Comment on "Weak Convergence to a Matrix Stochastic Integral with Stable Processes"

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Abstract. In this paper we identify a lacuna in a proof in the paper by M. Caner published in 1997 in this *Journal* concerning the weak limit behavior of various expressions involving heavy-tailed multivariate vectors and the convergence of stochastic integrals. In a later paper (Caner, 1998) uses results for these limit relations to formulate tests for cointegration with infinite variance errors.

Caner (1997) proved several important results (see Theorems 3, 4, and 5) on the limit behavior of various expressions constructed by means of independent, identically distributed (i.i.d.) random vectors in the domain of attraction of a multivariate stable law with an index $0 < \alpha < 2$. In a later paper (Caner, 1998) important results on cointegration are based on these limit relations. However, there is a lacuna in the proof of Theorem 2 in Caner (1997), and, as a consequence, all the above-mentioned results based on this theorem cannot be considered as strictly proven.

The main difficulty is the convergence of stochastic integrals. This difficulty is described in Paulauskas and Rachev (1998)(PR hereafter). In that paper, to prove the convergence of stochastic integrals, PR used Theorem 2.7 from Kurtz and Protter (1991) (KP hereafter) in which the so-called uniform tightness (UT) condition is the main condition. In Caner (1997), Theorem 2 is proven, yielding the same result as Theorem 2.7 from KP but without the UT condition. Roughly speaking, in Theorem 2 in Caner (1997) it is assumed that semi-martingale P_n is written as a sum of two components $P_n^{(1)}$ and $P_n^{(2)}$, where $P_n^{(1)}$ represents a jump process and $P_n^{(2)}$ is a locally square-integrable martingale. That is, unlike in the paper by KP, in Caner's decomposition there is no component of finite variation, A_n in the notation of KP. This can be seen from the beginning of the proof of Theorem 2 where Caner (1997, 518) states "This is a special case of Theorem 2.7 in Kurtz and Protter (1991), where $A_n = 0$". Then he claims that the weak convergence of $P_n^{(2)}$ implies the UT condition of Theorem 2.7 from KP, namely, that $\sup_n E[P_n^{(2)}]_{t\wedge 2a}$ is finite, where $[X]_t$ denotes the quadratic variation of X and a is an arbitrary number. But this is not true. Intuitively it is clear that boundedness in L_2 does not follow from weak convergence and it is not difficult to construct a counterexample. Here we provide the sketch of the construction of a counterexample.

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space and let $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ be a filtration generated by a Brownian motion. Let us take a sequence of $\mathcal{F}_1$-measurable random variables X_n and X such that the following properties hold:

$$\begin{aligned}
&\text{Property } 1 \ : \quad X_n \xrightarrow{L_1} X,\\
&\text{Property } 2 \ : \quad EX_n = EX = 0,\ EX_n^2 < \infty,\ EX^2 < \infty,\\
&\text{Property } 3 \ : \quad \sup_n EX_n^2 = \infty.
\end{aligned}$$

Let us take

$$P_n^{(2)}(t) = E(X_n|\mathcal{F}_t), \quad P^{(2)}(t) = E(X|\mathcal{F}_t).$$

Due to Property 2, $P_n^{(2)}(t)$ and $P^{(2)}(t)$ will be square-integrable martingales. Now applying Doob inequality and Property 1 we have that for any $\varepsilon > 0$

$$\mathbb{P}\Big\{ \sup_{0\leq t\leq 1} |P_n^{(2)}(t) - P^{(2)}(t)| > \varepsilon \Big\} \leq \varepsilon^{-1} E|P_n^{(2)}(1) - P^{(2)}(1)| = \varepsilon^{-1} E|X_n - X| \to 0,$$

therefore $P_n^{(2)} \xrightarrow{d} P^{(2)}$. Now we use the well-known representation of martingales generated by Brownian filtration, see, for example, Øksendal (2000), Theorem 4.3.4, which says that if $B(t)$ is a Brownian motion, $\{\mathcal{F}_t,\ t \geq 0\}$ is a filtration generated by this Brownian motion and $M(t)$ is an $\mathcal{F}_t$-martingale with $EM(t)^2 < \infty$ for all $t \geq 0$, then there exists a unique stochastic process $g(s),\ s \geq 0$ (satisfying natural measurability condition) such that for all $t \geq 0 \ \ E\int_0^t g(s)^2 ds < \infty$ and

$$M(t) = EM(0) + \int_0^t g(s)dB(s), \quad a.s.$$

Using this representation it is easy to show that

$$E[P_n^{(2)}]_1 = EX_n^2,$$

hence from Property 3 we see that the UT condition cannot be satisfied.

Therefore, the application of Theorem 2.7 from KP is not possible and Theorem 2 in Caner (1997), which is subsequently used in Theorems 3, 4, and 5 and then applied in Caner (1998), is not proven. Moreover, because KP, as well as other authors who have investigated the convergence of stochastic integrals, have shown that the UT condition in some sense is necessary for the convergence of stochastic integrals (see Paulauskas and Rachev (1998) where it is discussed), one is bound to believe that Theorem 2 in Caner (1997) is false. On the other hand, the results of Theorem 3 in Caner (1997) in which innovations are i.i.d. random variables, although based on Theorem 2 (which as we argued above is not proven and most probably false), is true. The reason is that PR verified the UT condition even in a more general situation. In Caner (1997) there are strong assumptions of symmetricity, independence of coordinates of innovations, and the same index of stability for all coordinates. All conditions of Theorem 2.7 in KP are verified without these assumptions in PR; therefore, coordinates of a limit stable vector are allowed to have different exponents and the normalization by a diagonal matrix is used. This gives flexibility in the application of this important result.

The principal purpose of this comment in addition to pointing out the gap in the proof of Theorem 2 in Caner (1997) is to warn researchers working in econometrics about the difficulties associated with the convergence problems for stochastic integrals and possible misuse of the continuous mapping theorem. In the case of finite variance and continuous limiting Brownian motion there was no such problems: stochastic integrals were continuous functionals and convergence was obtained by the standard continuous mapping theorem. The situation changed when econometricians began dealing with heavy-tailed distributions and Lévy processes as limits. In probability theory around the 1970s, it was observed that stochastic integrals with integrator and integrand having jumps can be discontinuous functionals and convergence of such stochastic integrals requires special attention. Powerful tools to cope with this problem were introduced in the works of T. Kurtz, P. Protter, A.

Jakubowski, J. Mémin and other researchers working in the area of stochastic analysis and stochastic differential equations.

It seems that the first correct application of these tools in econometrics was in PR, which, unfortunately, was published in an applied probability journal. We looked at recent papers in econometrics dealing with similar problems and found several recent works where strict proofs are available. For example, Zarepour and Roknossadati (2008) using results from PR and Mittnik, Paulauskas, and Rachev (2001) improved Theorem 3 in Caner (1997) in the above-mentioned direction: as in PR they allow dependence between and different exponents for coordinates, and use matrix normalization. Also a recent doctoral dissertation by Ferstl (2009) should be mentioned where substantially the results and proofs from PR are used.

We conclude with one more difficulty encountered with heavy-tailed distributions when considering the convergence of sums $a_n^{-1}\sum_{j=1}^{[nt]} X_j$ formed by linear processes $X_j = \sum_{i=0}^{\infty} \varepsilon_{i-j}$. It was demonstrated in Avram and Taqqu (1992) that in the case of linear processes generated by heavy-tailed i.i.d. innovations the above written sum generally does not converge in the usual Skorohod topology J_1 and therefore another topology M_1 should be used. (Here it is worth mentioning that A.V. Skorohod in his famous 1956 paper on the space of cadlag functions introduced four different topologies J_1, J_2, M_1, M_2, adapted for convergence of various types of stochastic processes, but in the literature the main topology used is J_1.) Taking into account that in the KP paper the traditional Skorohod topology J_1 is used, one must deal with the case of linear processes very carefully.

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