AR(1)-ARCH(1) model to Italian mortality data taken from the University of California, Berkeley "Human Mortality Database" available from 1922 to 2006.[4] We choose an opportune range of data (from 1960 to 2006) in order to have a reliable and complete data set. In Figure 1 we can observe the surface of the mortality data for years from 1960 to 2006 and ages from 0 to 94. However, we restrict the age from 40 to 94 in order to avoid the hump around age 0 and 39. In Figure 2 we report the surface of the dataset used in our analysis.

We divided the dataset into the following three subsamples:

- *Subsample 1*: From 1960 to 2003 and from age 40 to 91.
- *Subsample 2*:- From 2004 to 2006 and from age 40 to 91.

[4] Available at www.mortality.org.

- *Subsample 3*: Year 2004 to 2006 and from ages 92 to 94.

It is from s*ubsample 1* that we estimate the parameters of the models. Using subsample 2 we compare the Lee-Carter and AR-ARCH models. Recall that the classical Lee-Carter model can forecast only in one direction (i.e., the time). The AR(1)-ARCH(1) models with different innovations are compared using subsample 3. That is, we forecast in two directions, time and age in order to assess which of the two models performs the best.

Our empirical analysis involves the following three steps:

Step 1: Estimate Lee-Carter model and discuss how to model the time factor k_t. For three consecutive years, k_t and the force of mortality for the period 2004-2006 are forecasted.

Step 2: Fit the AR(1)-ARCH(1) model with Gaussian and *t*-innovations and forecast in the two dimensions the force of mortality for the period 2004-2006 and for ages 92-94.

Step 3: Compare the forecast from the Lee-Carter model, the AR(1)-ARCH(1) model with Gaussian and *t*-innovations, and the actual data on the subsample 2 and investigate the forecasting capacity of AR(1)-ARCH(1) models with different innovations for subsample 3 for the period 2004-2006 and for the ages 92-94.

We discuss the results of these three steps below.

Step 1

We fit the Lee-Carter model using the methodology presented in Section 3. In Figure 3 we report the estimate of temporal level of mortality $\hat{k}_t$. We tested for the presence of the unit root by applying the Adjusted Dickey-Fuller (ADF) test and found that we could not reject a null hypothesis for the presence of the unit root for different lags and different significance levels (from 0.01 to 0.05). We model $\hat{k}_t$ with a random walk with drift. Using equation (4), we estimate the drift θ =-0.8876. We observe that the residuals of the model are substantially i.i.d. Gaussian (no autocorrelation is revealed, the residual mean is 0, the standard deviation is 1.3356, and the kurtosis is 3.4706). We tested for the presence of the unit root in the residuals and we could reject the null hypothesis. Starting from the observation for the last year, we forecasted three years $\hat{k}_{t+i\Delta t}$ with i=1,2,3 and we reconstructed the forecasted matrix.

Step 2

Our analysis of the data indicates the presence of a nonlinear deterministic trend in both directions — through time and age. To remove the trend, we determined the coefficients of a polynomial of degree 2 that fits the data in a least squares sense. Then we analyzed the de-trended data. We tested for the presence of the unit root and we could always reject the null hypothesis for both directions. However, we observed the presence of a single jump on the diagonal of the force of mortality matrix. Considering the jump as a possible outlier, we smoothed the de-trended data in order to remove the jump. Smoothing the data implies removing part of the kurtosis but the time series characteristics remain unchanged. In order to have robust results, we continued our analysis using both the de-trended original data and the de-trended smoothed data.

In Table 1 we report the main statistics (volatility, skewness, and kurtosis) and the *p*-value of the Kolmogorov-Smirnov and the Jarque-Bera tests for the de-trended original data We From the kurtosis figures we see that almost 30% of the time series are leptokurtic and for 20% we reject the null hypothesis of a sample drawn from a normal distribution.

Inspecting the data's auto-correlogram and the squared data, we detect occasionally the presence of autocorrelations and heteroskedasticity, especially when we consider the age process $\{\mu_t\}_x$ with x=1,…,A.

We fit an AR(1)-ARCH(1) model assuming alternatively a Gaussian innovation and a *t*-student innovation. In Table 2 we report the estimate of the AR(1)-ARCH(1) for the age process $\{\mu_t\}_x$ and in Table 3 the time process $\{\mu_x\}_t$ with *t*-student innovation process.

In each row of Table 2 we report for a different year (with t = 1960,..,2003) the coefficients, the asymptotic *t*-statistics, the order of the integrated process, and the estimate of degrees of freedom of the innovation process. We observe that the AR coefficients are significantly different from zero for 31 of the 44 years. However, we cannot reject the null hypothesis for the ARCH coefficient for most of the age process. The estimated degrees of freedom reveal a non-Gaussian innovation process for 70% of the time series.

In each row of Table 3 we report the result for different ages, with x = 40,.., 91 obtaining similar results as reported in Table 2. The AR coefficients are significantly different from zero for 16 of the 52 ages and we cannot reject the null hypothesis for the ARCH coefficient for most of the age processes. The estimated degrees of freedom confirm a non-Gaussian innovation process for 46% of the ages.

For the period 2004-2006, we forecast the mortality rate using equation (7) under the two different distributional assumptions.

Step 3

Finally, we tested *ex-post* the ability of each model to forecast future Italian mortality rates. In Figure 4 we report the actual data, the Lee-Carter forecast, and the AR(1)-ARCH(1) forecast with *t* innovations (the 95% confidence levels is shown in red).

We analyzed the forecasts for three consecutive years for ages ranging from 40-91. In order to compare the models, we constructed a modified version of Theil's *U* index.

Theil's *U* index is a measure for forecasting quality in an out-of-sample analysis.[5]. It can be interpreted as the mean-squared error of the proposed forecasting model divided by the mean-squared error of a naïve prediction model used as a benchmark. For stationary time series, the naïve model is generally given by either the previous observation or a no-change model. In our analysis, we modified the index using as a naïve model the linear deterministic trend estimated for each age *x*.

Index's values less than unity show an improvement over the simple naïve forecast. In order to have a value of the index for each year with t = 2004,..,2006, we computed

$$TheilU_t = \sqrt{\frac{\sum_{x=40..91}\left(\mu_{x,t} - \hat{\mu}_{x,t}\right)^2}{\sum_{x=40..91}\left(\mu_{x,t} - p_x(t)\right)^2}} \qquad with \qquad t = 2004..2005$$

In Table 4 we report Theils' *U* index for each of the three years for both the de-trended data and the smoothed data, as well as the index for the overall period. We observe that in the forecasting period, the introduction of the econometric model lead to a more accurate forecast with compared to the Lee-Carter model. Theil's *U* index for the Lee- Carter model ranges between 0.60 and 0.63. In contrast, the AR(1)-ARCH(1) ranges between 0.434 and 0.431 when we consider a Gaussian innovation, and between 0.434 and 0.430 when we consider a *t*-student innovation. This means that the sum of the squared errors of the AR(1)-ARCH(1) model is less than the sum of the squared errors of the Lee-Carter model of the naïve model. These results

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$$TheilU = \sqrt{\frac{\sum_{t=1..T}\left(y_t - \hat{F}_t\right)^2}{\sum_{t=1..T}\left(y_t - y_{t-1}\right)^2}}$$

where F_t and y_t stand for a pair of predicted and observed values, with $t=1,\ldots,T$ and y_0 is the last observation used for the estimation.

strongly suggest that the AR-ARCH model with *t*-student innovation is superior to the Lee-Carter model in forecasting.

If we consider the index for each single year, we observe that the forecasting ability decreases with the increase of the time period forecast. Once again, AR(1)-ARCH(1) model assuming a *t*-student innovation provides the best fit to both the original data and the smoothed data set.

Finally, we forecast for each year the force of mortality for ages 92 to 94, and on the enlarged dataset, we fit again model (7) obtaining the forecast for years 2004-2006 and ages 92-94. In Table 5 report the results of Theil's *U* index. We do not provide the Lee-Carter model values since this model does not provide forecasts for out-of-sample ages. The AR(1)-ARCH(1) with *t*-student innovations confirm its superiority using both the original and the smoothed data.

In Figure 5, we show the actual data, the AR(1)-ARCH(1) forecast with *t*-student innovation (in red the 95% confidence levels) for ages 92,93, and 94.

5. Conclusions

In this paper, we propose two AR(1)-ARCH(1) models for forecasting the mortality rate and compare their forecasts to the classical Lee-Carter model. We find that an AR(1)-ARCH(1) model with *t*-student innovations provides the best fit among the models investigated because it is able to capture the non-Gaussian behavior of the dynamics associated with the time and age processes. Moreover, this model is capable of enlarging the mortality matrix in two dimensions: time and age.

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Figure 1 Surface of the mortality data for the years from 1960 to 2006 and the ages from 0 to 94: Italian $\hat{k}_t$ estimation (1960-2003)

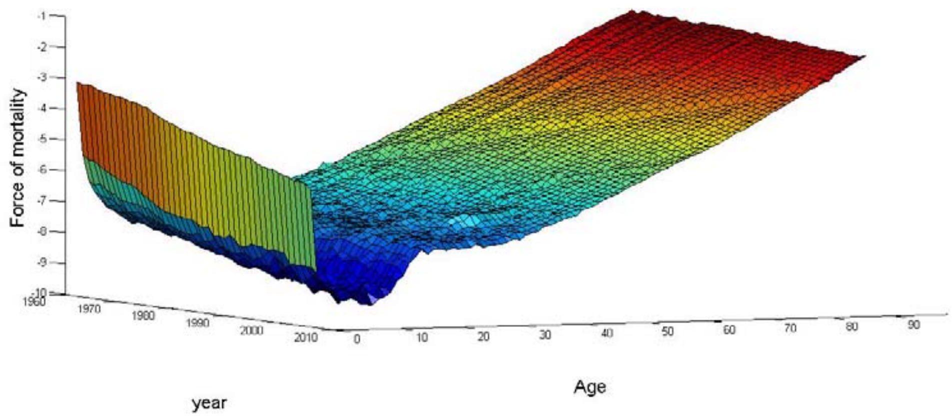
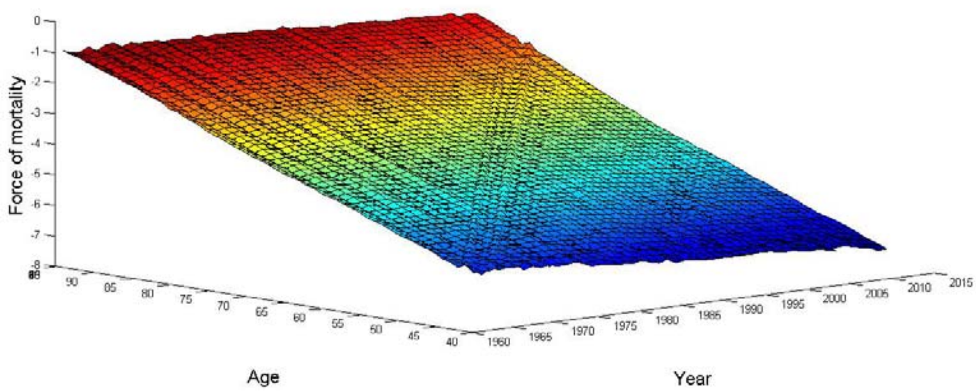


Figure 2 Surface of the mortality data from year 1960 to year 2006 and from age 40 to age 94



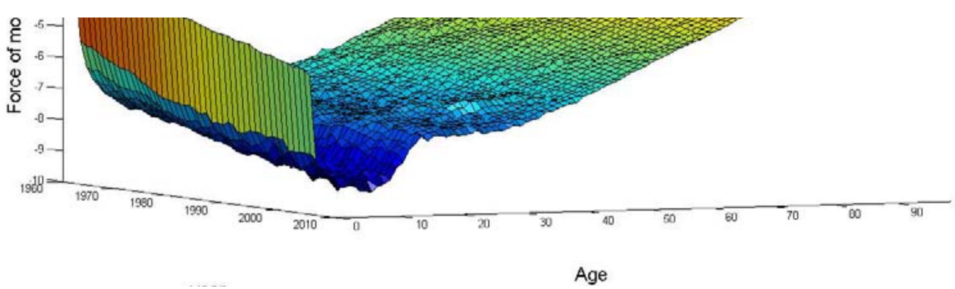


Figure 3 Estimation of the empirical level $\hat{k}_t$ of mortality for the Italian population (1960-2003) for the Lee-Carter model: Italian $\hat{k}_t$ estimation (1960-2003)

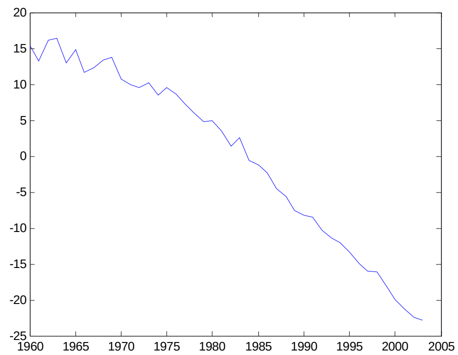


Figure 4 Forecasts for each age of the force of mortality for the years from 2004 to 2006 for the Italian population

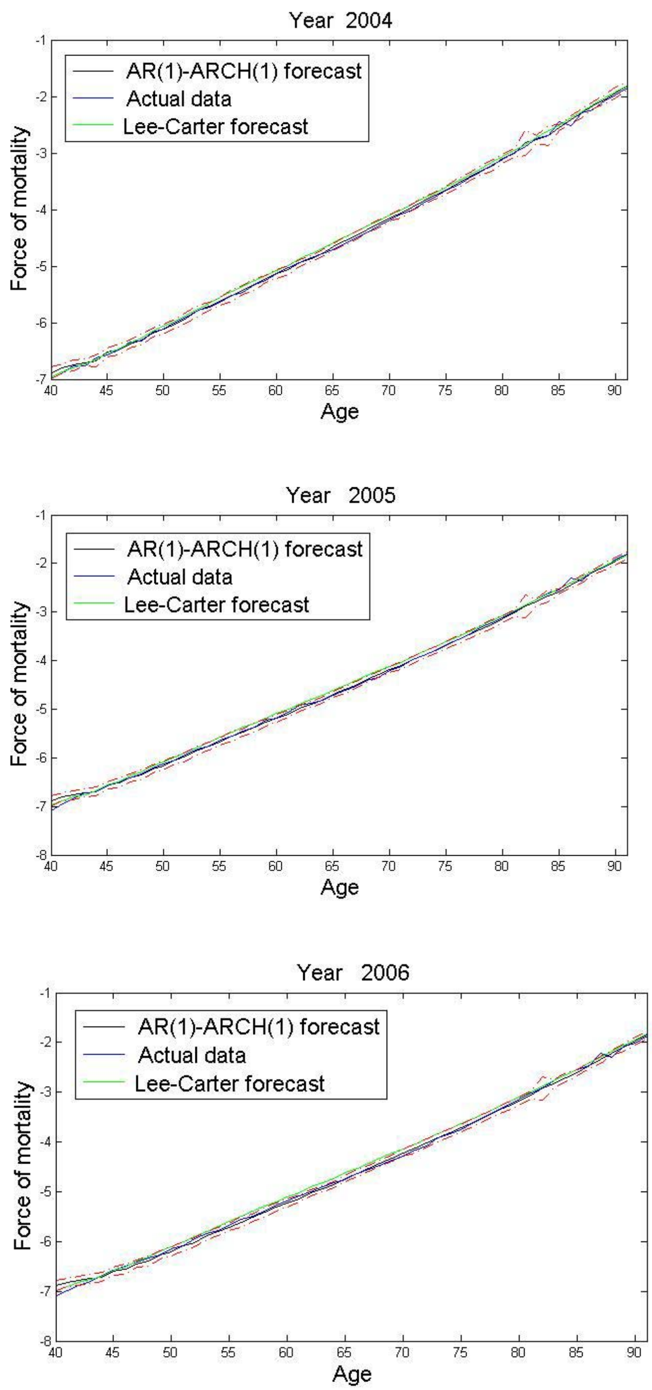


Figure 5 Forecasts for each year of the force of mortality of the ages from 92 to 93 for the Italian population (1960-1996)

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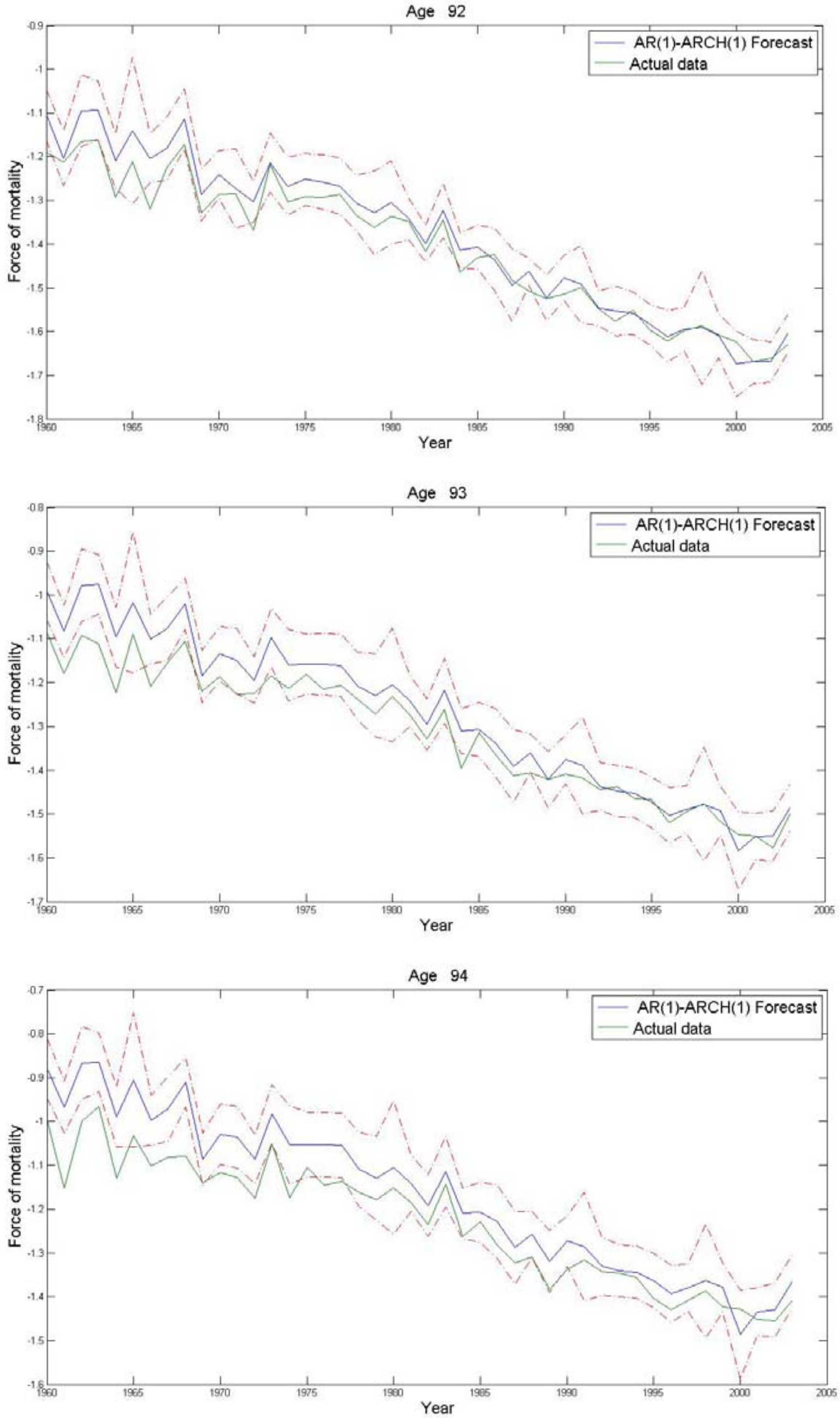

Table 1: Summary statistics for the de-trended original data: Volatility, skewness, kurtosis,and *p*-value of the Kolmogorov-Smirnov (KS) and Jarque-Bera (JB) tests

Age	Volatility	Skewness	Kurtosis	KS	JB	Year	Volatility	Skewness	Kurtosis	KS	JB
40	0.0545	0.2364	2.1904	0.4683	0.2952	1960	0.0369	0.6606	4.6509	0.4461	0.009
41	0.0449	0.2235	3.0461	0.1295	0.8184	1961	0.0341	-0.099	2.1055	0.3686	0.2653
42	0.0469	-0.7317	3.8788	0.3175	0.044	1962	0.0357	-0.0259	2.886	0.873	0.977
43	0.0465	-0.7542	2.8817	0.062	0.068	1963	0.0329	-0.4868	4.4115	0.3688	0.031
44	0.0497	-0.1561	2.1416	0.612	0.3158	1964	0.0333	-1.241	6.0731	0.047	0
45	0.0436	-0.2558	2.3223	0.3327	0.3862	1965	0.0341	-0.8129	6.3005	0.104	0.002
46	0.0499	-0.3423	3.1173	0.7608	0.5547	1966	0.031	-0.3074	7.8924	0.046	0
47	0.0442	-0.0542	2.6599	0.8158	0.8863	1967	0.0298	-0.5566	7.5367	0.046	0
48	0.0526	-0.4141	2.6357	0.4922	0.3179	1968	0.0294	-0.3224	4.0329	0.7151	0.09
49	0.0448	-0.4942	3.3496	0.4072	0.2148	1969	0.03	0.1332	2.7042	0.8209	0.8284
50	0.0492	-0.1043	2.3596	0.2808	0.5779	1970	0.0308	-0.2414	4.5049	0.2609	0.045
51	0.0427	-0.0904	2.9286	0.895	0.96	1971	0.0332	0.1771	4.3671	0.1839	0.061
52	0.0478	-0.3012	2.5616	0.019	0.49	1972	0.0317	0.3199	3.1364	0.3855	0.5367
53	0.0481	-0.2535	2.3335	0.863	0.4021	1973	0.0339	0.3319	3.3585	0.8049	0.4304
54	0.042	-0.0839	2.0913	0.053	0.2839	1974	0.0434	0.0404	3.5727	0.3803	0.6552
55	0.0402	-0.124	2.5984	0.3322	0.7842	1975	0.0389	-0.0962	2.7133	0.978	0.8781
56	0.0402	-0.2131	2.9675	0.8263	0.8221	1976	0.0469	0.2841	3.1677	0.43	0.6304
57	0.0397	0.5245	3.4687	0.2416	0.1507	1977	0.0444	0.1256	3.4115	0.5567	0.75
58	0.0421	0.0521	2.6613	0.1567	0.8652	1978	0.0454	0.3355	3.0441	0.8528	0.518
59	0.031	0.4282	2.744	0.107	0.3562	1979	0.0507	0.0357	2.901	0.4473	0.984
60	0.0334	0.3647	3.062	0.2958	0.5107	1980	0.0402	-0.1313	3.0954	0.5219	0.919
61	0.0333	0.4922	3.3297	0.7041	0.1954	1981	0.0465	0.0982	2.1656	0.523	0.3147
62	0.0341	0.3731	4.283	0.1906	0.055	1982	0.0443	-0.3195	2.3804	0.5712	0.2825
63	0.031	0.2959	2.7495	0.8508	0.6128	1983	0.0475	-0.1848	2.513	0.7293	0.5912
64	0.0335	0.5782	5.4177	0.1251	0.006	1984	0.0447	-0.3353	2.5769	0.1374	0.3908
65	0.0312	0.8264	4.3275	0.5904	0.015	1985	0.044	-0.166	2.6879	0.7887	0.7755
66	0.0365	-0.0875	3.5048	0.7162	0.7495	1986	0.0409	-0.4727	2.8878	0.599	0.2252
67	0.033	0.4937	3.9055	0.1934	0.087	1987	0.0457	-0.6235	3.2209	0.4538	0.086
68	0.0311	0.2019	3.8352	0.6139	0.314	1988	0.0422	-0.6966	3.5651	0.2689	0.061
69	0.0348	-0.8489	4.9173	0.8378	0.013	1989	0.0395	-0.8691	3.6002	0.01	0.024
70	0.0363	-0.3654	3.8958	0.1387	0.15	1990	0.035	-0.722	3.8258	0.4139	0.036
71	0.0428	-0.3791	3.9915	0.044	0.112	1991	0.0394	-0.665	3.2364	0.1322	0.061
72	0.0409	-0.1129	4.4169	0.031	0.076	1992	0.038	-0.2413	3.6386	0.3413	0.3688
73	0.0404	0.1063	3.8093	0.2696	0.4035	1993	0.0306	-0.9123	3.813	0.01	0.024
74	0.0368	-0.3921	3.7892	0.015	0.166	1994	0.0313	-0.2734	3.4862	0.4309	0.468
75	0.0358	-0.0565	2.9666	0.3901	0.985	1995	0.0345	0.0182	2.933	0.5267	0.994
76	0.0414	-0.2928	2.5219	0.005	0.4703	1996	0.0342	-0.1955	2.9948	0.3603	0.8302
77	0.039	0.0065	2.2134	0.7341	0.4597	1997	0.0291	-0.0089	4.0299	0.8484	0.1813
78	0.0397	0.1761	2.8176	0.97	0.8711	1998	0.0324	0.0311	4.4427	0.032	0.066
79	0.0443	0.2769	2.8151	0.905	0.6869	1999	0.0289	0.0788	3.6316	0.2825	0.5667
80	0.0381	0.5471	2.8509	0.071	0.1678	2000	0.0298	0.4661	4.7319	0.2766	0.029
81	0.0397	0.4446	2.4832	0.041	0.2102	2001	0.0275	0.7482	5.3856	0.1266	0
82	0.0392	0.5894	3.2435	0.1746	0.1307	2002	0.0303	0.6644	6.1049	0.042	0.001
83	0.038	0.9263	4.4409	0.2429	0.009	2003	0.0352	-0.0406	3.9717	0.7307	0.1995
84	0.044	1.0817	5.6958	0.6565	0.002						
85	0.0293	0.1282	2.1947	0.3431	0.3915						
86	0.0331	0.3655	2.9399	0.2733	0.5283						
87	0.0364	0.25	2.1361	0.5628	0.2547						
88	0.0302	0.3329	2.6321	0.807	0.4907						

89	0.0366	0.4375	3.1215	0.05	0.3438							
90	0.0403	0.0458	2.6087	0.555	0.852							
91	0.0365	-0.002	2.2239	0.789	0.4697							

Table 2 Estimate of the coefficients of the AR(1)-ARCH(1) and *t*-statistic for the age process

	Coefficients			*t*-statistic			Order of	
	AR	K	ARCH	AR	K	ARCH	Integration	DoF
1960	0.5551	0.0009	0.3168	4.1632	2.6459	1.0145	0	4.26
1961	0.6732	0.0006	0.3478	4.4001	2.3682	1.6719	0	200.00
1962	0.4449	0.0017	0.0374	3.8016	0.5097	0.113	0	2.87
1963	0.2997	0.0012	0	2.0886	1.5234	0	0	3.52
1964	0.3267	0.0008	0.3498	2.0593	1.0009	0.9351	0	3.29
1965	0.3044	0.0004	0.8587	1.4664	2.095	2.316	0	32.04
1966	-0.0647	0.0007	0.1914	-0.3731	2.4059	0.8676	0	4.19
1967	-0.003	0.0004	0.7002	-0.0157	1.3643	1.1981	0	3.75
1968	-0.2368	0.0006	0.2079	-1.1574	2.6832	0.8323	0	200.00
1969	0.5807	0.0004	0.5648	3.9567	3.0356	1.3918	0	200.00
1970	0.463	0.0003	0.9336	2.7411	1.9429	2.3648	0	46.97
1971	0.5104	0.001	0.2213	3.6849	1.3312	0.6811	0	3.29
1972	0.7135	0.0004	0.503	5.7466	2.6137	1.1324	0	5.81
1973	0.7164	0.0005	0.5772	6.7885	2.0005	1.6313	0	7.49
1974	0.6564	0.0009	0.6704	5.3089	1.3675	1.0191	0	3.77
1975	-0.5886	0.0007	0.5523	-5.2615	1.2995	0.7502	1	3.01
1976	-0.48	0.0008	0.4965	-3.0173	1.0745	0.9452	1	4.63
1977	0.6227	0.0009	0.3506	5.8841	2.2275	1.0383	0	4.70
1978	0.7241	0.0009	0.6105	6.8865	1.0384	0.9394	0	3.73
1979	0.6434	0.0023	0	6.623	0.7557	0	0	2.84
1980	0.674	0.0022	0.8915	6.5425	0.2514	0.2532	0	2.29
1981	0.8447	0.0006	0.576	8.7416	2.1177	1.8918	0	10.58
1982	0.76	0.0004	0.956	9.5644	1.5157	2.0198	0	7.86
1983	0.7917	0.0009	0.498	8.0418	1.7635	1.086	0	3.89
1984	-0.4195	0.0004	0.6465	-2.1382	2.8617	1.657	1	20.49
1985	0.7721	0.0006	0.626	7.3012	2.087	1.6266	0	10.75
1986	-0.5657	0.0005	0.8534	-4.2558	1.6524	1.3296	1	4.72
1987	0.6116	0.0018	0	5.4436	0.9125	0	0	2.95
1988	0.8674	0.0002	0.9797	13.3185	1.5791	2.0029	0	200.00
1989	0.6811	0.0007	0.5684	5.8052	1.3494	1.4543	0	5.12
1990	0.7519	0.0004	0.5956	4.7835	2.152	2.2279	0	200.00
1991	0.6338	0.0019	0.6526	4.8055	0.3918	0.3826	0	2.47
1992	-0.4155	0.0004	0.9535	-3.5888	2.8195	1.9381	1	200.00
1993	0.1216	0.0006	0.3534	0.5442	2.333	1.5989	0	200.00
1994	0.3509	0.0005	0.462	1.8854	1.4857	1.4735	0	200.00
1995	0.3229	0.0006	0.5309	1.7187	2.491	1.5455	0	200.00
1996	0.163	0.0009	0.2158	0.798	2.8864	1.0747	0	200.00
1997	0.3574	0.0005	0.3155	1.81	2.2213	1.3965	0	9.08
1998	0.0525	0.0043	0	0.357	0.1367	0	0	2.17
1999	-0.1914	0.0006	0.2258	-1.1342	1.7558	0.7496	0	7.39
2000	-0.5386	0.0006	1	-6.5546	1.9244	1.7248	2	7.01
2001	-0.0742	0.0006	0.155	-0.498	2.0404	0.5031	0	4.14
2002	-0.0843	0.0005	0.5291	-0.4829	1.6145	0.9993	0	3.92
2003	0.7216	0.0004	0.657	5.3077	1.6838	1.7748	0	5.74

Table 3: Estimate of the coefficients of the AR(1)-ARCH(1) and *t*-statistic for the time process

	Coefficients			*t*-statistic			Order of	
	AR	K	ARCH	AR	K	ARCH	integration	DoF
40	0.246	0.0027	0	1.42	1.5183	0	0	200.00
41	0.2364	0.0019	0	1.30	3.7588	0	0	52.31
42	0.0836	0.0021	0	0.44	3.0905	0	0	11.18
43	0.0181	0.0009	0.6559	0.08	1.1355	1.4028	0	200.00
44	0.3535	0.0016	0.264	2.24	2.0176	0.7314	0	200.00
45	0.5841	0.0014	0.0208	3.79	2.8693	0.1134	0	7.41
46	0.3407	0.0021	0.1194	1.42	1.8459	0.5083	0	5.67
47	0.6056	0.001	0.3943	3.41	2.3355	1.1337	0	14.66
48	0.3249	0.0023	0.0748	1.47	3.3803	0.3029	0	45.72
49	0.5568	0.0011	0.3629	3.65	2.3166	0.8418	0	8.99
50	0.5282	0.0018	0.216	3.18	1.9425	0.8243	0	6.71
51	-0.4168	0.0004	0.9313	-2.99	1.3262	1.8086	1	200.00
52	0.5876	0.0012	0.4196	3.97	1.3021	1.4476	0	200.00
53	0.6089	0.0013	0.2742	3.28	2.4372	0.9261	0	33.18
54	0.3337	0.0008	0.5386	2.02	0.9116	0.949	0	200.00
55	0.3506	0.0013	0.0944	1.85	2.9849	0.5655	0	29.74
56	0.3658	0.0024	0.1355	2.14	0.5665	0.3706	0	3.09
57	0.3625	0.0014	0.03	2.13	1.367	0.0644	0	6.74
58	0.322	0.0026	0.081	1.84	0.4613	0.1876	0	3.18
59	0.5328	0.0009	0.4878	3.21	0.559	0.6089	0	3.16
60	0.3132	0.0014	0.039	1.91	0.657	0.1126	0	3.36
61	-0.0857	0.0009	0.1719	-0.38	1.7134	0.7147	0	200.00
62	0.3907	0.0005	0.608	2.27	2.3432	1.5716	0	16.90
63	-0.1389	0.0009	0.0664	-0.77	2.0028	0.2689	0	200.00
64	-0.236	0.0009	0.0573	-1.20	2.1101	0.4132	0	5.00
65	0.0065	0.001	0	0.04	2.1692	0	0	5.76
66	-0.1029	0.0012	0.0713	-0.60	1.5324	0.3191	0	5.89
67	-0.1468	0.001	0.0603	-0.86	2.1857	0.3589	0	5.97
68	-0.121	0.001	0	-0.65	1.4145	0	0	5.43
69	-0.2778	0.0011	0	-2.05	2.9634	0	0	5.81
70	0.131	0.0007	0.5453	0.67	1.7881	1.2199	0	7.15
71	0.0076	0.0018	0.0477	0.05	1.8581	0.1725	0	5.22
72	-0.3869	0.0006	0.999	-2.53	1.4631	1.5806	1	6.44
73	0.137	0.0015	0.0906	0.66	2.2773	0.2813	0	7.62
74	0.0208	0.0008	0.6165	0.10	1.3158	1.2098	0	6.75
75	0.1008	0.001	0.1694	0.47	2.2954	0.5788	0	200.00
76	0.117	0.0016	0.0528	0.55	1.3574	0.2061	0	200.00
77	0.2902	0.0012	0.1187	1.36	1.6868	0.4484	0	200.00
78	0.1893	0.0012	0.194	0.90	1.6716	0.6816	0	200.00
79	-0.1116	0.0017	0.0928	-0.53	1.4648	0.4316	0	200.00
80	0.2034	0.0013	0.0727	1.09	2.0724	0.1736	0	30.87
81	-0.1171	0.0015	0	-0.54	1.7299	0	0	200.00
82	0.6146	0.0005	0.9926	3.71	1.2966	1.3705	0	200.00
83	0.2189	0.0015	0	1.36	1.2512	0	0	4.30
84	-0.2456	0.0013	0.3009	-1.08	2.2237	1.1343	0	200.00
85	0.0342	0.0008	0	0.20	2.2784	0	0	200.00
86	0.0034	0.0011	0	0.02	3.4303	0	0	200.00
87	-0.007	0.0009	0.2824	-0.04	1.5564	0.5364	0	200.00

88	0.0356	0.0009	0	0.19	3.3206	0	0	200.00
89	-0.0333	0.0012	0.0728	-0.17	2.7571	0.2933	0	28.30
90	0.0383	0.0011	0.3713	0.24	1.4181	0.9215	0	200.00
91	-0.0189	0.0013	0	-0.11	1.7902	0	0	200.00

Table 4 Theil's *U* index for the three different years for the de-trended data and the smoothed data and the index for the overall period

	Original data			Smoothed data			Original data	Smoothed data
Model	2004	2005	2006	2004	2005	2006	2004-2006	2004-2006
Lee Carter	0.6050	0.6101	0.6564	0.5865	0.5885	0.634	0.6267	0.6061
AR-ARCH Gaussian Innovation	0.3634	0.4644	0.4652	0.3863	0.4586	0.4438	0.4349	0.4314
AR-ARCH *t*-student Innovation	0.3622	0.4644	0.4642	0.3873	0.4565	0.4414	0.4340	0.4300

Table 5 Theil's *U* index for the three different years for the de-trended data and the smoothed data, and the index for the overall period using the enlarged sample set

	Original data			Smoothed data			Original data	Smoothed data
Model	2004	2005	2006	2004	2005	2006	2004-2006	2004-2006
AR-ARCH Gaussian Innovation	0.3845	0.4622	0.464	0.4144	0.4567	0.4469	0.4395	0.4401
AR-ARCH *t*-student Innovation	0.3760	0.4630	0.4634	0.4147	0.4545	0.4442	0.4371	0.4384