APPROXIMATION OF AGGREGATE AND EXTREMAL LOSSES WITHIN THE VERY HEAVY TAILS FRAMEWORK

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Abstract

The loss distribution approach (LDA) is one of the three advanced measurement approaches (AMA) to the Pillar I modelling proposed by Basel II in 2001. In this paper, one possible approximation of the aggregate and maximum loss distribution in the extremely Low Frequency/High Severity case, i.e. the case of infinite mean of the loss sizes and loss inter-arrival times. In this study independent but not identically distributed losses are considered. The minimum loss amount is considered increasing over time. Monte Carlo simulation algorithm is given and several quantiles are estimated. The same approximation is in place for modelling the maximum and aggregate worldwide economy losses caused by very rare and very extreme events such as 9/11, the Russian rouble crisis, and the U.S. subprime mortgage crisis.

Key words: very-heavy tails, extremal process, self-similar process, aggregate loss distribution, maximum loss distribution

1 Introduction

A stock market crash, a huge disaster, occurring simultaneously on most of the stock markets of the world as witnessed in October 1987 would amount to the quasi-instantaneous evaporation of trillions of dollars. How large might a possible stock market crash be tomorrow? Extreme asset price movements appear to be more pronounced recently and have major consequences for an economy's financial stability and monetary policy. The aggregate and maximum losses caused by such very rare but disastrous events should be modelled.

Another area of finance where a similar type of event needs to be quantified is in operational risk modelling. Large operational losses as a result of accounting scandals, corporate fraud, and rogue trading, to name just a few, have received increasing attention. The frequency of severe losses, with more than 100 instances of losses at financial institutions exceeding $100 million has caused many financial institutions to try to explicitly model operational risk to determine their economic capital for risk-based capital purposes. As financial institutions have begun to collect loss data and use it to manage operational risk, bank regulators have increased their expectations for measuring and modelling operational risk. Under the current U.S. rules for implementing the Basel Accord, large internationally active banks will be expected to use internal models to estimate capital for unexpected operational losses. The loss distribution approach (LDA) is the most sophisticated and the most favoured by regulators. LDA is the most accurate from a statistical perspective as it utilizes the exact distribution of historic losses (both frequency and severity) and is based on an individual bank's internal loss data. It suggests an actuarial-type model for the aggregated operational losses. Existing empirical evidence suggests that the general pattern of operational loss severity data is characterized by high kurtosis, severe right skewness, and a very heavy right tail created by several outlying events. The empirical evidence reported in several studies support this claim (see Cruz [7], Medova [12], Moscadelli [14], Embrechts et al. [10], De Fontnouvelle et al. [8], Chernobai et al. [5],[6], and Nešlehová et al. [15]). One approach uses Extreme Value Theory (EVT) to fit a Generalized Pareto Distribution (GPD) to extreme losses exceeding a high pre-specified threshold (see Embrechts et al. [9], Embrechts et al. [10], Chavez-Demoulin and Embrechts [3], and Nešlehová et al. [15]). Another approach to calibrate operational losses is to fit a parametric family of distributions, such as the lognormal, Weibull, gamma, and Pareto distribution.[1]

The main purpose of the paper is to provide approximations of the maximum and aggregate loss processes assuming (1) the loss amounts follow Pareto distributions and (2) the minimum amount of the consecutive extreme losses increases over time. The aggregate and maximum loss processes are transformed properly and weak limits of these transformations are derived. The limit processes can be used for approximating the initial ones. Some important properties of the approximating processes are stated and an efficient method for simulation is provided. Using Monte Carlo simulation we estimate several quantiles of the

[1] A review of the different approaches used in operational risk modelling is given in Chernobai et al. [4].

processes at time $t = 1$, i.e. one calendar year and compare the quality of the estimates. As an intermediate step we obtain a new member of the class of the so-called Sato processes.

This paper is organized as follows. Section 2 provides a description of the model for loss sizes and loss inter-arrival times and the approximation method used. Then the limits of the so-called accompanying processes are obtained and the limit approximating processes in the general case (which is the main result of the paper) are derived. Section 3 gives a method for simulating the approximating processes at any given time $t > 0$. Section 4 reports the results from a simulation study. Section 5 concludes the paper and summarizes the findings.

2 Model Setup

Usually an actuarial-type model is based on a sequence $\{(T_k, X_k), \quad k = 1, 2, \ldots\}$ of loss arrival times T_k and loss amounts X_k. The loss amounts are assumed mutually independent and identically distributed and independent of loss arrivals. In this model we assume that the severities are independent but not identically distributed. Let $X_k, \quad k = 1, 2, \ldots$ be independent Pareto distributed random variables with

$$\mathrm{P}(X_k > x) = \left(\frac{c(k)}{x}\right)^{\alpha}, x \geq c(k),$$

where $c(k) = Ck^{\delta}, 0 < \alpha < 1, C > 0$ and $\delta > 0$. In this way $c(k) \uparrow \infty$. The sequence $\{X_k, k = 1, 2, \ldots\}$ represents the loss amounts. The constant C controls the minimum value of the loss and the constant δ controls the growth rate of this minimum value over time. Let Y_k be i.i.d. positive r.v.'s representing the operational losses inter-arrival times with distribution function $G(x)$. Define the loss arrival times by

$$T_0 = 0, \quad T_n = \sum_{k=1}^{n} Y_k, \quad n = 1, 2, 3, \ldots,$$

and the associated counting process $N(t) = \max\{n : T_n \le t < T_{n+1}\}$.

Suppose Y_k belong to the domain of attraction of a stable law with index of stability $\gamma \in (0,1)$, i.e. $1 - G(x) = x^{-\gamma} L(x), x \to \infty$ with $\gamma \in (0,1)$ and $L(.)$ slowly varying at infinity. The aggregate loss up to time $t > 0$ is then defined by

$$Z(t) = \sum_{k=1}^{N(t)} X_k, \quad t > 0.$$

Note that in this very-heavy tailed case the maximum loss process

$$M(t) = \bigvee_{k=1}^{N(t)} X_k, \quad t > 0$$

determines the behavior of the aggregate loss process $Z(t)$. Both processes and their relationship is investigated in this paper.

2.1 Approximation method

In order to find approximations for $Z(t)$ and $M(t)$ we transform the loss arrival times and loss sizes in such a way that the number of losses in a fixed interval gets larger and the loss sizes get smaller. It is well known there exists a sequence $b(n) > 0$ such that

$$T_n(\cdot) := \frac{\sum_{k=1}^{[n\cdot]} Y_k}{b(n)} \Rightarrow D_\gamma(\cdot),$$

where $b(n)$ is determined from the relation $n(1 - G(b(n)) \to \frac{1}{\Gamma(1-\gamma)}$, and can be chosen as $b(n) \sim n^{1/\gamma} L_b(n)$. The process $D_\gamma(t)$ is a $\gamma-$stable Levy motion, where the Laplace transform of $D = D_\gamma(1)$ is given by $\mathbb{E}e^{-\lambda D} = e^{-\lambda^\gamma}, \quad \lambda > 0$. Since $b(.)$ is regularly varying with index $1/\gamma > 0$, Theorem 1.5.12, [1] gives that there exists a regularly varying function $\tilde{b}$ with index γ such that $b(\tilde{b}(c)) \sim c, c \to \infty$. (For two positive functions f, g $f \sim g$ means $f(c)/g(c) \to 1$ as $c \to \infty$.) The two functions b and $\tilde{b}$ are connected through the following relation

$$b(\tilde{b}(c)) \sim \tilde{b}(b(c)) \sim c \quad \text{as} \quad c \to \infty. \tag{2.1}$$

Now the normalized counting process $N(nt)/\tilde{b}(n), t > 0$ is weakly convergent to the hitting time process $\Lambda(t) = \inf\{u : D_\gamma(u) > t\}$ of D_γ, i.e.

$$\frac{N(n\cdot)}{\tilde{b}(n)} \Rightarrow \Lambda(\cdot). \tag{2.2}$$

As an inverse of D_γ , the process Λ is γ-self-similar, for $t > 0$, $\Lambda(t) \stackrel{\mathrm{d}}{=} (D/t)^{-\gamma}$ and its probability density is given by $f_t(x) = t\gamma^{-1}x^{-1-1/\gamma}g_\gamma(tx^{-1/\gamma})$, where g_γ is the probability density of D. The process $\{\Lambda(t)\}_{t\geq 0}$ does not have stationary and independent increments (see Meerschaert and Scheffler [13]).

Consider the sequence of point processes

$$\tilde{\mathcal{N}}_n = \{\left(T_{nk} = \frac{T_k}{b(n)}, X_{nk} = \frac{X_k}{a(n)}\right), k = 1, 2, \dots\}, \tag{2.3}$$

where $a(n) = \left(\frac{n^{\alpha\delta+1}}{\alpha\delta+1}\right)^{1/\alpha}$, $n \geq 1$. Define a sequence of extremal processes

$$\tilde{Y}_n(t) = \bigvee_{k=1}^{N_n(t)} X_{nk} = \bigvee_{T_{nk}\leq t} \frac{X_k}{a(n)}, \quad t > 0, \quad n = 1, 2, 3, \dots, \tag{2.4}$$

and a sequence of sum processes

$$\tilde{S}_n(t) = \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{T_k\leq b(n)t} \frac{X_k}{a(n)}, \quad t > 0, \quad n = 1, 2, 3, \dots, \tag{2.5}$$

where $N_n(t) = \max\{k : T_{nk} \leq t\} = \max\{k : T_k \leq b(n)t\} = N(b(n)t)$. Our goal is to find weak limits of the sequences $\tilde{S}_n$ and $\tilde{Y}_n$. These limit processes will be used for approximating the initial ones.

The problem is solved in two steps. The first one is to find limits of the following sequence of processes

$$Y_n(t) = \bigvee_{k=1}^{k_n(t)} X_{nk} = \bigvee_{k/n\leq t} \frac{X_k}{a(n)}, \quad t > 0, \quad n = 1, 2, 3, \dots, \tag{2.6}$$

and respectively the sequence of sum processes

$$S_n(t) = \sum_{k=1}^{k_n(t)} X_{nk} = \sum_{k/n\leq t} \frac{X_k}{a(n)}, \quad t > 0, \quad n = 1, 2, 3, \dots, \tag{2.7}$$

associated with the point processes

$$\mathcal{N}_n = \{(t_{nk}, X_{nk}), k = 1, 2, \ldots\} = \left\{\left(\frac{k}{n}, \frac{X_k}{a(n)}\right), k = 1, 2, \ldots\right\}, \quad n = 1, 2, \ldots$$

and $k_n(t) = \max\{k : t_{nk} \leq t\} = \max\{k : k \leq nt\} = [nt]$. After that we move forward to find the limits of $\tilde{S}_n$ and $\tilde{Y}_n$ using the results obtained in the first step and a relation between the counting process $N_n(t)$ and the counting function $k_n(t)$. We call the processes with deterministic time points S_n, Y_n and $\mathcal{N}_n$ accompanying to the processes with random time points $\tilde{S}_n, \tilde{Y}_n$ and $\tilde{\mathcal{N}}_n$, respectively. This scheme is followed by Pancheva and Jordanova [16], [17].

2.2 Accompanying Processes

In order to find the weak limits of the sequences (2.6) and (2.7), we use a theorem (Functional Extremal Criterion) stated in the work of Pancheva et al. [19]. The theorem gives a relationship between the convergence of a sequence of extremal processes and a sequence of sum processes associated with the same uniformly negligible triangular array (u.n.t.a.). We find directly the limit of the sequence (2.6) and after that check a technical condition stated in the theorem which together with the convergence of the extremal processes (2.6) gives the convergence of the sum processes (2.7).

The sequence of point processes $\{\mathcal{N}_n, n \geq 1\}$ forms a u.n.t.a. since $k/n \to \infty$, as $k \to \infty$ and $(k+1)/n - k/n \to 0, n \to \infty$ and the space components X_k (see Definition 1 in Pancheva et al. [19]) satisfy the asymptotic negligibility condition

$$\sup_{k \leq n} \mathrm{P}\left(\frac{X_k}{a(n)} > x\right) = \sup_{k \leq n}\left(\frac{c(k)}{a(n)x}\right)^{\alpha} = \left(\frac{c(n)}{a(n)x}\right)^{\alpha} \to 0, n \to \infty. \tag{2.8}$$

The distribution function $f_n(t, x)$ of the extremal process $Y_n(t)$ is given by

$$f_n(t, x) = \mathrm{P}(Y_n(t) < x) = \prod_{k/n \leq t} \mathrm{P}\left(\frac{X_k}{a(n)} < x\right) = \prod_{k/n \leq t}\left(1 - \left(\frac{c(k)}{a(n)x}\right)^{\alpha}\right)$$

$$= \exp\left(\sum_{k/n\le t}\log\left(1-\left(\frac{c(k)}{a(n)x}\right)^{\alpha}\right)\right) \sim \exp\left(-\sum_{k/n\le t}\left(\frac{c(k)}{a(n)x}\right)^{\alpha}\right).$$

The last relation is due to (2.8). For the sum in the right-hand side of the above expression we have

$$\sum_{k/n\le t}\left(\frac{c(k)}{a(n)x}\right)^{\alpha} = \left(\frac{C}{a(n)x}\right)^{\alpha}\sum_{k/n\le t} c(k)^{\alpha}$$

[and taking into account the form of $c(k)$ and $a(n)$ and Theorem 1.15.11, [1] we get]

$$\sim \frac{C^{\alpha}}{x^{\alpha}}\frac{\alpha\delta+1}{n^{\alpha\delta+1}}\sum_{k/n\le t} k^{\alpha\delta} \sim \frac{C^{\alpha}}{x^{\alpha}} \times \frac{\alpha\delta+1}{n^{\alpha\delta+1}} \times \frac{[nt]^{\alpha\delta+1}}{\alpha\delta+1} \to \frac{C^{\alpha}}{x^{\alpha}} t^{\alpha\delta+1}, \quad n\to\infty.$$

Now letting $n\to\infty$, we obtain the limit

$$f_n(t,x) = \mathrm{P}(Y_n(t) < x) \to f(t,x) = \exp(-x^{-\alpha}C^{\alpha}t^{\alpha\delta+1}), \quad t>0, \quad x>0.$$

which is a distribution function of an extremal process Y generated by a Poisson point process with mean measure

$$\mu((0,t]\times[x,\infty)) = \frac{C^{\alpha}t^{\alpha\delta+1}}{x^{\alpha}} \text{ for } \quad x>0, \quad t>0.$$

It is well known that the convergence of the one-dimensional distributions of extremal processes implies convergence of all finite dimensional distributions. Moreover, the monotonicity of the sample paths of these processes gives weak convergence $Y_n \Rightarrow Y$. The lower curve of the limit extremal process is identically zero. Also for $t>0$ the r.v. $Y(t)$ follows Frechet distribution.

The process Y is self-similar with exponent $H = \delta + 1/\alpha$, since

$$\mathrm{P}(Y(\lambda t) < x) = \exp\left(-\frac{C^{\alpha}(\lambda t)^{\alpha\delta+1}}{x^{\alpha}}\right) = \exp\left(-\frac{C^{\alpha}t^{\alpha\delta+1}}{\left(\frac{x}{\lambda^{\delta+1/\alpha}}\right)^{\alpha}}\right)$$

$$= \mathrm{P}\left(Y(t) < \frac{x}{\lambda^{\delta+1/\alpha}}\right) = \mathrm{P}\left(\lambda^{\delta+1/\alpha}Y(t) < x\right).$$

The last relation implies $\{Y(\lambda t)\}_{t>0} \overset{\mathrm{d}}{=} \{\lambda^{H}Y(t)\}_{t>0}$. Furthermore,

$$\mathrm{P}(Y(t_2) < x) = \mathrm{P}\left(Y(t_1) \vee U_m(t_1,t_2] < x\right) = \mathrm{P}\left(Y(t_1) < x\right)\mathrm{P}\left(U_m(t_1,t_2] < x\right),$$

where $U_m(t_1, t_2]$ is the max-increment of Y in the interval $(t_1, t_2]$. From the above equality follows that the distribution of this max-increment is

$$\mathrm{P}\left(U_m(t_1,t_2] < x\right) = \frac{\mathrm{P}\left(Y(t_2) < x\right)}{\mathrm{P}\left(Y(t_1) < x\right)} = \exp\left[-\frac{C^\alpha(t_2^{\alpha\delta+1} - t_1^{\alpha\delta+1})}{x^\alpha}\right]$$

which means that the process Y does not have homogeneous max-increments.

In order to prove the convergence of the sequence (2.7) and find the limit process we are going to use the functional extremal criterion stated in Pancheva et al. [19]. The convergence of the corresponding sequence of extremal processes (2.6) is already established, hence we are going to verify the condition

$$\lim_{n\to\infty} \sum_{k/n\le t} \mathbb{E}\left(\frac{X_k}{a(n)} I\{\frac{X_k}{a(n)} \le h\}\right) \to \int_0^t \int_0^h x\mu(ds, dx), n \to \infty. \tag{2.9}$$

Fix $h > 0$ and note that

$$\mathbb{E}\left(\frac{X_k}{a(n)} I\{\frac{X_k}{a(n)} \le h\}\right) = \int_0^h x d\mathrm{P}(X_k/a(n) < x)$$

$$= \int_0^h x \left(\frac{c(k)}{a(n)}\right)^\alpha \alpha x^{-\alpha-1} dx = \alpha \left(\frac{c(k)}{a(n)}\right)^\alpha \frac{h^{1-\alpha}}{1-\alpha}.$$

Thus

$$\sum_{k/n\le t} \mathbb{E}\left(\frac{X_k}{a(n)} I\{\frac{X_k}{a(n)} \le h\}\right) = \sum_{k/n\le t} \alpha \left(\frac{c(k)}{a(n)}\right)^\alpha \frac{h^{1-\alpha}}{1-\alpha}$$

$$= \frac{\alpha}{1-\alpha} h^{1-\alpha} \sum_{k/n\le t} \left(\frac{c(k)}{a(n)}\right)^\alpha \to \frac{\alpha}{1-\alpha} h^{1-\alpha} C^\alpha t^{\alpha\delta+1}, \text{ as } n \to \infty.$$

On the other hand

$$\int_0^t \int_0^h x\mu(ds, dx) = C^\alpha \int_0^t \int_0^h x\alpha(\alpha\delta+1)s^{\alpha\delta} x^{-\alpha-1} dx ds$$

$$= C^\alpha t^{\alpha\delta+1} \frac{\alpha}{1-\alpha} h^{1-\alpha} < \infty,$$

and therefore (2.9) is satisfied.

Now using the functional extremal criterion one can state that $S_n \Rightarrow S$ where the characteristic function of $S(t), t > 0$ has the following form

$$\text{(2.10)} \qquad \mathbb{E}e^{izS(t)} = \exp\left(\int_0^t \int_0^\infty (e^{izx}-1)\mu(ds,dx)\right)$$
$$= \exp\left(C^\alpha t^{\alpha\delta+1}\int_0^\infty (e^{izx}-1)\alpha x^{-\alpha-1}dx\right).$$

This is the characteristic function at time $t > 0$ of a self-similar additive process generated by self-decomposable law of positive r.v. $X = S(1)$ with characteristic function

$$\text{(2.11)} \qquad \mathbb{E}e^{izX} = \exp\left[\int_0^\infty (e^{izx}-1)\frac{C^\alpha \alpha x^{-\alpha}}{x}dx\right],$$

In this case the Hurst exponent is again $H = \delta + 1/\alpha$.

The process $S(t)$ has independent increments and the characteristic function of the increment $U_s(t_1,t_2] = S(t_2) - S(t_1), \quad t_1 < t_2$ is given by

$$\mathbb{E}e^{izU_s(t_1,t_2]} = \mathbb{E}e^{izS(t_2)}/\mathbb{E}e^{izS(t_1)}$$
$$= \exp\left[C^\alpha\left(t_2^{\alpha\delta+1} - t_1^{\alpha\delta+1}\right)\int_0^\infty (e^{izx}-1)\alpha x^{-\alpha-1}dx\right].$$

The above relations show that the process does not have homogeneous increments. So the process $S(t)$ is a self-similar additive process or Sato process.

2.3 General Case

Let us return to the sequence of point processes (2.3) and the associated sequences of extremal and sum processes (2.4) and (2.5). These extremal processes can be rewritten as

$$\tilde{Y}_n(t) = \bigvee_{T_k \le b(n)t} \frac{X_k}{a(n)} = \bigvee_{k \le N(b(n)t)} \frac{X_k}{a(n)}$$
$$= \bigvee_{k \le \frac{N(b(n)t)}{n} n} \frac{X_k}{a(n)} = Y_n\left(\frac{N(b(n)t)}{n}\right).$$

According to the relation (2.1) and the convergence (2.2) we see that

$$\text{(2.12)} \qquad \theta_n(\cdot) := \frac{N(b(n)\cdot)}{n} \sim \frac{N(b(n)\cdot)}{\tilde{b}(b(n))} \Rightarrow \Lambda(\cdot).$$

In addition to the above convergence, there is a relation between the counting function $k_n(t)$ and counting process $N_n(t)$ given by

$$k_n(\theta_n(t)) = [N(b(n)t)] = N(b(n)t) = N_n(t).$$

Using the fact $Y_n \Rightarrow Y$, as $n \to \infty$ and the continuity of composition theorem one gets

$$\tilde{Y}_n(\cdot) = Y_n\left(\frac{N(b(n)\cdot)}{n}\right) \Rightarrow Y \circ \Lambda(\cdot) =: \tilde{Y}(\cdot).$$

On the other hand,

$$\tilde{Y}_n(t) = \bigvee_{T_k \le b(n)t} \frac{X_k}{a(n)} = \bigvee_{k \le N(b(n)t)} \frac{X_k}{a(N(b(n)t))} \times \frac{a(N(b(n)t))}{a(n)}$$

$$= \frac{a(N(b(n)t))}{a(n)} \bigvee_{k \le N(b(n)t)} \frac{X_k}{a(N(b(n)t))} = \frac{a(N(b(n)t))}{a(n)} Y_{N(b(n)t)}(1).$$

Taking into account that $Y_n(1) \stackrel{d}{\to} Y(1)$, as $n \to \infty$, the form of the sequence $a(.)$ and (2.2) it is easily verified that

$$\frac{a(N(b(n)t))}{a(n)} = \left(\frac{N(b(n)t)}{n}\right)^H \sim \left(\frac{N(b(n)t)}{\tilde{b}(b(n))}\right)^H \stackrel{d}{\to} \Lambda(t)^H, \quad t > 0.$$

In this way we proved that

$$\tilde{Y}(t) := Y(\Lambda(t)) \stackrel{\mathrm{d}}{=} \Lambda(t)^H X \stackrel{\mathrm{d}}{=} t^{\gamma H} \frac{X}{D^{\gamma H}} =: t^{\gamma H} \tilde{X}, \tag{2.13}$$

where X and $\Lambda(t)$ are independent and X is self-decomposable r.v. with characteristic function given by (2.11). Moreover, note that (2.13) implies

$$\tilde{Y}(ct) \stackrel{\mathrm{d}}{=} (ct)^{\gamma H} \tilde{X} \stackrel{\mathrm{d}}{=} c^{\gamma H} \tilde{Y}(t).$$

The last relation proves that the process $\tilde{Y}(t)$ is self-similar with exponent equal to γH. However, in contrast with Y, the subordinated process $\tilde{Y}$ does not have independent increments, since Λ does not have independent increments. Since Y does not have homogeneous increments, the process $\tilde{Y}$ does not have homogeneous increments too. The distribution function of $\tilde{Y}(t), t > 0$ is given by

$$\mathrm{P}(\tilde{Y}(t) < x) = \mathrm{P}(Y(\Lambda(t)) < x) = \int_0^\infty \mathrm{P}(Y(s) < x) d\mathrm{P}(\Lambda(t) < s)$$

$$= \int_0^\infty C^\alpha s^{\alpha\delta+1} x^{-\alpha} \frac{t}{\gamma} s^{-1-1/\gamma} g_\gamma(ts^{-1/\gamma}) ds = \frac{C^\alpha t x^{-\alpha}}{\gamma} \int_0^\infty s^{\alpha\delta-1/\gamma} g_\gamma(ts^{-1/\gamma}) ds.$$

Consider the sequence of sum processes

$$\tilde{S}_n(t) = \sum_{k=1}^{N_n(t)} X_{nk} = \sum_{T_k \le b(n)t} \frac{X_k}{a(n)}, \quad t > 0, \quad n = 1, 2, 3 \ldots$$

For this sequence of processes the conditions of Theorem 8, [19] are satisfied and one can state the convergence

$$\tilde{S}_n(\cdot) \Rightarrow S \circ \Lambda(\cdot) =: \tilde{S}(\cdot).$$

The characteristic function of the limit process $\tilde{S}(t)$ at time $t > 0$ has the form

$$\mathbb{E} e^{iz\tilde{S}(t)} = \mathbb{E} \exp\left[C^\alpha \Lambda(t)^{\alpha\delta+1} \int_0^\infty \left(e^{izx} - 1\right) \alpha x^{-\alpha-1} dx\right]$$

$$= \int_0^\infty \exp\left[C^\alpha s^{\alpha\delta+1} \int_0^\infty \left(e^{izx} - 1\right) \alpha x^{-\alpha-1} dx\right] d\mathrm{P}(\Lambda(t) < s)$$

$$= \int_0^\infty \exp\left[C^\alpha s^{\alpha\delta+1} \int_0^\infty \left(e^{izx} - 1\right) \alpha x^{-\alpha-1} dx\right] \frac{t}{\gamma} s^{-1-1/\gamma} g_\gamma(ts^{-1/\gamma}) ds.$$

Moreover, the characteristic functions of $\tilde{S}(ct)$ and $c^{\gamma H}\tilde{S}(t)$ coincide, thus, the process $\tilde{S}$ is also self-similar with exponent γH. The above results can be summarized in the following

Proposition 2.1. *Let the sequence of point processes (2.3) and associated with it sequences of extremal (2.4) and sum processes (2.5) be given. There exist weak limits $\tilde{Y}$ and $\tilde{S}$ of the two sequences, respectively. Both limit processes are self-similar with common self-similarity exponent. For $t > 0$, the distribution function of $\tilde{Y}(t)$ is given by*

$$\mathrm{P}(\tilde{Y}(t) < x) = \frac{C^\alpha t x^{-\alpha}}{\gamma} \int_0^\infty s^{\alpha\delta-1/\gamma} g_\gamma(ts^{-1/\gamma}) ds,$$

and the characteristic function of $\tilde{S}(t)$ is given by

$$\mathbb{E}e^{iz\tilde{S}(t)} = \int_0^\infty \exp\left[C^\alpha s^{\alpha\delta+1} \int_0^\infty \left(e^{izx}-1\right)\alpha x^{-\alpha-1}dx\right] \frac{t}{\gamma} s^{-1-1/\gamma} g_\gamma(ts^{-1/\gamma})ds.$$

g_γ is the probability density function of a one sided $\gamma-$stable r.v. D with Laplace transform $\mathbb{E}e^{-\lambda D} = e^{-\lambda^\gamma}, \quad \lambda > 0$.

3 Simulation

In the previous section the limit processes $\tilde{Y}$ and $\tilde{S}$ were described in terms of their distribution function and characteristic function, respectively. These expressions are difficult to use since both depend on the probability density g_γ which is not known in closed form except for $\gamma = 1/2$ (Levy distribution). In this section we provide a method for simulating the processes $\tilde{Y}$ and $\tilde{S}$ at a given fixed time $t > 0$. In order to do that we are going to use the properties of the accompanying processes Y and S, and the time process Λ.

From (2.10) immediately follows the Laplace transform of $S(t)$ for $t > 0$ is

$$\mathbb{E}e^{-\lambda S(t)} = \exp\left(C^\alpha t^{\alpha\delta+1} \int_0^\infty (e^{-\lambda x}-1)\alpha x^{-\alpha-1}dx\right).$$

Hence for the increment is in place

$$\mathbb{E}e^{-\lambda U_s(t_1,t_2]} = \exp\left[C^\alpha \left(t_2^{\alpha\delta+1} - t_1^{\alpha\delta+1}\right) \int_0^\infty (e^{-\lambda x}-1)\alpha x^{-\alpha-1}dx\right].$$

The above two equations and the form of the Laplace transform of an $\alpha-$stable subordinator $S_\alpha(\sigma, 1, 0)$ (see Definition 1.1.6 and Proposition 1.2.12, [21]) lead to the following

Proposition 3.1. *For $t > 0$ the r.v. $S(t)$ follows the stable distribution $S_\alpha(\sigma, 1, 0)$ with*

$$\sigma = C \times \left(t^{\alpha\delta+1} \times \Gamma(1-\alpha) \times \cos(\pi\alpha/2)\right)^{1/\alpha}.$$

Moreover, for $0 < t_1 < t_2$, the r.v. $U_s(t_1, t_2]$ follows the stable distribution

$S_\alpha(\sigma, 1, 0)$ *with*

$$\sigma = C \times \left(\left(t_2^{\alpha\delta+1} - t_1^{\alpha\delta+1}\right) \times \Gamma(1-\alpha) \times \cos(\pi\alpha/2)\right)^{1/\alpha}.$$

Since for $t > 0$, the distribution function $f(t, x) = \exp(-x^{-\alpha} C^\alpha t^{\alpha\delta+1})$, the r.v. $Y(t)$ follows the Frechet distribution with scale parameter equal to $Ct^{\delta+1/\alpha}$ and shift parameter equal to zero.

Using Proposition 3.1 and the stochastic representation $\Lambda(t) \stackrel{\mathrm{d}}{=} (D/t)^{-\gamma}$ we can give two expressions for the r.v.'s $\tilde{S}(t)$ and $\tilde{Y}(t)$, for fixed $t > 0$.

Proposition 3.2. *For the random variables $\tilde{S}(t), t > 0$ and $\tilde{Y}(t), t > 0$ the following stochastic representations hold:*

$$\tilde{S}(t) \stackrel{\mathrm{d}}{=} (D/t)^{-\gamma H} S_1,$$
$$\tilde{Y}(t) \stackrel{\mathrm{d}}{=} (D/t)^{-\gamma H} Y_1.$$

The random variable D follows the stable distribution$S_\gamma(\sigma_\gamma, 1, 0)$ with $\sigma_\gamma = (\cos(\pi\gamma/2))^{1/\gamma}$, the random variable S_1 follows the stable distribution $S_\alpha(\sigma_\alpha, 1, 0)$ with $\sigma_\alpha = C \times (\Gamma(1-\alpha) \times \cos(\pi\alpha/2))^{1/\alpha}$, and the random variable Y_1 follows the Frechet distribution with scale parameter $C > 0$ and location parameter 0.

Proposition 3.2 provides an efficient method for simulation of the random variables $\tilde{S}(t), t > 0$ and $\tilde{Y}(t), t > 0$. This is extremely important since the quality of the higher quantiles estimates strongly depends on the number of scenarios generated. This statement is confirmed by the simulation reported in the following section.

4 Monte Carlo Quantile Estimates

In this section the results from a simulation study are presented. We generate N scenarios for the approximating processes at time $t = 1$, i.e. one year.

The period of one year is chosen since it is a regulatory requirement for operational risk estimation. Based on these simulations, we estimate several quantiles, including 99.9%, of the aggregate and maximum loss distributions. The experiment is repeated 200 times in order to find the mean and the variance of the estimates. The importance of the number of scenarios will be presented using two different values of N and comparing the results.

Tables 1 and 2 contain the means and standard deviations of the quantile estimates of the aggregate and maximum loss distributions. The value of the tail indices α and γ is 0.95, $\delta = 0.0001$. The parameter C is chosen as \$1 million because this is the lower threshold for external databases. The results are based on 200 samples with size 10,000, i.e. $N = 10{,}000$. As expected, the estimates of the most extreme quantiles in Tables 1 and 2 have very high standard deviations. It is clear that the estimate for the 99.9% quantile based on 10,000 scenarios is unreliable.

Taking advantage of the expressions in Proposition 3.2 we generate 10 million scenarios in approximately 25 seconds using Pentium M 2.0GHz processor. The algorithm is based on the Chambers-Mallows-Stuck method for stable random variables generation (see Chambers et al. [2]).

The results obtained using $N = 10$ million scenarios are given in Tables 3 and 4 for the aggregate and maximum loss, respectively. The improvement is significant for the $q_{99.9\%}$ estimate of the aggregate loss distribution. The standard deviation of this estimate is equal to 1.030% of its mean; that is, 10 million scenarios provide a low variance estimate given the input parameters of the model.

Although the estimate of the 99.9% quantile based on 10 million scenarios has a small variance, it depends on the input parameters. Hence, biases in the estimates of the model parameters can lead to a bias in the estimated quantile. The less the value of α, the heavier the tail of the losses is. On the other hand, the less the value of γ, the rarer the individual losses are. Figure 1 gives the

99.9% quantile as a function of the two parameters α and γ. Note that in the region where α is small and γ is high the difference in the quantiles is more pronounced. Figure 2 shows a part of the surface given in Figure 1 since the area where $\alpha > 0.7$ looks like a plane compared to the area where $\alpha < 0.7$.

Figures 3 and 4 show the sensitivity of the 99.9% quantile with respect to α given the value of γ is 0.90. Again two different scales for α are chosen for better visualization. The figures indicate that one should be very careful in estimation of the parameter α since its value seriously affects the 99.9% quantile.

5 Conclusions

This paper develops new approximations of the maximum and aggregate loss processes when the losses are very rare but their amount is very large. The assumption that the loss amounts are i.i.d. is relaxed and these amounts are assumed to follow a Pareto distribution with different parameters. In this way, the minimum amount of the consecutive losses increases over time. The method is based on time and space changes of the original maximum and aggregate loss processes. The weak limits of the sequences of transformed processes are obtained. The limits are found in two steps. First the limits of the sequences of the so-called accompanying processes are achieved. Then the limits of the initial sequences are obtained by subordination of the respective accompanying sequences limits with a random time change.

Simulation algorithms for the approximating processes at any fixed $t > 0$ are derived. Monte Carlo simulations for both the aggregate loss and extremal loss processes at time $t = 1$ are generated. Estimates for several quantiles of the aggregate and maximum losses are compared. It turns out that the Monte Carlo method used gives fast and efficient higher quantile estimates when 10 million simulations are drawn.

References

[1] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, (1987).

[2] J.M. Chambers, C.L. Mallows and B.W. Stuck, *A method for simulating stable random variables*, J. Amer. Statist. Assoc. 71, 340–344, (1976).

[3] V. Chavez-Demoulin and P. Embrechts, *Advanced extremal models for operational risk*, Tech. rep., ETH Zurich, (2004).

[4] A. Chernobai, S. Rachev and F. Fabozzi, *Operational risk: A guide to Basel II capital requirements, models, and analysis*, Wiley Finance Series, John Wiley & Sons, Hoboken, New Jersey, (2007).

[5] A. Chernobai, C. Menn, S. Rachev and S. Trück, *Estimation of the operational value-at-risk in the presence of minimum collection thresholds*, Tech. Rep., University of California, Santa Barbara, (2005a).

[6] A. Chernobai, C. Menn, S. Rachev and S. Trück, *A note on the estimation of the frequency and severity distribution of operational losses*, Mathematical Scientist, 30 (2), (2005b).

[7] M. Cruz, *Modeling, Measuring and Hedging Operational Risk*, John Wiley & Sons, Chichester ,New York, (2002).

[8] P. De Fontnouvelle, E. Rosengren and J. Jordan, *Implications of alternative operational risk modelling techniques*, Tech. rep., Federal Reserve Bank of Boston and FitchRisk, (2004).

[9] P. Embrechts, H. Furrer, and R. Kaufmann, *Quantifying regulatory capital for operational risk*, Derivatives Use, Trading, and Regulation 9 (3), 217-233, (2003).

[10] P. Embrechts, P. Kaufmann and G. Samorodnitsky, *Ruin theory revisited: Stochastic models for operational risk*, Risk Management for Central Bank Foreign Reserves (Eds. C. Bernadell et al), European Central Bank, Frankfurt, 243-261, (2004).

[11] P. Embrechts, C. Kluppelberg and T. Mikosch, *Modelling Extremal Events: for Insurance and Finance*, Springer-Verlag, Berlin, (1997).

[12] E. Medova, *Operational risk capital allocation and integration of risks*, Tech. rep.,Center for Financial Research, University of Cambridge,(2002).

[13] M. M. Meerschaert and H.-P. Scheffler, *Limit theorems for continuous-time random walks with infinite mean waiting times*, J. Appl. Prob. 41, 623–638, (2004).

[14] M. Moscadelli, *The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee*, Technical Report 517, Banca d'Italia, (2004).

[15] J. Nešlehová, P. Embrechts and V. Chavez-Demoulin, *Infinite-mean models and the LDA for operational risk*, Journal of Operational Risk, 1(1), 3–25, (2006).

[16] E. Pancheva and P. Jordanova, *A functional extremal criterion*, Journal of Math. Sci., Vol.121, No. 5,2636–2644, (2004).

[17] E. Pancheva and P. Jordanova, *Functional transfer theorems for maxima of i.i.d. random variables*, Comptes rendus de l'Acade'mie bulgare des Sciences, Vol. 57, No. 8. 9–14, (2004).

[18] E. Pancheva, I. Mitov and Z. Volkovich, *Sum and extremal processes over explosion area*, Comptes rendus de l'Acade'mie bulgare des Sciences, Vol. 59, No. 12. 19–26, (2006).

[19] E. Pancheva, I. Mitov and Z. Volkovich, *Relationship between extremal and sum processes generated by the same point process*, (working paper)

[20] S. Rachev and S. Mittnik, *Stable Paretian Models in Finance*, John Wiley & Sons, New York, (2000).

[21] G. Samorodnitsky M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall/CRC, (1994).

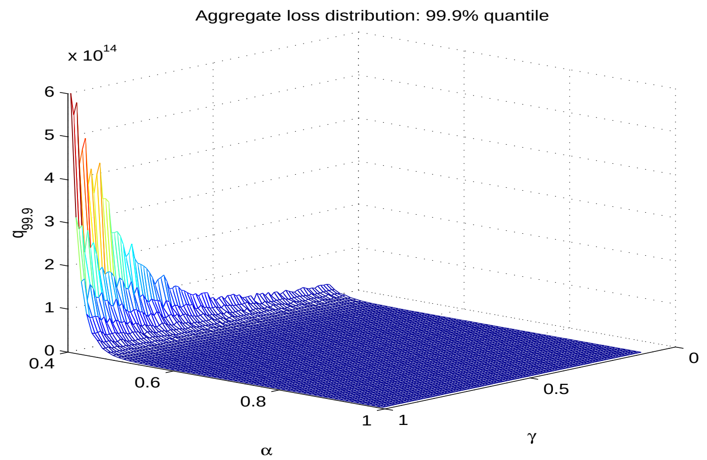

Figure 1: The 99.9% aggregate loss quantile as a function of α and γ.

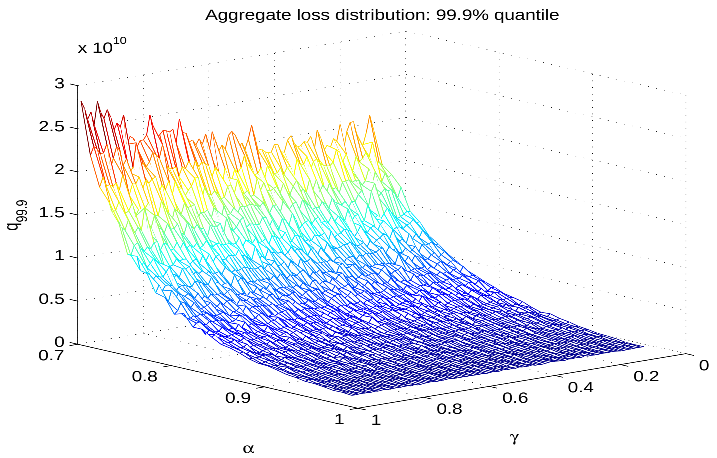

Figure 2: The 99.9% aggregate loss quantile as a function of α and γ.

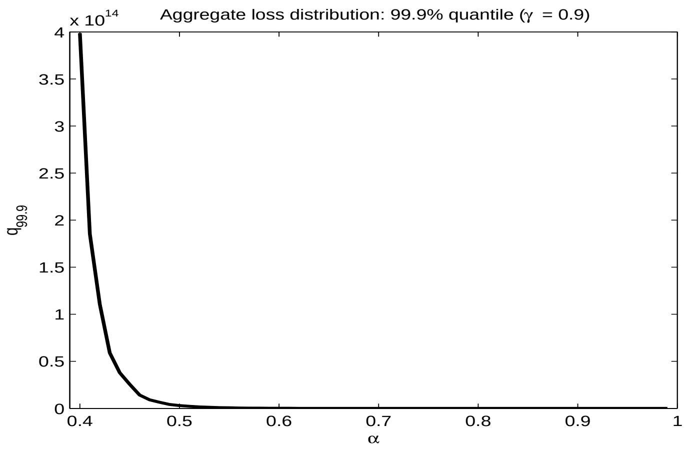

Figure 3: The 99.9% aggregate loss quantile as a function of α for $\gamma = 0.90$.

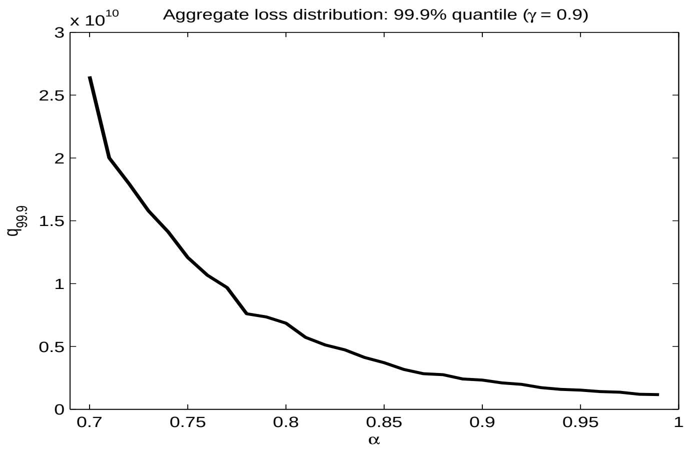

Figure 4: The 99.9% aggregate loss quantile as a function of α for $\gamma = 0.90$.

Table 1: Aggregate loss quantiles ($ million): $N = 10{,}000$.

Quantile	Mean ($\hat{m}$)	Std. ($\hat{\sigma}$)	$\hat{\sigma}/\hat{m}$
$q_{99\%}$	157.824	15.068	9.548%
$q_{99.5\%}$	301.803	40.540	13.433%
$q_{99.9\%}$	1651.119	594.926	36.032%

Table 2: Maximum loss quantiles ($ million): $N = 10{,}000$.

Quantile	Mean ($\hat{m}$)	Std. ($\hat{\sigma}$)	$\hat{\sigma}/\hat{m}$
$q_{99\%}$	5.383	0.132	2.455%
$q_{99.5\%}$	6.315	0.198	3.142%
$q_{99.9\%}$	8.428	0.392	4.656%

Table 3: Aggregate loss quantiles ($ million): $N = 10$ million.

Quantile	Mean ($\hat{m}$)	Std. ($\hat{\sigma}$)	$\hat{\sigma}/\hat{m}$
$q_{99\%}$	156.070	0.444	0.285%
$q_{99.5\%}$	296.770	1.296	0.437%
$q_{99.9\%}$	1499.200	15.449	1.030%

Table 4: Maximum loss quantiles ($ million): $N = 10$ million.

Quantile	Mean ($\hat{m}$)	Std. ($\hat{\sigma}$)	$\hat{\sigma}/\hat{m}$
$q_{99\%}$	5.396	0.004	0.075%
$q_{99.5\%}$	6.300	0.0059	0.094%
$q_{99.9\%}$	8.447	0.0138	0.164%