RISK MANAGEMENT AND DYNAMIC PORTFOLIO SELECTION WITH STABLE PARETIAN DISTRIBUTIONS

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ABSTRACT: This paper assesses stable Paretian models in portfolio theory and risk management. We describe investor's optimal choices under the assumption of non-Gaussian distributed equity returns in the domain of attraction of a stable law. In particular, we examine dynamic portfolio strategies with and without transaction costs in order to compare the forecasting power of discrete-time optimal allocations obtained under different stable Paretian distributional assumptions. Finally, we consider a conditional extension of the stable Paretian approach to compute the value at risk and the conditional value at risk of heavy-tailed return series.

KEY WORDS: Stable Paretian distributions, multi-period portfolio choice, value at risk, conditional value at risk, dynamic portfolio strategies.

JEL CLASSIFICATION: G11, G14, C61

1. Introduction

In this paper we propose some stable Paretian models for optimal portfolio selection and quantify the risk of a given portfolio. After examining the multi-period optimal portfolio problems under different distributional assumptions, we propose an *ex-ante* and an *ex-post* empirical comparison between the stable Paretian approach and a moment-based one. We then discuss how to use the stable Paretian model to compute the value at risk (VaR) and the conditional value at risk (CVaR) of a given portfolio.

It is well-known that asset returns are not uniquely determined by their mean and variance. Numerous empirical studies, beginning with the works of Mandelbrot (1963a, 1963b, 1967) and Fama (1963, 1965a, 1965b), have refuted the commonly accepted view that financial returns are normally distributed.[1] In this paper, we examine the implications of different distributional hypotheses for dynamic portfolio strategies of investors. In particular, we compare the performance of dynamic strategies based on a stable Paretian model and on a moment-based model.

The literature on multi-period portfolio selection has focused on maximizing expected utility functions of terminal wealth and/or multi-period consumption. In contrast to the focus of classical multi-period approaches, we generalize the mean-variance analysis suggested by Li and Ng (2000), giving a three-parameter formulation of optimal dynamic portfolio selection. These alternative multi-period approaches are consistent with the

[1] See Rachev and Mittnik (2000) and the reference therein.

admissible optimal portfolio choices of risk-averse investors. In particular, we develop analytical optimal portfolio policies for the multi-period mean-dispersion-skewness formulation. In order to compare a moment-based three-parameter portfolio model and the stable Paretian dynamic model, we analyze several investment allocation problems.

The primary contribution of the empirical comparison is the analysis of the impact of the distributional assumptions on multi-period asset allocation decisions. Thus, we propose two alternative performance comparisons between multi-period portfolio policies obtained under different distributional assumptions. For this purpose, we analyze some allocation problems for non-satiable risk-averse investors with different risk-aversion coefficients. We determine the *ex-ante* and *ex-post* multi-period efficient frontiers given by the minimization of the dispersion measures. Each investor, characterized by his/her utility function, will prefer the model which maximizes his/her expected utility on the efficient frontier. The portfolio policies obtained with this methodology represent the optimal choices for the different approaches for an investor. Therefore, we examine the differences in optimal strategies for an investor under the stable and the moment-based distributional hypothesis.

In addition, we propose an *ex-ante* and an *ex-post* comparison between the parametric-portfolio selection models proposed assuming that no short sales and transaction costs are allowed. Thus we assess these models considering that every week each investor recalibrates his/her portfolio in order to maximize his/her expected utility on a three-parametric efficient frontier. Finally, we present a conditional asymmetric stable-fund separation model to compute the VaR and the CVaR of a given portfolio.

2. Three parameters portfolio selection models without short sale constraints

In this section, we analyze a discrete-time extension of the Li and Ng (2000) problem. In particular,we consider the optimal allocation among $n+1$ assets: n of those assets are risky assets with stable distributed risky returns $z_{t_j} = [z_{1,t_j}, ..., z_{n,t_j}]'$on the time period $[t_j, t_{j+1})$ and the $(n+1)th$ asset is risk-free with returns $z_{0,t}$ for $t = t_0, t_1, ..., t_{T-1}$.

Let W_{t_j} be the wealth of the investor at the beginning of the period $[t_j, t_{j+1})$, and let x_{i,t_j} $i = 1, ..., n$; $t_j = t_0, t_1, ..., t_{T-1}$ (with $t_0 = 0$ and $t_i < t_{i+1}$) be the amount invested in the *i-th* risky asset at the beginning of the period $[t_j, t_{j+1})$. x_{0,t_j}; $t_j = 0, t_1, ..., t_{T-1}$ is the amount invested in the risk-free asset at the beginning of the period $[t_j, t_{j+1})$.

Li and Ng (2000) have proposed an analytical solution to the dynamic mean-variance portfolio selection problem when the vectors of risky returns z_t are statistically independent. In their analysis they assume that the amounts invested in the assets at the beginning of each period $[t_j, t_{j+1})$ $t_j = t_1, ..., t_{T-1}$ could be random variables. In contrast to the Li-Ng proposal, we assume that the multi-period portfolio policies in the risky assets $x_{t_j} = [x_{1,t_j}, ..., x_{n,t_j}]'$ for any j, are deterministic variables of the problem and the wealth invested in the risk-free return at time t_j is given by

$W_{t_j} - x'_{t_j} e$ where $e = [1, ..., 1]'$ (and, clearly, it is a random variable). In the following analysis, we assume that the wealth process is uniquely determined by three parameters as in the model proposed by Ortobelli et al (2004): mean, dispersion, and skewness. In particular, we assume:

a) the initial wealth $W_0 = \sum_{i=0}^{n} x_{i,0}$ is known and the vectors of returns $z_t = [z_{1,t}, ..., z_{n,t}]'$ are i.i.d. [2] of any time $t = t_0, t_1, ..., t_{T-1}$;

b) the returns z_t follow the model

$$z_{i,t} = \mu_{i,t} + b_{i,t} Y_t + \varepsilon_{i,t} \tag{1}$$

where $Y_t \sim S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$ is an α_2-stable Paretian distributed asymmetric equity return ($\beta_Y \neq 0$, $\alpha_2 > 1$) independent of α_1-stable sub-Gaussian distributed vectors of residuals $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$; ($\alpha_1 > 1$), which are statistically independent of any $t = t_0, t_1, ..., t_{T-1}$. Observe

[2] We implicitly assume that the length of the periods of analysis, $t_{j+1} - t_j$, is constant varying j, for this reason we assume that the vectors of returns z_t are also identically distributed. When we consider daily or weekly returns, we can adopt either continuously compounded returns $z_{i,t} = \ln\left(\frac{S_{i,t+1}}{S_{i,t}}\right)$ or the returns $z_{i,t} = \frac{S_{it+1} - S_{it}}{S_{it}}$ (where $S_{i,t}$ is the price of the *i-th* asset at time t). As a matter of fact, daily or weekly continuously compounded returns approximate well enough the returns $\frac{S_{it+1} - S_{it}}{S_{it}}$ and we generally do not observe material differences in the portfolio strategies obtained with the two alternative definitions (see, among others, Biglova et al. (2004)). In addition, the empirical evidence shows that daily or weekly returns are very often in the domain of attraction of stable laws (see Rachev and Mittnik (2000) and the reference therein).

that the assumption that the vector of residual ε_t is elliptically distributed as an α_1-stable sub-Gaussian implies that the vector of returns $z_t = \mu_t + b_t Y_t + \varepsilon_t$ describes a three-fund separation model (see Ross (1978) and Simaan (1993)).

Under these assumptions, the vector of returns z_t admits the following characteristic function $\Phi_{z_t}(u) = E(\exp(iu'z))$:

$$\exp\left(-\left(u'Qu\right)^{\alpha_1/2} - \left|u'b_t\sigma_Y\right|^{\alpha_2}\left(1 - i\left(u'b_t\sigma_Y\right)^{\langle\alpha_2\rangle}\beta_Y \tan\frac{\pi\alpha_2}{2}\right) + iu'\mu_t\right)$$

where $x^{\langle\alpha\rangle} = sgn(x)\left|x\right|^{\alpha}$, $b_t = [b_{1,t}, ..., b_{n,t}]'$, $\mu_t = E(z_t)$, Q is the definite positive dispersion matrix associated at the vector of residuals $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$ at time t, and σ_Y, β_Y are respectively the scale and the skewness parameter of the centered equity return Y (independent of ε_t). Considering that the wealth at each time is given by

$$W_{t_{k+1}} = \sum_{i=0}^{n} x_{i,t_k}(1 + z_{i,t_k}) =$$
$$= (1 + z_{0,t_k})W_{t_k} + x'_{t_k}p_{t_k} \quad k = 0, 1, 2, ..., T-1$$

where $p_{t_i} = [p_{1,t_i}, ..., p_{n,t_i}]'$ is the vector of excess of returns $p_{k,t_i} = z_{k,t_i} - z_{0,t_i}$, then we can write the final wealth as follows:

$$W_{t_T} = W_0 \prod_{k=0}^{T-1}(1 + z_{0,t_k}) + \sum_{i=0}^{T-2} x'_{t_i}p_{t_i} \prod_{k=i+1}^{T-1}(1 + z_{0,t_k}) + x'_{t_{T-1}}p_{t_{T-1}} \qquad (2)$$

for any fixed initial wealth W_0. Since the multi-period portfolio policies in the risky assets x_{t_j} are deterministic variables, then the mean of the final wealth W_{t_T} is given by

$$E(W_{t_T}) = W_0 \prod_{k=0}^{T-1}(1 + z_{0,t_k}) +$$
$$+ \sum_{i=0}^{T-2} x'_{t_i}E(p_{t_i}) \prod_{k=i+1}^{T-1}(1 + z_{0,t_k}) + x'_{t_{T-1}}E(p_{t_{T-1}}).$$

Moreover, considering that the final wealth is determined by the relationship given by (2) and the vectors of returns follow the stable law given by (1), then the final wealth W_{t_T} maintains the same distributional structure of the returns:

$$W_{t_T} = W_0 \prod_{k=0}^{T-1}(1+z_{0,t_k}) + \sum_{i=0}^{T-2} x'_{t_i} E(p_{t_i}) \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) +$$
$$+ x'_{t_{T-1}} E(p_{t_{T-1}}) + Y \left(\sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}} \right) +$$
$$+ \sum_{i=0}^{T-2} x'_{t_i} \varepsilon_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} \varepsilon_{t_{T-1}} = E(W_{t_T}) + A_x Y + \Psi_x$$

where $A_x = \sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}}$ is a deterministic variable, while $\Psi_x = \sum_{i=0}^{T-2} x'_{t_i} \varepsilon_{t_i} \prod_{k=i+1}^{T-1}(1+z_{0,t_k}) + x'_{t_{T-1}} \varepsilon_{t_{T-1}}$ is the sum of α_1-stable independent random variables. Therefore, the final wealth W_{t_T} is a linear combination of two independent stable laws Y (α_2-stable distributed) and Ψ_x that is α_1-stable sub-Gaussian distributed with null mean and dispersion $\sigma_{(x'_{t_i}\varepsilon_{t_i})}$ defined by

$$\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} = \sum_{i=0}^{T-2} \left(x'_{t_i} Q x_{t_i}\right)^{\alpha_1/2} \left(\prod_{k=i+1}^{T-1} (1+z_{0,t_k}) \right)^{\alpha_1} + \left(x'_{t_{T-1}} Q x_{t_{T-1}} \right)^{\alpha_1/2}.$$

Recall that all risk-averse investors (i.e., investors with concave utility functions) prefer the return X to the return Z if and only if X dominates Z in the sense of Rothschild-Stiglitz (see Rothschild and Stiglitz (1970)) or equivalently if and only if $E(X)=E(Z)$ and

$$\int_{-\infty}^{v} \Pr(X \le s)\, ds \le \int_{-\infty}^{v} \Pr(Z \le s)\, ds$$

for every real v. Let W_x and W_y be two admissible final wealths determined respectively by the portfolio policies x_{t_j} and y_{t_j}. Suppose that under the assumptions of model (1), W_x and W_y have the same mean $E(W_x) = E(W_y)$

and the same parameter $A_x = A_y$. Then we have the following equality in distribution (conditioned at $Y = u$) for any real u any

$$X_{/Y=u} = \frac{W_x - E(W_x) - A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} \frac{\Psi_x}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} \frac{\Psi_y}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \stackrel{d}{=} S_{\alpha_1}(1,0,0).$$

Let's suppose that $\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)} > \sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}$. Then, W_y dominates W_x in the sense of Rothschild-Stiglitz because for every real v :

$$\begin{gathered}
\int_{-\infty}^{v} \left(\Pr\left(W_y \le s\right) - \Pr\left(W_x \le s\right)\right) ds = \\
= \int_{-\infty}^{v} \int_R \left(\Pr\left(X \le \tfrac{s-E(W_y)-A_y u}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) - \right. \\
\left. - \Pr\left(X \le \tfrac{s-E(W_y)-A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) \right) f_Y(u) du \; ds = \\
= \int_R \int_{-\infty}^{v} \left(\Pr\left(X \le \tfrac{s-E(W_y)-A_y u}{\sigma_{\left(y'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) - \right. \\
\left. - \Pr\left(X \le \tfrac{s-E(W_y)-A_x u}{\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}} \middle| Y = u \right) \right) ds f_Y(u) du \le 0
\end{gathered}$$

where f_Y is the density of Y. Therefore, the non-dominated portfolio policies are obtained by minimizing the residual dispersion $\sigma_{\left(x'_{t_i}\varepsilon_{t_i}\right)}$ for some fixed mean $E(W_x)$ and parameter B_x. Thus, when unlimited short sales are allowed, any risk-averse investor will choose one of the multi-portfolio policy solutions of the following optimization problem for some m, v, and W_0:

$$\begin{gathered}
\min_{\left\{x_{t_j}\right\}_{j=0,1,\dots,T-1}} \tfrac{1}{2}\sigma^{\alpha_1}_{\left(x'_{t_i}\varepsilon_{t_i}\right)} \\
\text{s. t. } E(W_{t_T}) = m; \\
\sum_{i=0}^{T-2} x'_{t_i} b_{t_i} \prod_{k=i+1}^{T-1} (1 + z_{0,t_k}) + x'_{t_{T-1}} b_{t_{T-1}} = v
\end{gathered} \quad . \tag{3}$$

Imposing the first-order conditions on the Lagrangian

$$L(x_{t_j}, \lambda_1, \lambda_2) = \frac{1}{2}\sigma^{\alpha_1}_{\left(x'_{t_i}\varepsilon_{t_i}\right)} - \lambda_1(E(W_{t_T}) - m) - \lambda_2(A_x - v)$$

all the multi-portfolio policy solutions of problem (3) are given by:

$$\begin{aligned} x_{t_j} &= \left(\tfrac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} \frac{\left(\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)' Q^{-1}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}}}{B_{j+1}} \times \\ &\times Q^{-1}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right) \\ &\forall j=0,1,...,T-2 \\ x_{t_{T-1}} &= \left(\left(\lambda_1 E(p_{t_{T-1}})+\lambda_2 b_{t_{T-1}}\right)' Q^{-1}\left(\lambda_1 E(p_{t_{T-1}})+\lambda_2 b_{t_{T-1}}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}} \times \\ &\times \left(\tfrac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} Q^{-1}\left(\lambda_1 E(p_{t_{T-1}})+\lambda_2 b_{t_{T-1}}\right), \end{aligned} \tag{4}$$

where $B_i = \prod_{k=i}^{T-1}(1+z_{0,t_k})$ and λ_1, λ_2 are uniquely determined by the following relations

$$\begin{array}{c} \sum_{i=0}^{T-2} x'_{t_i} b_{t_i} B_{i+1} + x'_{t_{T-1}} b_{t_{T-1}} = v \\ \sum_{i=0}^{T-2} x'_{t_i} E(p_{t_i}) B_{i+1} + x'_{t_{T-1}} E(p_{t_{T-1}}) = m - W_0 B_0 \end{array}.$$

Moreover, we can represent the dispersion of final wealth residual Ψ_x as a function of the Lagrangian coefficients λ_1, λ_2, i.e.,

$$\sigma^{\alpha_1}_{(x'_{t_i}\varepsilon_{t_i})} = \sum_{j=0}^{T-1}\left(\left(\frac{2}{\alpha_1}\right)^2 \left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)' Q^{-1}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)\right)^{\frac{\alpha_1}{2(\alpha_1-1)}}.$$

Besides, the wealth invested in the risk-free asset at the beginning of the period $[t_k, t_{k+1})$ is the deterministic wealth $W_0 - x'_0 e$ in t_0, while, for any $k \geq 1$, it is given by the random variable $W_{t_k} - x'_{t_k} e$, where $W_{t_1} = (1+z_{0,0})W_0 + x'_0 p_0$ and for any $j \geq 2$

$$W_{t_j} = W_0 \prod_{k=0}^{j-1}(1+z_{0,t_k}) + \sum_{i=0}^{j-2} x'_{t_i} p_{t_i} \prod_{k=i+1}^{j-1}(1+z_{0,t_k}) + x'_{t_{j-1}} p_{t_{j-1}}$$

In particular, when the vector $\varepsilon_t = [\varepsilon_{1,t}, ..., \varepsilon_{n,t}]'$ is Gaussian distributed (i.e., $\alpha_1 = 2$), we obtain the following analytical solution to the optimization problem (3)

$$x_{t_j} = \frac{(m-W_0B_0)A-vD}{B_{j+1}(AC-D^2)} Q^{-1}E(p_{t_j}) + \frac{vC-(m-W_0B_0)D}{B_{j+1}(AC-D^2)} Q^{-1}b_{t_j}$$

$$\forall j = 0, 1, ..., T-2 \tag{5}$$

$$x_{t_{T-1}} = \frac{(m-W_0B_0)A-vD}{AC-D^2} Q^{-1}E(p_{t_{T-1}}) + \frac{vC-(m-W_0B_0)D}{AC-D^2} Q^{-1}b_{t_{T-1}} ,$$

where

$$A = \sum_{i=0}^{T-1} b'_{t_i} Q^{-1} b_{t_i},$$

$$B_i = \prod_{k=i}^{T-1} (1 + z_{0,t_k}),$$

$$C = \sum_{i=0}^{T-1} E(p_{t_i})' Q^{-1} E(p_{t_i})$$

$$\text{and } D = \sum_{i=0}^{T-1} E(p_{t_i})' Q^{-1} b_{t_i}.$$

We obtain the portfolio policies given by (5) even when the vector of residuals ε_t is elliptical distributed with finite variance and the index Y is an asymmetric random variable with finite third moment. Under this assumption, the variance of final wealth residual Ψ_x is a function of m and v. That is:

$$\sigma^2_{\left(x'_{t_i}\varepsilon_{t_i}\right)} = \frac{A\left(m - W_0B_0\right)^2 + v^2C - 2v\left(m - W_0B_0\right)D}{AC - D^2}.$$

We call this approach that assumes residuals with finite variance the *moment-based approach*, in order to distinguish it from the stable Paretian one with $\alpha_1 < 2$. In both cases (stable non-Gaussian and moment-based approaches), the three-fund separation property holds because the multi-portfolio policies in the risky assets x_{t_j} are spanned by vectors $Q^{-1}E(p_{t_j})$, $Q^{-1}b_{t_j}$ for any time t_j. Moreover, simple empirical applications of these formulas show that the implicit term structure $z_{0,t}$ for $t = t_0, t_1, ..., t_{T-1}$ could determine major differences in the portfolio weights of the same strategy and different periods. As a matter of fact, when the interest rates implicit in the term structure are growing (decreasing), investors are more (less) attracted to invest in the risk-free asset in future periods.

As discussed by Simaan (1993) and Ortobelli et al (2005), when we consider a three-fund separation model, the solution of any allocation problem depends on the choice of the asymmetric random variable Y. Clearly, one should expect that the optimal allocation will differ when one assumes that asset returns are in the domain of attraction of a stable law or that they depend on a three-moment model. In order to examine the impact of these different distributional assumptions, in the next section we compare the performance of the two models.

3. A comparison among parametric dynamic strategies

In this section, we evaluate and compare the performances between the fund separation portfolio models previously presented. In particular, we propose an *ex-ante* and an *ex-post* comparison between the stable non-Gaussian and the moment-based approaches. In this comparison, we assume dynamic portfolio choice strategies either when short sales are allowed or when transaction cost constraints and no short sales are allowed.

For both comparisons, we assume that investors recalibrate their portfolio weekly. Thus, we analyze optimal dynamic strategies during a period of 25 weeks among a risk-free asset proxied by the 30-day Eurodollar CD (and offering a rate of one-month Libor), and 25 developed country stock market indices. The stock indices are those that are or have been part of the MSCI World Index in the last 20 years.[3] The historical returns for all of the stock indices covered the period January 1993 to May 2004. We split the historical return data series into two parts. The first part (January 1993 - December 2003) is used to estimate the model parameters; the second part (December 2003-May 2004) is used to verify *ex-post* the impact of the forecasted allocation choices.

We consider as the benchmark index Y the centered MSCI World Index and we assume initial capital W_0 equal to 1. Hence, we use weekly returns (where each week consists of five trading days) taken from 25 risky returns included in the MSCI World Index. Therefore, using the notation of the previous section, we assume as risk-free weekly returns z_{0,t_i} $t_0 = 12/08/2003, ..., t_{24} = 5/24/2004$ the observed one-month Libor (see Table 1).

3.1 *Comparison between three-fund separation models without portfolio constraints*

In our comparison, we assume that unlimited short sales are allowed, and we approximate optimal solutions to different expected utility functions. In particular, we assume that each investor maximizes one from among the following five expected utility functions:

[3] They are: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hong Kong, Ireland, Italy, Japan, Malaysia, Netherlands, New Zealand, Norway, Portugal, Singapore, South African Gold Mines, Spain, Sweden, Switzerland, on the United Kingdom, and the United States.

1) $\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} E\left(\log(W_T)\right)$

2) $\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} -E\left(\exp(-\gamma W_T)\right)$ with $\gamma = 1, 5, 7, 17;$

3) $\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} E\left(\frac{W_T^c}{c}\right)$ with $c = -1.5, -2.5;$

4) $\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} E(W_T) - cE\left(\left|W_T - E(W_T)\right|^{1.3}\right)$ with $c = 1,\ 2.5;$

5) $\max_{\{x_{t_j}\}_{j=0,1,\ldots,T-1}} E(W_T) - cE\left(\left|W_T - E(W_T)\right|^{2}\right)$ with $c = 1,\ 2.5.$

Observe that when the returns are in the domain of attraction of a stable law, with $1 < \alpha_1, \alpha_2 < 2$, the above expected utility functions could be infinite. However, assuming that the returns are truncated far enough, those formulas are formally justified by pre-limit theorems (see Klebanov et al. (2000) and Klebanov et al. (2001)), which provide the theoretical basis for modeling heavy-tailed bounded random variables with stable distributions. On the other hand, the investor can always approximate his/her expected utility, since he/she works with a finite amount of data. We assume the vectors of returns $z_{t_j} = [z_{1,t_j}, ..., z_{25,t_j}]'$ are statistically independent and follow the model given by (1).

In the model we need to estimate several parameters: the index of stability α_1, the mean μ, the dispersion matrix Q, and the vector $b_t = [b_{1,t}, ..., b_{25,t}]'$. In order to simplify our empirical comparison, we assume the index of stability α_1, the vector mean $\mu = E(z_t)$, and the vector b_t are constant over the time t. We estimate α_1 to be equal to the mean of 10,000 indexes of stability computed with the maximum likelihood estimator (MLE) of random portfolios of the residuals $\widetilde{\varepsilon} = \widetilde{z} - \widehat{b}Y$, i.e., $\alpha_1 = \frac{1}{10000}\sum_{k=1}^{10000} \alpha_{(k)}$ =1.8007 where $\alpha_{(k)}$ is the index of stability of a random portfolio $(x^{(k)})'\widetilde{\varepsilon}$. The estimator of μ is given by the vector $\widehat{\mu}$ of the sample average. Then, we consider as factor Y the centralized MSCI World Index return. Regressing the centered returns $\widetilde{z}_i = z_i - \widehat{\mu}_i$ $(i = 1, ..., 25)$ on Y, we write the following estimators[4] for $b = [b_1, ..., b_{25}]'$ and Q:

$$\widehat{b}_i = \frac{\sum_{k=1}^{N} Y^{(k)} \widetilde{z}_i^{(k)}}{\sum_{k=1}^{N} \left(Y^{(k)}\right)^2}; \quad i = 1,...,25, \tag{6}$$

$$\text{and} \;\; \widehat{Q} = [\widehat{q}_{i,j}]$$

where

$$\widehat{q}_{j,j} = \left(A(p) \frac{1}{N} \sum_{k=1}^{N} \left| \widetilde{\varepsilon}_j{}^{(k)} \right|^p \right)^{\frac{2}{p}},$$

$$\widehat{q}_{i,j} = \frac{1}{2} \left(\left(A(p) \frac{1}{N} \sum_{k=1}^{N} \left| \widetilde{\varepsilon}_i{}^{(k)} + \widetilde{\varepsilon}_j{}^{(k)} \right|^p \right)^{\frac{2}{p}} - \widehat{q}_{j,j} - \widehat{q}_{i,i} \right)$$

$p \in (0, \alpha_1)$, $A(p) = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2^p \Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{p+1}{2}\right)}$, and $\widehat{\varepsilon}^{(k)} = \widetilde{z}^{(k)} - \widehat{b}Y^{(k)}$ is the sample residual vectors. The entries of the dispersion matrix derive from the moment method suggested by Property 1.2.17 in Samorodnitsky and Taqqu (1994) (see also Ortobelli et al (2004)). In addition, arguing along the same lines as Rachev (1991), Götzenberger et al (2001), and Tokat et al (2003), we can explain and prove the asymptotic properties of this estimator. We assume that parameter p is equal to the mean of optimal $\widehat{p}_i$ that minimizes the average of the distance between the moment-dispersion estimator of residuals $\widetilde{z}_{i,t} - b_{i,t}Y_t$ and its maximum likelihood stable estimate (see Table 1).

[4] See Kim, Rachev, Samorodnitsky and Stoyanov (2005) for a discussion of the best estimators of vector b when a heavy-tailed series is assumed.

Theoretically, the optimal p must be near zero for stable distributions (see Rachev (1991)). However, if we approximate $\widetilde{\varepsilon}_i$ with a stable distribution, the optimal $p \in (0, \alpha)$ depends on the historical series of observations $\left\{\widetilde{\varepsilon}_i^{(k)}\right\}_{k=1}^{N}$. According to the analysis proposed by Lamantia et al. (2006), we consider the optimal $\widehat{p_j}$ that minimizes the average of distance between $\widehat{q}_{j,j}(p) = \left(A(p)\frac{1}{N}\sum_{k=1}^{N}\left|\widetilde{\varepsilon_j}^{(k)}\right|^p\right)^{1/p}$ (which we call *moment-dispersion estimator*) and the MLE $\overline{v}_{j,j}$ of dispersion. That is,

$$\widehat{p_j} = \arg\left(\min_p \frac{1}{T}\sum_{t=1}^{T}\left|\widehat{q}_{jj,t/t-1}(p) - \overline{v}_{j,j}\right|\right), \quad j = 1, ..., 25.$$

In Table 1 we report the MLE stable parameters of the historical return series, the estimated vector $\widehat{b}$, and the optimal $\widehat{p_j}$ of weekly return series between January 1993 to December 2003. Here, we adopt the common parameter $\widehat{p} = \frac{1}{25}\sum_{j=1}^{25}\widehat{p_j} \simeq 0.60812$.

In order to compare the different models, we use (in a multi-period context) the same algorithm proposed by Giacometti and Ortobelli (2004) and Ortobelli et al (2005). Thus, first we consider the optimal strategies for different levels of the mean and skewness. Second, we select the portfolio strategies on the efficient frontiers that maximize some parametric expected utility functions for different risk-aversion coefficients. Then, we compare the performance of the stable Paretian and of moment-based approaches for each optimal allocation proposed.

Therefore, considering N i.i.d. observations $z^{(i)}$ $(i = 1, \ldots, N)$ of the vector $z_t = [z_{1,t}, z_{2,t}, ..., z_{25,t}]'$, the main steps in our comparison are the following:

Step 1 Consider the optimal portfolio strategies

$$x_j(\lambda_1,\lambda_2) = \left(\frac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} \frac{\left(\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)' Q^{-1}\left(\lambda_1 E(p_{t_j})+\lambda_2 b_{t_j}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}}}{B_{j+1}} \times$$
$$\times Q^{-1}\left(\lambda_1 E(p_{t_j}) + \lambda_2 b_{t_j}\right) \qquad \forall j = 0,1,...,23$$
$$x_{24}(\lambda_1,\lambda_2) = \left((\lambda_1 E(p_{t_{24}}) + \lambda_2 b_{t_{24}})' Q^{-1}\left(\lambda_1 E(p_{t_{24}}) + \lambda_2 b_{t_{24}}\right)\right)^{\frac{2-\alpha_1}{(\alpha_1-1)2}} \times$$
$$\times \left(\frac{2}{\alpha_1}\right)^{\frac{1}{(\alpha_1-1)}} Q^{-1}\left(\lambda_1 E(p_{t_{24}}) + \lambda_2 b_{t_{24}}\right),$$

that generate the efficient frontier.

Step 2 Choose a utility function u with a given coefficient of aversion to risk.

Step 3 Compute for every multi-period efficient frontier

$$\max_{\lambda_1,\lambda_2} \frac{1}{N}\sum_{i=1}^{N} u\left(W_{25}^{(i)}\right).$$

where $W_{25}^{(i)} = \prod_{k=0}^{24}(1+z_{0,k}) + \sum_{j=0}^{23} x_j'(\lambda_1,\lambda_2)p_j^{(i)} \prod_{k=j+1}^{24}(1+z_{0,k}) + x_{24}'(\lambda_1,\lambda_2)p_{24}^{(i)}$ is the *i-th* observation of the final wealth and $p_t^{(i)} = [p_{1,t}^{(i)},...,p_{n,t}^{(i)}]'$ is the *i-th* observation of the vector of excess returns $p_{k,t}^{(i)} = z_{k,t}^{(i)} - z_{0,t}$ relative to the *t-th* period. In particular, we implicitly assume the approximation:

$$\frac{1}{N}\sum_{i=1}^{N} u\left(W_{25}^{(i)}\right) \approx E\left(u\left(W_{25}^{(i)}\right)\right).$$

and that $\{x_j(\lambda_1,\lambda_2)\}_{j=0,1,...,24}$ are the optimal portfolio strategies given by (4).

Step 4 Repeat steps 2 and 3 for every utility function and for every risk-aversion coefficient.

Using these steps, we obtain the results reported in Table 2 with the approximated maximum expected utility and the *ex-post* final wealth. In order to emphasize the differences in the optimal portfolio composition, we employ the following notation:

a) $x_{t_j}^{stable}$ $t_j = t_0, t_1, ..., t_{24}$ the optimal portfolio policies that realize the maximum expected utility assuming the stable Paretian model;

b) $x_{t_j}^{moment}$ $t_j = t_0, t_1, ..., t_{24}$ the optimal portfolio policies that realize the maximum expected utility assuming the moment-based approach.

Then we consider the half average of the absolute difference between the portfolio compositions at each time t_j, i.e.:

$$\frac{1}{50}\sum_{j=0}^{24}\sum_{k=1}^{25}\left|x_{k,t_j}^{stable} - x_{k,t_j}^{moment}\right|. \quad (7)$$

This measure points out how much the portfolio composition for each recalibration changes in terms of the mean.

Table 2 summarizes the comparison between the fund-separation approaches discussed above. In particular, it shows that the *ex-ante* optimal solutions that maximize the expected utility functions are always on the mean-dispersion-skewness frontier of the stable Paretian model and investors increase their performance when they use the stable Paretian model. Only in two cases do we observe that the *ex-post* final wealth of the moment-based model is higher than the stable Paretian one. Moreover, we observe substantial differences in the optimal portfolio composition. Considering that the two models, moment-based and stable Paretian, are based on a different risk perception of the residuals, this empirical comparison suggests that the residuals have a strong impact on the portfolio selection decisions made by investors.

3.2 *Comparison between three-fund separation models with portfolio constraints*

Now we will compare dynamic strategies with constant and proportional transaction costs of 0.2%[5] when short sales are not permitted. In particular, we compare:

a) the *ex-post* final wealth sample paths of investors who maximize one of the five utility functions listed in Section 3.1;

b) the *ex-ante* maximum expected utility obtained at each time t_j for the following three optimization problems:

1) $\max -E(\exp(-X))$;
2) $\max E(X) - E\left(|X - E(X)|^{1.3}\right)$;
3) $\max E(X) - 2.5E\left(|X - E(X)|^{2}\right)$.

We assume that the returns follow the three-fund separation model given by (1) and that each investor recalibrates his/her portfolio weekly starting from 12/08/2003 till 5/24/2004. In order to describe the different portfolio strategies considering transaction costs and short sale constraints,we have to determine the optimal choices of the investors at each time t_j. Thus, at each time t_j, we have to solve two different optimization problems: the first to fit the efficient frontier with transaction cost constraints and the second to determine the optimal expected utility on the efficient frontier. In particular, considering N observations $z^{(i)}$ $(i = 1, \ldots, N$) of the vector

[5] The transaction costs generally change for different countries. Here we fix some indicative transaction costs often used by institutional investors in Italy.

$z_t = [z_{1,t}, z_{2,t}, ..., z_{25,t}]'$, the main steps of our comparison are summarized in the following algorithm:

Step 1 We choose a utility function u with a given coefficient of aversion to risk.

Step 2 At time t_0=12/08/2003, we fit the three-parameter efficient frontiers corresponding to the different distributional hypothesis: moment-based and stable Paretian approaches. Therefore, we fit 5,000 optimal portfolio weights x_{t_0} varying the weekly mean $m_W \geq z_{0,0}$ =0.0002924 and the index of skewness b^* in the following quadratic programming problem:

$$\begin{array}{c} \min_{x_{t_0}} x'_{t_0} Q x_{t_0} \quad \text{subject} \quad \text{to} \\ x'_{t_0}\mu + (1 - x'_{t_0} e) z_{0,0} = m_{W_{t_0}} \\ x'_{t_0} b_t = b^*, \quad 0 \leq x'_{t_0} e \leq 1 \\ \text{and } x_{i,t_0} \geq 0, \quad i = 1, ..., n \end{array}, \tag{8}$$

where $e = [1, ..., 1]'$ and $W_{t_0} = x' z_t + (1 - x'_{t_0} e) z_{0,0}$.

We assume that over time t the vector mean $\mu = E(z_t)$ and the dispersion matrix Q of the residuals are constant. Then, for each efficient frontier, we have to determine the portfolio weights x_{t_0} that maximize the expected utility given by the solution to the following optimization problem

$$\max_{x_{t_0}} \frac{1}{N} \sum_{i=t_0-N}^{t_0} u\left(x'_{t_0} z^{(i)} + (1 - x'_{t_0} e) z_{0,0}\right)$$
$$\text{subject} \quad \text{to}$$
$$x_{t_0} \text{ are optimal portfolio of the efficient frontier.}$$

Thus given

$$x^*_{t_0} = \arg(\max_{x_{t_0}\text{belongs to the efficient frontier}} (E(u(x'_{t_0} z_t + (1 - x'_{t_0} e) z_{0,0}))))$$

the *ex-post* final wealth at time 5/31/2004 is obtained by $W_1 = W_0(1 + \left(x^*_{t_0}\right)' z^{(t_1)} + (1 - e' x^*_{t_0}) z_{0,1} - 0.002)$ where 0.002 is the fixed proportional transaction costs for unity of wealth invested.

In order to determine the optimal portfolio strategies in the other periods, we have to take into account that the investor pays proportional transaction costs of 0.2% on the absolute difference of the changes of portfolio compositions. Thus, at time t_k (after k weeks) we fit 5,000 optimal portfolio weights x_{t_k} varying the weekly mean $m \geq z_{0,t_k}$ and the index of skewness b^* in the following optimization problem:

$$\begin{array}{c} \min_{x_{t_k}} x'_{t_k} Q x_{t_k} \quad \text{subject} \quad \text{to} \\ m = E(X(x_{t_k})) \\ x'_{t_k} b_t = b^*, \quad 0 \leq x'_{t_k} e \leq 1 \\ \text{and } x_{i,t_k} \geq 0, \quad i = 1, ..., 25 \end{array},$$

where $X(x_{t_k}) = x'_{t_k} z_{t_k} + (1 - x'_{t_k} e) z_{0,t_k} - t.c.(x_{t_k})$ and $t.c.(x_{t_k})$ represents the transaction costs at time t_k of portfolio x_{t_k} which are given by

$$0.002 \left| (1 - x'_{t_k} e) - \frac{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k})}{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k}) + \sum_{i=1}^{25} x_{i,t_{k-1}} (1 + z_i^{(t_k)})} \right| +$$
$$+ 0.002 \sum_{i=1}^{25} \left| x_{i,t_k} - \frac{x_{i,t_{k-1}} (1 + z_i^{(t_k)})}{(1 - x'_{t_{k-1}} e)(1 + z_{0,t_k}) + \sum_{i=1}^{25} x_{i,t_{k-1}} (1 + z_i^{(t_k)})} \right|,$$

where $x_{i,t_{k-1}}(1 + z_i^{(t_k)})$ is the percentage of wealth invested on the *i-th* stock at time t_{k-1}capitalized at time t_k.

Therefore, for each efficient frontier (the moment-based and stable Paretian ones), we have to determine the optimal portfolio weights

$$x^*_{t_k} = \arg(\max_{x_{t_k}\text{belongs to the efficient frontier}}(E(u(X(x_{t_k}))))).$$

Step 3 We compute the *ex-post* final wealth that is given by

$$W_{t_{k+1}} = W_{t_k}(1 + \left(x^*_{t_k}\right)' z^{(t_k+5)} + (1 - e'x^*_{t_k})z_{0,t_{k+1}} - t.c.\left(x^*_{t_k}\right))$$

where the transaction costs $t.c.\left(x^*_{t_k}\right)$ are defined above.

Step 4 We repeat steps 2 and 3 for every utility function and for every risk-aversion coefficient.

Observe that at each time t_k the investor's optimal choices are uniquely characterized by the mean, the dispersion, and the skewness. In particular, if we assume that $\alpha = \alpha_1 = \alpha_2$, the vector of returns is jointly α-stable distributed and every centered portfolio $\widetilde{z}_{p,t_k} = \sum_{i=1}^n x_{i,t_k}\widetilde{z}_{i,t_k}$ admits the stable distribution $S_\alpha\left(\sigma_{p,t_k}, \beta_{p,t_k}, 0\right)$ where $\sigma_{p,t_k} = \left(\left(x'_{t_k}Qx_{t_k}\right)^{\alpha/2} + \left|x'_{t_k}b\sigma_Y\right|^\alpha\right)^{1/\alpha}$ is the volatility and $\beta_{p,t_k} = \frac{\left|x'_{t_k}b\sigma_Y\right|^\alpha sgn(x'_{t_k}b)\beta_Y}{\left(x'_{t_k}Qx_{t_k}\right)^{\alpha/2}+\left|x'_{t_k}b\sigma_Y\right|^\alpha}$ is the portfolio skewness. Thus, we can represent the investor's optimal choices in terms of the mean $E(x'_{t_k}z + (1 - x'_{t_k}e)z_{0,t_k} - t.c.(x_{t_k}))$, the dispersion σ_{p,t_k}, and the portfolio skewness β_{p,t_k}. Similarly, when we consider the moment-based model, the optimal portfolio choices can be represented in terms of the mean, the standard deviation, and the Fisher skewness parameter: $\frac{E\left((\widetilde{z}_{p,t_k}-E(\widetilde{z}_{p,t_k}))^3\right)}{E\left((\widetilde{z}_{p,t_k}-E(\widetilde{z}_{p,t_k}))^2\right)^{3/2}}$.

Figure 1 shows the efficient frontiers of the two models valued at time t_0. As we should expect, in both cases the optimal choices are represented by a plane curved. First of all, we could observe that, at each time t_j, the *ex-ante* expected utility obtained with the stable Paretian approach is always greater than that obtained with moment-based model, and this result holds for any utility function (see Table 3).

Table 4 summarizes the final wealth obtained at time 5/31/2004 by the different expected utility maximizer. Even in this comparison we consider the distance given by (7) between the portfolio compositions at each time t_j. Then we observe significant differences in the optimal portfolio compositions (more than 27%), although these differences are lower than those obtained when unlimited short sales are allowed. However, the *ex-post* comparison shows that the final wealths obtained with the stable Paretian model are almost always greater than those obtained with the moment-based model. Practically, as shown in Figure 2, we observe that in many of the cases studied the stable Paretian portfolio strategy dominates the moment-based one. The figure shows the *ex-post* final wealth sample path of an investor with utility function $u(x) = \frac{-x^{-1.5}}{1.5}$. Thus, this performance analysis confirms and emphasizes the importance of properly evaluating the residual distribution behavior in the fund-separation portfolio models.

4. VaR and CVaR models with conditional stable distributed returns

In this section, we consider the conditional stable Paretian approach proposed by Lamantia, et al. (2006) to value the risk of a given portfolio. In particular, we assume the centered index return $Y_t \sim S_\alpha(\sigma_{Y_t}, \beta_{Y_t}, 0)$,

$(t=1,2,...)$ asymmetric α-stable distributed $(\beta_Y \neq 0)$ and independent of residual vectors

$$\widetilde{z}_t - bY_t = [\sigma_{11,t/t-1}\varepsilon_{1,t}, ..., \sigma_{nn,t/t-1}\varepsilon_{n,t}]'.$$

Furthermore, we assume that the conditional distribution of the centered continuously compounded return vector $\widetilde{z}_{t+1} = [\widetilde{z}_{1,t+1}, ..., \widetilde{z}_{n,t+1}]'$ is jointly α-stable with characteristic function

$$\Phi_{\widetilde{z}_{t+1}}(u) = \exp\left(-\left(\left(u'Q_{t+1/t}u\right)^{\alpha/2} + \left|u'b\sigma_Y\right|^{\alpha}\right) \times \right.$$
$$\left. \times \left(1 - i\frac{\left|u'b\sigma_Y\right|^{\alpha} sgn(u'b)\beta_Y}{\left(u'Q_{t+1/t}u\right)^{\alpha/2} + \left|u'b\sigma_Y\right|^{\alpha}} \tan\left(\frac{\pi\alpha}{2}\right)\right)\right)$$

In contrast to Lamantia, et al. (2006), we suggest an alternative evolution of the residual dispersion matrix $Q_{t+1/t}$ that is justified by Property 2.7.16 in Samorodnitsky and Taqqu (1994). That is, the centered continuously compounded returns $\widetilde{z}_{i,t}$ are generated as follows

$$\widetilde{z}_{i,t+1} = b_i Y_{t+1} + \sigma_{ii,t+1/t}\varepsilon_{i,t+1} = \left(\sigma^{\alpha}_{ii,t+1/t} + \left|b_i\sigma_Y\right|^{\alpha}\right)^{\frac{1}{\alpha}} X_{i,t+1}$$

$$\sigma^p_{ii,t+1/t} = (1-\lambda)\left|\widetilde{z}_{i,t} - b_iY_t\right|^p A(p) + \lambda\sigma^p_{ii,t/t-1}$$

$$B_{ij,t+1/t}(p) = (1-\lambda)\left(\widetilde{z}_{i,t} - b_iY_t\right)\left(\widetilde{z}_{j,t} - b_jY_t\right)^{\langle p-1\rangle} A(p) + \lambda B_{ij,t/t-1}(p)$$

$$\sigma^2_{ij,t+1/t} = B_{ij,t+1/t}(p)\sigma^{2-p}_{jj,t+1/t}$$

where $A(p) = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2^p\Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{p+1}{2}\right)}$.

The conditional distribution of the residual vector is sub-Gaussian α-stable and for any i and t, $\varepsilon_{i,t} \sim S_\alpha(1,0,0)$ and $X_{i,t} \sim S_\alpha\left(1, \frac{\left|b_i\sigma_Y\right|^{\alpha} sgn(b_i)\beta_Y}{\left(\sigma^{\alpha}_{ii,t/t-1} + \left|b_i\sigma_Y\right|^{\alpha}\right)}, 0\right)$. λ is a parameter (decay factor) that regulates the weighting on past co-variation parameters. The vector $b = [b_1, b_2, ..., b_n]'$ is estimated using the estimator given by (6). The forecast time $t+1$ stable scale parameter of the *i-th* residual is given by:

$$\sigma_{ii,t+1/t} = (E_t(|\widetilde{z}_{i,t+1} - b_i Y_{t+1}|^p)A(p))^{1/p} \simeq$$
$$\simeq \left(A(p)(1-\lambda)\sum_{k=0}^{K} \lambda^{K-k} \left|\widetilde{z}_{i,t-K+k} - b_i Y_{t-K+k}\right|^p\right)^{1/p}.$$

The time $t+1$ stable covariation parameter between the *i-th* and the *j-th* residual is defined by $\sigma^2_{ij,t+1/t}$ and

$$B_{ij,t+1/t}(p) = A(p)E_t\left(\left(\widetilde{z}_{i,t+1} - b_i Y_{t+1}\right)\left(\widetilde{z}_{j,t+1} - b_j Y_{t+1}\right)^{\langle p-1\rangle}\right) \simeq A(p)\times$$
$$\times(1-\lambda)\sum_{k=0}^{K}\left(\lambda^{K-k}\left(\widetilde{z}_{i,t-K+k} - b_i Y_{t-K+k}\right)\left(\widetilde{z}_{j,t-K+k} - b_j Y_{t-K+k}\right)^{\langle p-1\rangle}\right).$$

Under these assumptions, the forecast $(1-\theta)\%$ VaR of portfolio

$$\widetilde{z}_{p,t} = w'\widetilde{z}_t = \sum_{i=1}^{n} w_i \widetilde{z}_{i,t}$$

in the period $[t, t+1]$ is given by the corresponding $(1-\theta)$ percentile of the α-stable distribution $S_\alpha\left(\sigma_{p,t+1/t}, \beta_{p,t+1/t}, 0\right)$, where

$$\sigma_{p,t+1/t} = \left(\left(w' Q_{t+1/t} w\right)^{\alpha/2} + |w' b \sigma_Y|^\alpha\right)^{1/\alpha}$$

is the forecast volatility and

$$\beta_{p,t+1/t} = \frac{|w' b \sigma_Y|^\alpha \, sgn(w'b)\beta_Y}{\left(w' Q_{t+1/t} w\right)^{\alpha/2} + |w' b \sigma_Y|^\alpha}$$

is the forecast skewness. Similarly the CVaR with confidence level $\theta\%$ of portfolio $\widetilde{z}_{p,t}$, denoted by

$$CVaR_{(1-\theta)\%}(\widetilde{z}_{p,t}) = E\left(\widetilde{z}_{p,t} / \widetilde{z}_{p,t} \le VaR_{(1-\theta)\%}\left(\widetilde{z}_{p,t}\right)\right),$$

can be simply computed considering the algorithms proposed by Stoyanov et al (2006). Although the stable VaR model has been recently tested and studied (see Lamantia et al (2006)), further analyses and empirical comparisons among different stable VaR and CVaR models are necessary.

5. Conclusions

In this paper, we examine a stable Paretian version of the three-fund separation model and propose VaR and CVaR models with stable distributed returns. We first discuss portfolio choice models considering returns with heavy-tailed distributions. In order to present heavy-tailed models that consider the asymmetry of returns, we examine a discrete time three-fund separation model where the portfolios are in the domain of attraction of a stable law. Second, we propose and then test an *ex-ante* and an *ex-post* comparison between dynamic stable portfolio strategies and those obtained by a moment-based fund separation approach. Our empirical comparison demonstrates that heavy tails of residuals can have a fundamental impact on the asset allocation decisions by investors. As a matter of fact, the stable Paretian model takes into account the heavy tails of residuals and we find that the stable Paretian model dominates the moment-based model in terms of expected utility and of the *ex-post* final wealths. Finally, we propose a conditional extension of the stable Paretian fund separation model in order to compute the VaR and CVaR of a given portfolio.

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	STABLE	PARAMETERS			Vector b	Optimal p	date	One-month Libor
	α	β	σ	μ				
World	1.8119	-0.4721	1.39E-02	1.06390E-03	//	//		
Australia	1.9039	-0.5761	1.81E-02	1.27704E-03	0.69915336	0.901	12/8/2003	0.00024366
Austria	1.8827	-0.9134	1.84E-02	4.90564E-04	0.41269641	0.791	12/15/2003	0.0002395
Belgium	1.7166	-0.3579	1.75E-02	7.93329E-04	0.82281788	0.435	12/22/2003	0.00023768
Canada	1.7403	-0.4494	1.73E-02	1.33854E-03	0.95577381	0.495	12/29/2003	0.00023611
Denmark	1.878	-0.1875	1.81E-02	2.44492E-03	0.62197582	0.739	1/5/2004	0.00023325
Finland	1.8049	-0.5025	3.57E-02	3.56638E-03	1.58221202	0.671	1/12/2004	0.00022909
France	1.8527	-0.3812	1.93E-02	1.35466E-03	1.07931915	0.741	1/19/2004	0.00022909
Germany	1.7496	-0.3245	2.06E-02	9.99059E-04	1.21313555	0.543	1/26/2004	0.00022909
Greece	1.8183	0.1071	2.89E-02	2.12436E-03	0.77665342	0.592	2/2/2004	0.00022909
Honk Kong	1.8557	-0.2997	2.73E-02	9.12564E-04	1.02073281	0.629	2/9/2004	0.00022909
Ireland	1.854	-0.495	1.89E-02	1.85463E-03	0.73384617	0.672	2/16/2004	0.00022779
Italy	1.8662	-0.1196	2.36E-02	1.84903E-03	0.94883386	0.751	2/23/2004	0.00022701
Japan	1.8665	0.458	2.25E-02	3.54105E-04	0.8410432	0.733	3/1/2004	0.00022909
Malaysia	1.4345	-0.0433	2.30E-02	9.21145E-04	0.73252637	0.231	3/8/2004	0.00022753
Netherlands	1.7183	-0.464	1.73E-02	1.06414E-03	1.04661998	0.459	3/15/2004	0.00022701
New Zealand	1.8319	-0.4512	2.08E-02	1.03758E-03	0.56151513	0.636	3/22/2004	0.00022701
Norway	1.7976	-0.5846	1.98E-02	1.13420E-03	0.77828204	0.573	3/29/2004	0.00022701
Portugal	1.873	-0.1199	2.05E-02	1.78560E-03	0.62436739	0.621	4/5/2004	0.00022909
Singapore	1.6793	0.012	2.19E-02	8.56174E-04	0.84932897	0.329	4/12/2004	0.00022909
South African Gold	1.7097	-0.2279	2.38E-02	1.29268E-03	0.84936104	0.397	4/19/2004	0.00022909
Spain	1.8885	-0.5767	2.22E-02	2.03309E-03	1.00178349	0.662	4/26/2004	0.00022909
Sweden	1.8617	-0.6198	2.70E-02	2.20147E-03	1.37302633	0.715	5/3/2004	0.00022909
Switzerland	1.8375	-0.4555	1.74E-02	2.18278E-03	0.8566863	0.648	5/10/2004	0.00022909
UK	1.8643	-0.4924	1.57E-02	1.02209E-03	0.83454662	0.723	5/17/2004	0.00022909
USA	1.7859	-0.3313	1.53E-02	1.52378E-03	1.02808878	0.516	5/24/2004	0.00022909

Table 1 Weekly one-month Libor, MLE stable parameters, OLS estimates of skewness vector ***b*,** and optimal values p_i assuming weekly return series between January 1993 and December 2003

Expected Utility	Stable Paretian model: Expected Utility	Stable Paretian model: Final Wealth	Moment-based model: Expected Utility	Moment-based model: Final Wealth	Difference between portfolio compositions $\frac{1}{50}\sum_{j=0}^{24}\sum_{i=1}^{25}\left\lvert x_{i,t_j}^{stable} - x_{i,t_j}^{moment}\right\rvert$
E(log(X))	**0.06748929**	**0.921348**	0.035949	0.89988926	1.223015
-E(exp(-X))	**-0.3434836**	**0.913607**	-0.35453	0.89254154	1.267922
-E(exp(-5X))	**-0.0061188**	**0.98823**	-0.006316	0.98406318	0.253439
-E(exp(-7X))	**-0.0008167**	**0.993622**	-0.000843	0.99059865	0.181591
-E(exp(-17X))	**-3.459E-08**	**1.001446**	-3.65E-08	0.99273461	0.114266
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**-0.6474572**	**0.996126**	-0.648086	0.96153605	0.340285
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**-0.38462**	**0.992376**	-0.38483	0.974308	0.238968
$E(X)-E(\lvert X-E(X)\rvert^{1.3})$	**1.00857878**	1.001703	1.007344	**1.00349768**	0.07409
$E(X)-2.5E(\lvert X-E(X)\rvert^{1.3})$	**1.0070141**	**1.006794**	1.006963	1.00678217	0.003309
$E(X)-E(\lvert X-E(X)\rvert^{2})$	**1.0346303**	**0.996126**	1.022109	0.94873114	0.539501
$E(X)-2.5E(\lvert X-E(X)\rvert^{2})$	**1.01851465**	0.982589	1.01301	**0.98365812**	0.220456

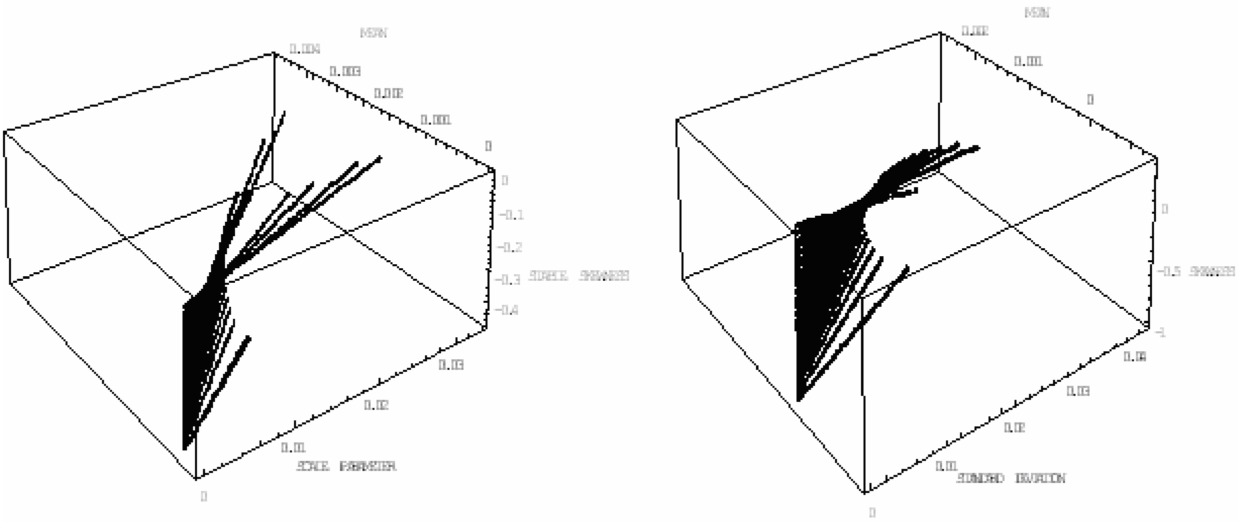
Table 2 Comparison on three parametric efficient frontiers and analysis of the models' performance. We maximize the expected utility on the *ex-ante* efficient frontiers considering weekly returns from January 1993 till December 2003 for 25 country equity market indices and 30-day Eurodollar CD. Moreover, we also consider the *ex-post* final wealth of the investor's choices.

Times	Stable Paretian Model			Moment-based model		
	Expected Utility -E(exp(-X))	**Expected Utility** E(X)- -E(|X-E(X)|$^{1.3}$)	**Expected Utility** E(X)-2.5E(|X- -E(X)|^{2})	**Expected Utility** -E(exp(-X))	**Expected Utility** E(X)- -E(|X-E(X)|$^{1.3}$)	**Expected Utility** E(X)-2.5E(|X- -E(X)|^{2})
12/8/2003	**-1.00061989**	**-0.001755038**	**-0.00137045**	-1.000635911	-0.001755067	-0.00137278
12/15/2003	**-0.9987222**	**0.000140813**	**0.00052715**	-0.99873854	0.000140782	0.000524703
12/22/2003	**-0.99869823**	**0.000139098**	**0.00053992**	-0.998717431	0.000139056	0.000536138
12/29/2003	**-0.99867117**	**0.000137654**	**0.00055454**	-0.998692649	0.000137603	0.000549765
1/5/2004	**-0.99863393**	**0.000134954**	**0.00057403**	-0.998657369	0.000134894	0.000568404
1/12/2004	**-0.99859893**	**0.000130949**	**0.00059177**	-0.998622842	0.000130886	0.000586184
1/19/2004	**-0.99859597**	**0.000130966**	**0.00059384**	-0.998618888	0.000130907	0.000588835
1/26/2004	**-0.99857558**	**0.000131052**	**0.00060551**	-0.99859984	0.000130986	0.000599828
2/2/2004	**-0.99859463**	**0.000130975**	**0.0005951**	-0.998616812	0.000130919	0.000590506
2/9/2004	**-0.99860609**	**0.000130929**	**0.00058893**	-0.998627144	0.000130879	0.000585062
2/16/2004	**-0.99857776**	**0.000129749**	**0.00060431**	-0.998599504	0.000129696	0.000600327
2/23/2004	**-0.99857306**	**0.000128994**	**0.00060693**	-0.998595557	0.000128938	0.00060262
3/1/2004	**-0.99859079**	**0.000130996**	**0.00059821**	-0.998612592	0.000130945	0.000594301
3/8/2004	**-0.99858053**	**0.000129487**	**0.0006036**	-0.9986017	0.000129437	0.000600095
3/15/2004	**-0.99863035**	**0.00012876**	**0.00057515**	-0.99864758	0.000128729	0.000573684
3/22/2004	**-0.99865496**	**0.00012866**	**0.00056151**	-0.998668045	0.000128647	0.0005614
3/29/2004	**-0.99866581**	**0.000128618**	**0.00055579**	-0.998677519	0.000128612	0.000554432
4/5/2004	**-0.99861368**	**0.000130899**	**0.00058536**	-0.998629271	0.000130875	0.000584969
4/12/2004	**-0.99859803**	**0.000130966**	**0.00059438**	-0.998616857	0.000130929	0.000592356
4/19/2004	**-0.99862434**	**0.000130859**	**0.00057989**	-0.998643882	0.000130819	0.000577312
4/26/2004	**-0.99863342**	**0.000130824**	**0.00057513**	-0.998690393	0.00013063	0.000552539
5/3/2004	**-0.99865624**	**0.000130732**	**0.00056259**	-0.998674346	0.000130699	0.000560597
5/10/2004	**-0.99868805**	**0.000130602**	**0.0005449**	-0.998704626	0.000130577	0.000543639
5/17/2004	**-0.99875516**	**0.000130319**	**0.00050667**	-0.998767256	0.000130311	0.00050644
5/24/2004	**-0.99874857**	**0.00013035**	**0.00051072**	-0.998760348	0.000130343	0.000510267

Table 3 *Ex ante* comparison on three parametric efficient frontiers. We maximize the expected utility on the *ex-ante* efficient frontiers considering weekly returns from January 1993 till December 2003 for 25 country equity market indices and 30-day Eurodollar CD.

Expected Utility	Stable Paretian Model Final Wealth	Moment-based model Final Wealth	Difference between portfolio composition $\frac{1}{50}\sum_{j=0}^{24}\sum_{i=1}^{25}\left\|x_{i,t_j}^{stable}-x_{i,t_j}^{moment}\right\|$
E(log(X))	**0.946068**	0.923961	0.275455
-E(exp(-X))	**0.9333206**	0.911717	0.248854
-E(exp(-5X))	**0.9952915**	0.991088	0.110664
-E(exp(-7X))	**0.997546**	0.994512	0.079008
-E(exp(-17X))	**1.0037667**	0.995008	0.051296
$\frac{-1}{1.5}E\left(X^{-1.5}\right)$	**0.9936615**	0.975512	0.116882
$\frac{-1}{2.5}E\left(X^{-2.5}\right)$	**1.0001669**	0.967061	0.188742
$E(X)-E(\|X-E(X)\|^{1.3})$	0.9998229	**1.001619**	0.031864
$E(X)-2.5E(\|X-E(X)\|^{1.3})$	1.0047192	**1.004798**	0.013961
$E(X)-E(\|X-E(X)\|^{2})$	**0.9774207**	0.960879	0.266129
$E(X)-2.5E(\|X-E(X)\|^{2})$	**0.9891482**	0.983203	0.098486

Table 4 Comparison of the *ex-post* final wealth (computed for the period 12/15/2003-5/31/2004) on the efficient frontiers. We compute the *ex-post* final wealth considering weekly returns for 25 country equity market indices and 30-day Eurodollar CD.



Mean-scale parameter-stable skewness efficient frontier with the risk-free asset

Mean-standard deviation-skewness efficient frontier with the risk-free asset

Figure 1: *Mean-Risk-Skewness efficient frontiers when the risk-free asset is allowed.*

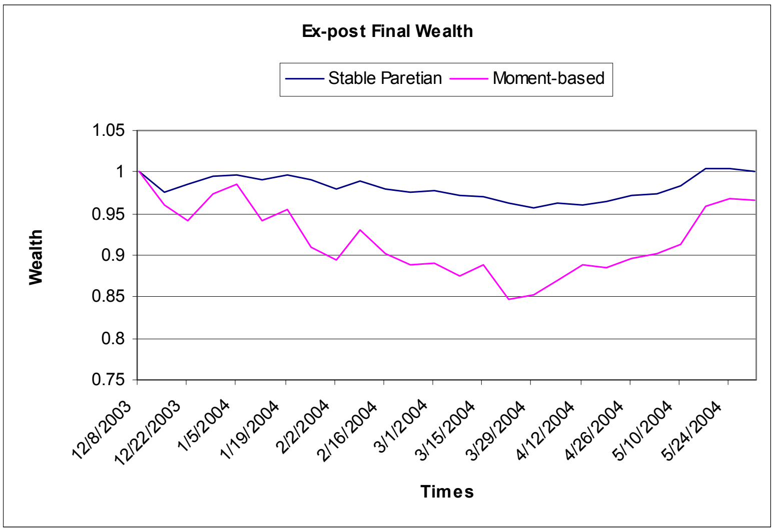


Figure 2: *Ex-post comparison of portfolio strategies of an investor with utility function* $u(x) = -x^{-1.5} / 1.5$.