The Theory of Orderings and Risk Probability Functionals

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Abstract

This paper studies and describes stochastic orderings of risk/reward positions in order to define in a natural way risk/reward measures consistent/isotonic to investors' preferences. We begin by discussing the connection among the theory of probability metrics, risk measures, distributional moments, and stochastic orderings. Then, we demonstrate how further orderings could better specify the investor's attitude toward risk. Finally, we extend these concepts in a dynamic context by defining and describing new risk measures and orderings among stochastic processes with and without considering the available information in the market.

Key words: Probability metrics, stochastic dominance, coherent risk measures, FORS orderings, Mellin transform, coherent and convex measures, RCLL processes.

JEL Classification: G11, C44, C61

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1. Introduction

In this paper we describe the admissible classes of probability functionals that are consistent with a given order of preferences. To classify the orderings and risk probability functionals, we distinguish between primary and compound probability functionals; between static and dynamic orderings/measures; between uncertainty and risk orderings/measures; between orderings and survival/dual orderings; and between bounded and unbounded orderings. By doing so, we present a general and unifying framework to understand the connections between the investor's preferences that are consistent to a given order and choice problem.

We first discuss the links between continua stochastic dominance orders, dual stochastic dominance rules based on Lorenz orders, and the different distributional moments of a portfolio of returns (see Fishburn (1976, 1980a) and Muliere and Scarsini (1989)). We tie together the consistency-isotonicity of risk and reward measures with the classical orderings. We study the properties that a probability functional must satisfy to solve optimal choice problems. The theory of probability functionals and metrics was developed by Zolotarev and his students to solve stability problems (see Rachev (1991) and the references therein). Furthermore, a strong connection exists between probability functionals and orderings (see, among others, Kakosyan et al (1987), Kalashnikov and Rachev (1988), and Rachev and Ruschendorf (1998, 1999)).

In the first part of this paper, we discuss the static approach to the theory of choice under risk and uncertainty. In particular, we are interested in the economic use of probability functionals to optimize choices for a given order of investors' preferences. From this discussion we propose a new set of orderings, risk and reward measures that are coherent to the investors' choices. The new probability functionals and orderings generalize those found in the literature and are strictly related to the theory of choice under uncertainty (see, among others, Von Neumann and Morgenstern (1953), Machina (1982), Yaari (1987) Gilboa and Schmeidler (1989), and Maccheroni et al (2005)) and to theory of probability functionals and metrics (see Rachev (1991) and the references therein). While the new orderings serve to further characterize and specify the investors' choices/preferences, the new risk measures should be used either to minimize the risk of a portfolio of financial assets or to minimize its distance to a given benchmark (see Rachev et al. (2005), Stoyanov et al (2006), and Ortobelli et al (2006)). We will call these new measures/orderings "FORS measures/orderings." We show how one can generate further orderings and measures by using the Mellin transform when applied to the fractional integral.

In the second part of the paper we extend the results to continuous time. Since 1999, several papers have extended the concept of risk to a dynamic framework (see, among others, Civitanic

Karatzas (1999), Artzner et al (2002), Frittelli and Rosazza Gianin (2004), Riedel (2004), Pflug and Ruszczynski (2004), Detlefsen, Scandolo (2005), and Cheridito et al (2004, 2005)). However, most of these studies focus on the monotony order (X>Y) among random variables and processes. We generalize the basic concepts of orderings in a dynamic context considering any admissible order of preferences. As discussed by Artzner et al (2002) and Detlefsen and Scandolo (2005), a dynamic setting requires (1) accounting for the availability of additional information, (2) monitoring risk continuously, and (3) searching for the occurrence of intermediate payoffs. In particular, we distinguish between two classes of dynamic probability functionals: one for random variables and one for stochastic processes. With dynamic probability functionals for random variables, we focus on the first two requirements, whereas probability functionals for stochastic processes concerns the third. Once the main properties of dynamic probability functionals are defined, we can explain the properties and characteristics of dynamic orderings.

In the next section, we examine continua and dual stochastic dominance and their connection with the distributional moments of portfolios. In section 3, we describe how to use probability functionals to define new orderings and portfolio risk measures. In section 4 we generalize the results to a multi-period context. Finally, we briefly summarize the results.

2. Continua and Dual Stochastic Dominance Theory

In this section, we study the stochastic orders in a complete probability space $(\Omega, \Im, P)$. By doing so, we take the perspective of an investor who wants to solve a portfolio selection problem.

In particular, we denote with $L^0(\Im)$ the space of all real valued random variables defined on $(\Omega, \Im, P)$ while $L^p(\Im) = \left\{ X : (\Omega, \Im, P) \to R \,/\, E\left(|X|^p\right) < +\infty \right\}$. Recall that X dominates Y with respect to n (integer) order stochastic dominance ($X \underset{n}{\geq} Y$) if and only if E(u(X))≥E(u(Y)) for every utility function u whose derivatives satisfy the inequalities $(-1)^{k+1}u^{(k)} \geq 0$ for k=1,…,n, if and only if for every real t we have $F_X^{(n)}(t) = \int_{-\infty}^{t} F_X^{(n-1)}(u)du \leq F_Y^{(n)}(t) = \int_{-\infty}^{t} F_Y^{(n-1)}(u)du$. Furthermore, we observe that for any m>n, $X \underset{n}{\geq} Y$ implies $X \underset{m}{\geq} Y$. In addition, we state that X dominates Y in the sense of Rothschild and Stiglitz ($X\ R-S\ Y$) if and only if E(u(X))≥E(u(Y)) for every concave utility function u, if and only if $X \underset{2}{\geq} Y$ and E(X)=E(Y) (see Rothschild and Stiglitz (1970)). This order is also called concave order in the ordering literature (see, among others, Shaked and Shanthikumar

(1993) and Muller and Stoyan (2002)). Moreover, all these relations can be easily generalized in continuous terms.

2.1 Continua, survival, bounded/unbounded stochastic dominance rules

Fishburn (1976, 1980a) considers continuous orders applied to bounded and unbounded random variables. In the following, we further characterize and generalize these orders. This extension is possible because the first lemma found in Fishburn (1976) is still valid following the Fubini-Tonelli Theorem (see also Miller and Ross (1993) and Zhang and Jin (1996)).

Lemma 1 *For all* $\alpha>0$, $v>0$ *such that* $w<z$; $z,w\in R$, $\int_w^z (z-x)^{v-1}(x-w)^{\alpha-1}dx = B(\alpha,v)(z-w)^{\alpha+v-1}$ *where* $B(\alpha,v)=\dfrac{\Gamma(\alpha)\Gamma(v)}{\Gamma(\alpha+v)}$ *and* $\Gamma(t)=\int_0^{+\infty} z^{t-1}e^{-z}dz$. *Assume the real space* (R,B_R,μ) *with the Borel sigma algebra* B_R *and the positive measure* μ *is sigma finite. Then*

$$\int_a^z (z-x)^{v-1}\left(\int_a^{x^-}(x-y)^{\alpha-1}d\mu(y)\right)dx = B(\alpha,v)\int_a^{z^-}(z-y)^{\alpha+v-1}d\mu(y),$$

$$\int_a^z (x-a)^{v-1}\left(\int_{x^+}^{z}(y-x)^{\alpha-1}d\mu(y)\right)dx = B(\alpha,v)\int_{a^+}^{z}(y-a)^{\alpha+v-1}d\mu(y)$$

for any $-\infty\le a<z\le+\infty$. *These relations are still valid if* α,v *are complex numbers with* $\operatorname{Re}\alpha,\operatorname{Re}v>0$ *and the* $B(\alpha,v)$ *is the beta function with complex arguments.*

We assume $F_X^{(1)}=F_X$ and $a=\inf\{x/F_X(x)>0\}$. Using the definition of fractional integral (see Erdelyi and McBride (1970) and Miller and Ross (1993)), we obtain for every real $\alpha>0$ and $\alpha\neq 1$

$$F_X^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\int_a^{t^-}(t-y)^{\alpha-1}dF_X(y)=\frac{E\left((t-X)_{++}^{\alpha-1}\right)}{\Gamma(\alpha)} \text{ for every } t>a\text{ , } F_X^{(\alpha)}(t)=0\text{, }\forall t\le a \qquad (1)$$

where $(t-x)_{++}^{\alpha-1}=(t-x)^{\alpha-1}I_{[x<t]}$ and $I_{[x<t]}$ is the indicator function equal to 1 if $x<t$, and equal to zero otherwise. Thus, $F_X^{(\alpha)}$ is a positive continuous function for $\alpha>1$; it is right continuous for $\alpha=1$ and left continuous for $\alpha\in(0,1)$. A slightly different definition (see Fishburn (1980a)) is necessary for $\alpha\in(0,1)$ in order to include the probability measures that satisfy $P(X=t_i)>0$ for some real numbers t_i. Analogously, we can use the survival function $\bar{F}_X^{(1)}(x)=P(X>x)=1-F_X(x)$ and we obtain for every real $\alpha>0$ and $\alpha\neq 1$:

$$\bar{F}_X^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (y-t)^{\alpha-1}dF_X(y)=\frac{E\left(\left(X-t\right)_+^{\alpha-1}\right)}{\Gamma(\alpha)} \text{ for every } t<b\text{, and } \bar{F}_X^{(\alpha)}(t)=0,\ \forall t\geq b \quad (2)$$

where $b=\sup\{x/F_X(x)<1\}$. In this case, $\bar{F}_X^{(\alpha)}(x)$ is right continuous $\forall\alpha\in(0,1]$ and it is continuous for $\alpha>1$. In particular, when *X* is a continuous random variable, $\bar{F}_X^{(\alpha)}(u)=F_{-X}^{(\alpha)}(-u)$ for every $u\in[a,b]$ and $\alpha>0$. From the lemma above, if μ is the probability measure obtained by the right continuous distribution function of *X*, $F_X^{(1)}(y)=F_X(y)=\mu(y)$ or by the survival function $\mu(y)=\bar{F}_X^{(1)}(y)$, we obtain

$$\bar{F}_X^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha-\nu)}\int_t^b (u-t)^{\alpha-\nu-1}\bar{F}_X^{(\nu)}(u)du \text{ for every } \alpha>\nu>0 \quad (3)$$

$$F_X^{(\alpha)}(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha-\nu)}\displaystyle\int_a^{t^-}(t-u)^{\alpha-\nu-1}F_X^{(\nu)}(u)du & \forall\alpha>\nu\geq 1\vee\forall 1>\alpha>\nu>0\\ \lim\limits_{t_n\searrow t}\dfrac{1}{\Gamma(\alpha-\nu)}\displaystyle\int_a^{t_n}\left(t_n-u\right)^{\alpha-\nu-1}F_X^{(\nu)}(u)du & \forall\alpha\geq 1>\nu>0\end{cases}. \quad (4)$$

We can define stochastic orders as it follows:

Definition 1 *For every* $\alpha>0$*, we state X dominates Y with respect to* α *bounded stochastic dominance order* ($X\underset{\alpha}{\overset{b}{\geq}}Y$) *iff* $F_X^{(\alpha)}(t)\leq F_Y^{(\alpha)}(t)$ *for every t belonging to* $supp\{X,Y\}\equiv[a,b]$ *where* $a,b\in\bar{R}$ *and* $a=\inf\{x/F_X(x)+F_Y(x)>0\}$, $b=\sup\{x/F_X(x)+F_Y(x)<2\}$. *We state that X strictly dominates Y with respect to the* α *bounded order* (*namely* $X\underset{\alpha}{\overset{b}{>}}Y$) *iff* $X\underset{\alpha}{\overset{b}{\geq}}Y$ *and* $F_X\neq F_Y$.

Moreover following Fishburn (1980a): for every $\alpha\geq 1$, *X* dominates *Y* with respect to the α stochastic dominance order ($X\underset{\alpha}{\geq}Y$) iff $F_X^{(\alpha)}(t)\leq F_Y^{(\alpha)}(t)$ for every real *t*. *X* strictly dominates *Y* with respect to the α order (namely $X\underset{\alpha}{>}Y$) iff $X\underset{\alpha}{\geq}Y$ and $F_X\neq F_Y$.

Since $X\underset{\alpha}{\geq}Y$ ($X\underset{\alpha}{\overset{b}{\geq}}Y$) is simply $X\underset{\alpha}{>}Y$ ($X\underset{\alpha}{\overset{b}{>}}Y$) plus the identity relation, we shall consider only $\underset{\alpha}{>}$ ($\underset{\alpha}{\overset{b}{>}}$) explicitly. As proven by Fishburn (1980a), it is equivalent to say that $X\underset{\alpha}{>}Y$ or $X\underset{\alpha}{\overset{b}{>}}Y$ if and only if $\alpha\in[1,2]$ and *X, Y* admit finite expected values. However, when $\alpha\notin[1,2]$, the orders $\underset{\alpha}{>}$, $\underset{\alpha}{\overset{b}{>}}$ do not generally coincide. Similarly we define the survival (un)bounded ($X\underset{sur\,\alpha}{\overset{a}{\geq}}Y$) order iff $\bar{F}_X^{(\alpha)}(t)\leq\bar{F}_Y^{(\alpha)}(t)$ for every *t* that belongs to $supp\{X,Y\}\equiv[a,b]$ (unbounded order: $X\underset{sur\,\alpha}{\geq}Y$ iff $\bar{F}_X^{(\alpha)}(t)\leq\bar{F}_Y^{(\alpha)}(t)$ for every real t). We prefer to concentrate on stochastic dominance orders because when $\alpha > 1$ $\overline{F}_X^{(\alpha)}(u) = E\left(\left(X-u\right)_+^{\alpha-1}\right)/\Gamma(\alpha) = F_{-X}^{(\alpha)}(-u)$ and the results are equivalent to those obtained for orders on random variable inverses.

Even if Definition 1 generalizes the orders proposed by Fishburn (1976, 1980a) to α bounded orders with $\alpha \in (0,1)$ that imply the first stochastic dominance, in many cases we cannot compare random variables respect to these orders. In particular, if $X \overset{b}{\underset{\alpha}{>}} Y$ with $\alpha \in (0,1)$, a point t$< \sup\left\{x / F_X(x) + F_Y(x) < 2\right\}$ such that $0 < P\left(Y = t\right) < P\left(X = t\right)$ cannot exist because in the right neighborhood of *t* we have $F_X^{(\alpha)}(t^+) > F_Y^{(\alpha)}(t^+)$. In addition, as it follows by this proposition, we cannot express α order $X \underset{\alpha}{>} Y$ for any $\alpha \in (0,1)$.

Proposition 1 *For any pair of bounded* (*from above or/and from below*) *random variables X and Y that are continuous on the extremes of their support, there is no* $\alpha \in (0,1)$ *such that* $F_X^{(\alpha)}(t) \le F_Y^{(\alpha)}(t)$ $\forall t \in supp(X,Y)$. *In addition, for any pair of random variables X and Y, there no* $\alpha \in (0,1)$ *such that* $F_X^{(\alpha)}(t) \le F_Y^{(\alpha)}(t)$ *for every real t.*

The proof of this proposition and of the results, as well as the four other propositions presented later in the paper, can be found in the appendix.

Due to this proposition we cannot compare random variables according to α bounded order with $\alpha \in (0,1)$ except in a few cases. However, although α bounded orders with $\alpha \in (0,1)$ are not applicable in many cases, they could serve to rank financial losses and truncated variables. This is why this generalization could be interesting from a financial point of view. Typically, for every pair of random variables *X* and *Y,* with density of probability such that $f_X(t) \le f_Y(t)$, $\forall t < M$ and $P(X \le M) = P(Y \le M) = 1$, we have $F_X^{(\alpha)}(t) \le F_Y^{(\alpha)}(t)$ $\forall t \in supp\left\{X,Y\right\}$, and $X \overset{M}{\underset{\alpha}{>}} Y$ for every $\alpha > 0$.

The following examples show the use of α order for truncated variables.

Example 1: Let Y_1 and Y_2 be two financial losses with truncated Gaussian distribution functions:

a) $F_{Y_1}(t) = \dfrac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2} dx$ for $t < \mu < +\infty$ is equal to a Gaussian $N(\mu,\sigma)$ and $F_{Y_1}(t) = 1$ for $t \ge \mu$;

b) $F_{Y_2}(t) = \dfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\left(\frac{x-\mu-\varepsilon}{2\sigma}\right)^2} dx$ for $t < \mu < +\infty$ is equal to a Gaussian $N(\mu+\varepsilon,\sigma)$ with $\varepsilon > 0$ and $F_{Y_1}(t) = 1$ for $t \geq \mu$.

Since the probability density of the two losses are $f_{Y_2}(t) \leq f_{Y_1}(t)$, $\forall t < \mu$ and $P(Y_1 \leq \mu) = P(Y_2 \leq \mu) = 1$, $F_{Y_2}^{(\alpha)}(t) \leq F_{Y_1}^{(\alpha)}(t)$ $\forall t \in supp\{Y_1, Y_2\}$, and $Y_2 \underset{\alpha}{\overset{\mu}{>}} Y_1$ for every $\alpha > 0$. Therefore all investors would prefer loss Y_1 to Y_2. □

Example 2: Let X_1 and X_2 be two random variables with distribution functions: $X_i = \begin{cases} -u_i & probability \;\; p_i \\ u_i & probability \;\; (1-p_i) \end{cases}$ where $u_i > 0; p_i \in (0,1)$ i=1,2. Then, for $\alpha = 1$ we obtain $F_{X_i}^{(\alpha)}(t) = \begin{cases} 0 & if \;\; t < -u_i \\ p_i & if \;\; t \in [-u_i, u_i) \\ 1 & if \;\; t \geq u_i \end{cases}$ and for every $\alpha \neq 1$, $\alpha > 0$ we have:

$$\bar{F}_{X_i}^{(\alpha)}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left((-u_i - t)^{\alpha-1} p_i + (1-p_i)(u_i - t)^{\alpha-1}\right) & if \;\; t < -u_i \\ \frac{1}{\Gamma(\alpha)}(u_i - t)^{\alpha-1}(1-p_i) & if \;\; t \in [-u_i, u_i) \\ 0 & if \;\; t \geq u_i \end{cases}, \quad F_{X_i}^{(\alpha)}(t) = \begin{cases} 0 & if \;\; t \leq -u_i \\ \frac{1}{\Gamma(\alpha)}(t+u_i)^{\alpha-1} p_i & if \;\; t \in (-u_i, u_i] \\ \frac{1}{\Gamma(\alpha)}\left((t+u_i)^{\alpha-1} p_i + (1-p_i)(t-u_i)^{\alpha-1}\right) & if \;\; t > u_i \end{cases}.$$

Therefore, when $p_2 < p_1$, $u_1 = u_2$ we can see that $F_{X_2}^{(\alpha)}(t) \leq F_{X_1}^{(\alpha)}(t)$ and $\bar{F}_{X_2}^{(\alpha)}(t) \geq \bar{F}_{X_1}^{(\alpha)}(t)$ $\forall t \in [-u_1, u_1]$ for every $\alpha > 0$ and $X_2 \underset{\alpha}{\overset{u_1}{>}} X_1$ and $-X_1 \underset{\alpha}{\overset{u_1}{>}} -X_2$ $\forall \alpha > 0$. If $p_2 \leq p_1$, $u_1 > u_2 > 0$, the random variables cannot be compared according to α order with $\alpha \in (0,1)$. However, $\forall \alpha \in (1,2)$, if $p_2 \leq p_1$, $u_1 > u_2 > 0$, $F_{X_2}^{(\alpha)}(t) \leq F_{X_1}^{(\alpha)}(t)$ $\forall t \in [-u_1, u_1]$ when $\dfrac{1}{2} \leq \dfrac{u_1 p_1^{1/(\alpha-1)}}{u_1 - u_2} - \dfrac{u_2 p_2}{u_1 - u_2}$ and for every $p_2 \geq p_1$, $u_1 > u_2 > 0$ $\bar{F}_{X_2}^{(\alpha)}(t) \leq \bar{F}_{X_1}^{(\alpha)}(t)$ $\forall t \in [-u_1, u_1]$ when $\dfrac{1}{2} \leq \dfrac{u_1(1-p_1)^{1/(\alpha-1)}}{u_1 + u_2} + \dfrac{u_2 p_2}{u_1 + u_2}$. For example, if $u_2 = 1; u_1 = 2$ and $p_2 = \dfrac{1}{2}$, then $X_2 \underset{\alpha}{\overset{u_1}{>}} X_1$ for $\alpha \in (1,2)$ when $p_1 \geq \left(\dfrac{1}{2}\right)^{\alpha-1}$ and $-X_1 \underset{\alpha}{\overset{u_1}{>}} -X_2$ for $\alpha \in (1,2)$ if $p_1 \leq 1 - \left(\dfrac{1}{2}\right)^{\alpha-1}$. □

In addition, equation (4) extends the use of functions $F_X^{(\alpha)}$. From a practical point of view, this has an immediate effect as shown in the following remark:

Remark 1 *The following hold:*

1) $X \underset{\alpha}{>} Y$ *implies* $X \overset{b}{\underset{\alpha}{>}} Y$. *These orders coincide if and only if* $\alpha \in [1,2]$. *Therefore every outcome of* $\overset{b}{\underset{\alpha}{>}}$ *order is true when* $\underset{\alpha}{>}$ *holds, but the converse it is not generally true if* $\alpha \notin [1,2]$.

2) Every $\alpha > v > 0$ *if* $F_X^{(v)}(t) \le F_Y^{(v)}(t)$ $\forall t \in supp\{X,Y\}$ *implies* $F_X^{(\alpha)}(t) \le F_Y^{(\alpha)}(t)$ $\forall t \in supp\{X,Y\}$. *In particular, the order* $X \overset{b}{\underset{v}{>}} Y$ *(* $X \underset{v}{>} Y$ *if* $v \ge 1$*) implies the order* $X \overset{b}{\underset{\alpha}{>}} Y$ *(* $X \underset{\alpha}{>} Y$ *).*

3) $X \overset{b}{\underset{1}{>}} Y$ *if and only if* $X_-^{(M)} \overset{M}{\underset{1}{\ge}} Y_-^{(M)}$, $X_+^{(M)} \overset{b}{\underset{1}{\ge}} Y_+^{(M)}$ *and at least one dominance is strict, where* $X = X_+^{(M)} + X_-^{(M)}$ *and* $X_+^{(M)} = XI_{[X \ge M]}; X_-^{(M)} = XI_{[X<M]}$ *for every* $M \in supp\{X,Y\}$. *In addition,* $X \overset{b}{\underset{\alpha}{\ge}} Y$ *with* $\alpha > 1$ *implies* $X_-^{(M)} \overset{M}{\underset{\alpha}{\ge}} Y_-^{(M)}$ *for any given* $M \in supp\{X,Y\}$.

4) $X \underset{\alpha}{\ge} Y$ *(* $X \overset{b}{\underset{\alpha}{\ge}} Y$ *) if and only if* $cX + t \underset{\alpha}{\ge} cY + t$ *(* $cX + t \overset{cb+t}{\underset{\alpha}{\ge}} cY + t$ *) for every* $t \in R$*, c>0,* $\alpha > 0$.

There is a strong connection between moments and stochastic orders as many authors have pointed out (see, among others, Fishburn (1980b) and O'Brien (1984)). The following proposition summarizes some of these results and provides necessary conditions on moments for α stochastic orders. It is interesting to observe that generally these implications do not always hold when we consider α bounded stochastic orders.

Proposition 2 *The following implications hold:*

a) *Suppose* $X \underset{\alpha}{>} Y$ *and the moments of X and Y through integer n are finite where* $n-1 < \alpha \le n$. *Then* $\left(E(X),...,E(X^n)\right) \ne \left(E(Y),...,E(Y^n)\right)$ *and* $(-1)^{k+1}E(X^k) > (-1)^{k+1}E(Y^k)$ *for minimum integer k at which* $E(X^k) \ne E(Y^k)$.

b) *If* $X \overset{b}{\underset{\alpha}{\ge}} Y$ *with* $\alpha > 1$ *implies*

$$\frac{1}{2}E\left(\left|X_1 - X_2\right|^{\alpha-1}\right) \le E\left(\left(X_1 - Y_1\right)_+^{\alpha-1}\right),\ E\left(\left(Y_1 - X_1\right)_+^{\alpha-1}\right) \le \frac{1}{2}E\left(\left|Y_1 - Y_2\right|^{\alpha-1}\right) \qquad (5)$$

where Y_1, Y_2 *are independent realizations of Y,* X_1, X_2 *are independent realizations of X and* X_1, Y_1 *are independent.*

As for integer orders, we can characterize stochastic orders with respect to a given class of utility functions. In particular, as the result of the previous Lemma and of Fishburn (1976, 1980a), we observe that $\underset{\alpha}{\geq}$ is a reflexive and transitive preorder, while $\underset{\alpha}{>}$ is a strict partial order (asymmetric and transitive) on space

$$\tilde{L}^{\alpha-1} = \begin{cases} X / E\left(|X|^{\alpha-1}\right) < +\infty & if \quad \alpha \neq 1, \alpha > 0 \\ all\, r.v. \ X & if \quad \alpha = 1 \end{cases}.$$

Moreover, for every pair of random variables $X, Y \in \tilde{L}^{\alpha-1}$, $X \underset{\alpha}{\geq} Y$ if and only if $E\left(u(X)\right) \geq E(u(Y))$ for all utility functions u that belong to

$$U_\alpha = \left\{ u(x) = c - \int_{x^+}^{+\infty} (y-x)^{\alpha-1} dv(y) \middle| c, x \in R; \text{ where } v \text{ is positive } \sigma - \text{finite measure } \int_{-\infty}^{+\infty} |y|^{\alpha-1} dv(y) < \infty \right\}.$$ [9]

In particular, for every random variable $X \in \tilde{L}^{\alpha-1}$, all utility functions $-\Gamma(\alpha)\bar{F}_X^{(\alpha)}(x) = -\int_{x^+}^{+\infty} \left(t-x\right)^{\alpha-1} dF_X(t) = -E\left(\left(X-x\right)_+^{\alpha-1}\right)$ belong to U_α. Similarly for every pair of random variables $X, Y \in \tilde{L}^{\alpha-1}$ with support on [a,b] ($a, b \in \bar{R}$), $X \overset{b}{\underset{\alpha}{\geq}} Y$ if and only if $E\left(u(X)\right) \geq E(u(Y))$ for all utility functions u belong to

$$U_\alpha^b = \left\{ u : [a,b] \to R \middle| u(x) = c - \int_{x^+}^{b} (y-x)^{\alpha-1} dv(y) - k(b-x)^{\alpha-1};\ c \in R, k \geq 0; \right.$$
$$\left. \text{where } v \text{ is positive } \sigma - \text{finite measure } \int_a^b |y|^{\alpha-1} dv(y) < \infty \right\}.$$

The classes U_α and U_α^b are closed under positive affine transformations and are sufficient to characterize the α stochastic order ($\underset{\alpha}{\geq}$, $\overset{b}{\underset{\alpha}{\geq}}$) although more general base classes could be used. On the other hand, Fishburn (1976, 1980a) and Muller (1997) prove that $U_\alpha \supseteq U_\beta$ $U_\alpha^b \supseteq U_\beta^b$ for every $1 \leq \alpha < \beta$ and the derivatives of $u \in U_\alpha$ (U_α^b) satisfy the inequalities $(-1)^{k+1} u^{(k)} \geq 0$ where k=1,…,n-1 for integer n such that $n-1 \leq \alpha < n$. The main advantage of using continua orders is given by their definitions in terms of moments. It is well known that portfolio returns exhibit heavy tails that do not always guarantee finite first moments. We apply α stochastic dominance orders to portfolios:

1) with $\alpha \neq 1$ only if all portfolios X belong to the $L^{\alpha-1}$ space (i.e. $L^{\alpha-1} = \left\{ X / E\left(|X|^{\alpha-1}\right) < +\infty \right\}$;

2) when α=1 (first-order stochastic dominance) no regularity conditions on moments are needed.

Thus, one can rank the investor's choices by using orderings $\underset{\alpha}{>}$ with $\alpha \in (1,2)$, even when the finite first moments cannot be guaranteed. The following definition considers orders that generalize the classic Rothschild-Stiglitz (R-S) order.

Definition 2 *We state that X dominates Y in the sense of* α *-(bounded)Rothschild-Stiglitz order* (*strict*) (α *-(bounded)R-S* (*strict*)) *when* $X \underset{\alpha}{\geq} Y$ ($X \overset{b}{\underset{\alpha}{\geq}} Y$, $X \underset{\alpha}{>} Y$, $X \overset{b}{\underset{\alpha}{>}} Y$) *and* $-X \underset{\alpha}{\geq} -Y$ ($-X \overset{-a}{\underset{\alpha}{\geq}} -Y$, $-X \underset{\alpha}{>} -Y$, $-X \overset{-a}{\underset{\alpha}{>}} -Y$).

We remark that in the literature an α -R-S order is also known as an α -concave order when α is an integer that is greater than or equal to 2. In particular, when α =2, we obtain the classic R-S order. Furthermore, (bounded) R-S order is strictly linked to the moment order. The following corollary summarizes some of the main implication regarding R-S type orderings.

Corollary 1 *The following implications hold:*

a) X α *-*(*bounded*)*R-S* (*strict*) *Y implies X* β *-(bounded)R-S* (*strict*) *Y for every* $\beta \geq \alpha$.

b) Suppose that X strictly α *-R-S Y and the moments of X and Y through integer n are finite where* $n-1 < \alpha \leq n$. *Then* $\left(E(X),...,E(X^n)\right) \neq \left(E(Y),...,E(Y^n)\right)$ *and* $E(X^k) < E(Y^k)$ *for the minimum even k at which* $E(X^k) \neq E(Y^k)$. *In particular, if X and Y are random variables with finite first moments, then X* α *-R-S Y implies E*(*X*)=*E*(*Y*).

c) X α *-*(*bounded*)*R-S* (*strict*) *Y if and only if* $dX + c$ α *-*(*bounded*)*R-S* (*strict*) $dY + c$ *for every* $c \in R$ *d>0, if and only if for every real* t ($\forall t \in supp(X,Y)$)

$$\Gamma(\alpha)F_X^{(\alpha)}(t) = E\left(\left(t-X\right)_+^{\alpha-1}\right) \leq E\left(\left(t-Y\right)_+^{\alpha-1}\right) = \Gamma(\alpha)F_Y^{(\alpha)}(t)\,;$$

$$\Gamma(\alpha)\overline{F}_X^{(\alpha)}(t) = E\left(\left(X-t\right)_+^{\alpha-1}\right) \leq E\left(\left(Y-t\right)_+^{\alpha-1}\right) = \Gamma(\alpha)\overline{F}_Y^{(\alpha)}(t),$$

(*and at least one inequality is strict for some t when the respective orders are strict*).

d) X α *R-S Y implies that* $E\left(\left|X_1-X_2\right|^{\alpha-1}\right) \leq E\left(\left|X_1-Y_1\right|^{\alpha-1}\right) \leq E\left(\left|Y_1-Y_2\right|^{\alpha-1}\right)$, *and* $E\left(\left|X-t\right|^{\alpha-1}\right) \leq E\left(\left|Y-t\right|^{\alpha-1}\right)$ *for every real t* (*that is strict for some t when the* α *- R-S order is strict*), *where* Y_1, Y_2 *are independent copies of Y,* X_1, X_2 *are independent copies of X and even* X_1, Y_1 *are independent.*

Clearly, α must be strictly greater than 1 in the definition, because $X \underset{1}{>} Y$ implies that $-Y \underset{1}{>} -X$ and we cannot have $E(X) > E(Y)$ and $-E(X) > -E(Y)$. In addition, we can compare bounded random variables in the sense of α-R-S order only when $\alpha \geq 2$, as it follows by the next proposition that summarizes some of the most important implications relative to R-S type orders.

Proposition 3 *The following implications hold:*

a) If X and Y are (below or above) bounded random variables with first moment finite it does not exist $\alpha \in (1,2)$ such that X α-(bounded)R-S (strict) Y.

b) Assume Y belongs to L^p with p>α. If X α -R-S Y and $E\left(|X|^r\right) = E\left(|Y|^r\right)$ for a given $r \in (\alpha - 1, p]$, then $F_X = F_Y$, otherwise X α-R-S strict Y implies $E\left(|X|^r\right) < E\left(|Y|^r\right)$ for every $r \in (\alpha - 1, p]$. In particular, a random variable $X \notin L^p$ cannot never α R-S dominates a random variable $Y \in L^p$.

c) If X and Y are symmetric with null mean X (bounded)R-S Y if and only if $X \underset{\alpha}{\geq} Y$ ($X \overset{b}{\underset{\alpha}{\geq}} Y$).

Observe that in example 2 if we assume that $p_2 = p_1 = 0.5$ and $u_1 > u_2 > 0$, then the random variables are symmetric with null mean. Thus X_2 α*-R-S* X_1 for any $\alpha \geq 2$ as a consequence of point c).

From the previous analysis, we deduce that the inequalities between absolute moments allow one to order portfolio uncertainty, coherently to different type of investors. Another immediate consequence is the next corollary..

Corollary 2 *If in the market there exist two portfolios X and Y with the same mean and dispersion $E\left(|X - E(X)|^r\right) = E\left(|Y - E(Y)|^r\right)$, then either one portfolio is redundant (because it has the same distribution as the other) or the two portfolio are not comparable in the sense of $(p+1)$-R-S order for any p<r.*

According to an operational definition of the *risk and uncertainty* that is perceived by investors (see among others, Rachev et al (2005) and Holton (2004)), the previous discussion suggests distinguishing the orderings with respect to (a) the uncertainty of different positions and (b) the investor's exposure to risk. Generally, R-S type orders serve to characterize the different degrees of portfolio uncertainty, whilst the stochastic "risk" orders also take into account the downside risk of portfolios. Clearly, this first distinction could have an important impact for investors.

Specifically, to select the set of admissible choices which are coherent to a given category of investors, we can consider the direct risk measures $\rho(X)$ (associated to random wealth X) that are *consistent* with the order relation ($\underset{\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$, α -(bounded) R-S); that is, $\rho(X) \leq \rho(Y)$ if X dominates ($\underset{\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$, α -(bounded) R-S) Y. Typically, we have that $\rho_{t,\alpha}(X) = E\left((t-X)_+^{\alpha-1}\right)$ is a risk measure consistent with $\underset{\alpha}{\geq}$ ($\overset{b}{\underset{\alpha}{\geq}}$) order for any fixed t (belonging to the support of all optimal portfolios). Similarly, the measures (also called *uncertainty measures*) $\tilde{\rho}_\alpha(X) = E\left(|X_1 - X_2|^{\alpha-1}\right)$ and $\tilde{\rho}_{t,\alpha}(X) = E\left(|t-X|^{\alpha-1}\right)$ are consistent with α *-R-S* (α -bounded R-S) order for any fixed t (belonging to the support of all optimal portfolios) and assuming X_1, X_2 are independent copies of X. The measures consistent with $\underset{\alpha}{\geq}$ ($\overset{b}{\underset{\alpha}{\geq}}$) orders also take into account downside risk, while the measures consistent with α *-R-S* (α -bounded R-S) orders discriminate between the different levels of uncertainty. Thus, as discussed by Ortobelli et al (2005), their use in portfolio choice problems changes.

Furthermore, we can order the choices considering the reward instead of the risk. According to the definition given by Rachev et al (2005) and De Giorgi (2005), we assume a reward measure to be any functional v defined on the portfolio returns that is isotonic with respect to a given stochastic risk order (for example: $\underset{\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$). Thus, when a given category of investors (e.g., non-satiable, non-satiable risk averse) prefers X to Y, then $v(X) \geq v(Y)$. On the other hand, Rachev et al (2005) Biglova et al (2004) have shown that the use of a reward-risk ratio could be important not only from a computational point of view, but also because it takes into account portfolio diversification. Any consideration that we do for measures consistent with some risk orderings can be extended to reward measures changing the sense of the inequality and considering a maximization problem instead of a minimization problem. That is, if $\rho(X)$ is a risk measure consistent with a risk ordering, then the measure $-\rho(X)$ is a reward measure isotonic with the same order. Thus if we characterize the consistency with respect to risk orderings (say $\underset{\alpha}{\geq}, \overset{b}{\underset{\alpha}{\geq}}$), we also implicitly characterize the isotonicity. For this reason in the following we place much more attention to the consistency with a given order.

2.2 Inverse stochastic dominance

Similar to classic stochastic dominance rules, we can describe stochastic dominance rules based on the left inverse of F_X given by $F_X^{-1}(p)=\inf\{x:\Pr(X\le x)=F_X(x)\ge p\}$ $\forall p\in(0,1]$ and $F_X^{-1}(0)=\lim_{p\searrow 0}F_X^{-1}(p)$. In particular, Muliere and Scarsini (1989) have defined inverse stochastic dominance order as follows: we say that *X* strictly dominates *Y* with respect to *n* (integer) inverse order stochastic dominance ($X\underset{-n}{>}Y$) if and only if

$$F_X^{(-n)}(t)=\int_0^t F_X^{(n-1)}(u)du\ge F_Y^{(-n)}(t)=\int_0^t F_Y^{(n-1)}(u)du \quad \forall t\in[0,1].$$

As for integer stochastic orders, even the above dual stochastic orders can be easily extended in continuous terms. Let us consider the unique completion of the σ-finite positive measure associated with F_X^{-1}, that on the half open intervals of the forms $[a,b)\subseteq[0,1]$ is given by:

$$\mu_X([a,b))=F_X^{-1}(b)-F_X^{-1}(a)=\int_a^b dF_X^{-1}(p);\ \mu_X(\{a\})=F_X^{-1}(a^+)-F_X^{-1}(a).$$

Then we can define the α dual functions:

$$F_X^{(-\alpha)}(p)=\frac{1}{\Gamma(\alpha)}\int_0^p (p-u)^{\alpha-1}dF_X^{-1}(u);\ \ \forall p\in[0,1];\ \alpha\ne 1,\ F_X^{(-1)}(p);\ \forall p\in[0,1], \qquad (6)$$

which are continuous for every $\alpha>1$ and left continuous for $\alpha\le 1$. Instead the functions

$$\bar{F}_X^{(-\alpha)}(p)=\frac{1}{\Gamma(\alpha)}\int_{p^+}^1 (u-p)^{\alpha-1}dF_X^{-1}(u)\ \ \forall p\in[0,1];\ \alpha\ne 1;\ \bar{F}_X^{(-1)}(p)=-F_X^{(-1)}(p);\ \forall p\in[0,1] \qquad (7)$$

are continuous for every $\alpha>1$, left continuous for $\alpha=1$ and right continuous for $\alpha<1$. In particular, when *X* is a continuous random variable, we have that $\bar{F}_X^{(-\alpha)}(p)=F_{-X}^{(-\alpha)}(1-p)\ \ \forall\alpha>0$. Moreover, as a consequence of Lemma 1 we obtain

$$F_X^{(-\alpha)}(p)=\frac{1}{\Gamma(\alpha-v)}\int_0^p (p-u)^{\alpha-v-1}F_X^{(-v)}(u)du\text{, for every } \alpha>v>0, \qquad (8a)$$

$$\bar{F}_X^{(-\alpha)}(p)=\begin{cases}\frac{1}{\Gamma(\alpha-v)}\int_{p^+}^1 (u-p)^{\alpha-v-1}\bar{F}_X^{(-v)}(u)du & \forall\alpha>v\ge 1\vee\forall 1>\alpha>v>0\\ \lim_{p_n\nearrow p}\frac{1}{\Gamma(\alpha-v)}\int_{p_n}^1 (u-p_n)^{\alpha-v-1}\bar{F}_X^{(-v)}(u)du & \forall\alpha>1>v>0\end{cases}. \qquad (8b)$$

In particular, when $\alpha=2$ we obtain the Lorenz curve $F_X^{(-2)}(p)=L_X(p)=\int_0^p F_X^{-1}(t)dt$. Thus the following definition extends the previous dual orders to continua orders.

Definition 3 *For every* $\alpha > 0$*, we say X dominates Y with respect to* α *dual* (*also called inverse*) *stochastic order* ($X \underset{-\alpha}{\geq} Y$) *iff* $F_X^{(-\alpha)}(t) \geq F_Y^{(-\alpha)}(t) \quad \forall t \in [0,1]$ *and we say that X strictly dominates Y with respect to* α *dual order* ($X \underset{-\alpha}{>} Y$) *iff* $X \underset{-\alpha}{\geq} Y$ *and* $F_X \neq F_Y$ *. We say that X dominates Y in the sense of dual* α*-Rothshild Stiglitz order* (*strict*) (*dual* α*-R-S* (*strict*)) *when* $X \underset{-\alpha}{\geq} Y$ ($X \underset{-\alpha}{>} Y$) *and* $-X \underset{-\alpha}{\geq} -Y$ ($-X \underset{-\alpha}{>} -Y$).

Similarly, we can define the survival order, that is $X \underset{sur\ -\alpha}{\geq} Y$ iff $\bar{F}_X^{(-\alpha)}(t) \leq \bar{F}_Y^{(-\alpha)}(t)$ for every t belonging to $[0,1]$. Considering that for $\alpha > 1$ we get $\bar{F}_X^{(-\alpha)}(p) = F_{-X}^{(-\alpha)}(1-p)$, thus the results obtained for survival dual orders (with $\alpha > 1$) are equivalent to those obtained for orders applied to the opposite of the random variables. From this definition we get that $F_X^{(-v)}(p)$ is a reward measure for any *p* belonging to (0,1). As for the α stochastic orders we can prove similar properties for the dual stochastic orders. In particular, it is well known that $\underset{1}{\geq}$ and $\underset{2}{\geq}$ orders are equivalent to the respective $\underset{-1}{\geq}$ and $\underset{-2}{\geq}$ orders. Therefore all the implications which are valid for $\underset{1}{\geq}$, $\underset{2}{\geq}$ ($\overset{b}{\underset{1}{\geq}}$, $\overset{b}{\underset{2}{\geq}}$) and 2-(bounded) R-S orders are still valid for the equivalent orders $\underset{-1}{\geq}$, $\underset{-2}{\geq}$ and dual 2-R-S orders. However, integer stochastic dominance orders greater than two are different by the respective dual orders (see, among others, Muliere and Scarsini (1989)). This is logical because inverse stochastic order is defined only on the support of the random variables (as $\overset{b}{\underset{\alpha}{\geq}}$ order but differently by $\underset{\alpha}{\geq}$ order). Thus there probably exists a correspondence between $\overset{b}{\underset{\alpha}{\geq}}$ and dual orders that will be the subject of future research.

On the other hand, we observe that the inverse stochastic orders previously defined can be extended to an unbounded inverse stochastic order as follows. Suppose that either $\left|F_X^{(-1)}(0)\right| < \infty$ or $\left|F_X^{(-1)}(1)\right| < \infty$ for X belonging to given class of random variables Λ. Then we extend $F_X^{(-1)}$ on all the real line R assuming $F_X^{(-1)}(u) = F_X^{(-1)}(0) \quad \forall u \leq 0$ and $F_X^{(-1)}(t) = F_X^{(-1)}(1) \quad \forall t \geq 1$. Moreover, we say X dominates Y with respect to the unbounded α inverse stochastic orders (unbounded $X \underset{-\alpha}{\geq} Y$) iff $F_X^{(-\alpha)}(u) \geq F_Y^{(-\alpha)}(u)$ for every $u \in R$, where $F_X^{(-\alpha)}(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u^-} (u-t)^{\alpha-1} dF_X^{(-1)}(t)$.

Many of the considerations done for stochastic dominance orders can be repeated even for the dual orders, and in the next remark we summarize the main properties of these orders.

Remark 2 *The following implications hold:*

1) Unbounded $X \underset{-\alpha}{\geq} Y$ *, implies* $X \underset{-\alpha}{\geq} Y$ *. In addition, for every* $\beta \geq \alpha$ *, (unbounded)* $X \underset{-\alpha}{\geq} Y$ *implies (unbounded)* $X \underset{-\beta}{\geq} Y$ *and X dual (unbounded)* α *-R-S Y implies X dual (unbounded)* β *-R-S Y.*

2) $X \underset{-\alpha}{\geq} Y$ *if and only if* $cX + t \underset{-\alpha}{\geq} cY + t$ *for every* $t \in R$ *, c>0,* $\alpha > 0$ *. If* $X \underset{-\alpha}{\geq} Y$ *with* $\alpha > 1$ *implies* $X_{-}^{(M)} \underset{-\alpha}{\geq} Y_{-}^{(M)}$ *, for any given* $M \in R$ *.*

3) For every $\alpha > 1$ *and for every* $X, Y \in \Lambda_{(\alpha)} =: \left\{ X / \left| F_X^{(-\alpha)}(x) \right| < \infty \ \forall x \in (0,1) \right\}$ *we have that* $X \underset{-\alpha}{\geq} Y$ *if and only if* $\int_0^1 \phi(x) dF_X^{-1}(x) \leq \int_0^1 \phi(x) dF_Y^{-1}(x)$ *for every* $\phi \in V^\alpha$ *where*

$$V^\alpha = \Big\{ \phi(x) = -\int_{x^+}^1 (s-x)^{\alpha-1} d\tau(s) - k(1-x)^{\alpha-1} \Big| k \geq 0; \tau \text{ is a } \sigma-\text{finite positive measure s.t. } \forall X \in \Lambda_{(\alpha)}$$
$$\text{the function } |s-x|^{\alpha-1} \text{ is } d\tau(s) \times dF_X^{-1}(x) \text{ integrable in } [0,1]\times[0,1] \Big\}.$$

4) *For every* $\alpha > 1$ *and for every* $X, Y \in \Lambda_{(\alpha)} =: \left\{ X / \left| F_X^{(-\alpha)}(x) \right| < \infty \ \forall x \in R \right\}$ *we have that unbounded* $X \underset{-\alpha}{\geq} Y$ *if and only if* $\int_{-\infty}^{+\infty} \phi(x) dF_X^{-1}(x) \leq \int_{-\infty}^{+\infty} \phi(x) dF_Y^{-1}(x)$ *for every* $\phi \in UV^\alpha$ *where*

$$UV^\alpha = \Big\{ \phi(x) = -\int_{x^+}^{+\infty} (s-x)^{\alpha-1} d\tau(s) \Big| \tau \text{ is a } \sigma-\text{finite positive measure s.t. } \forall X \in \Lambda_{(\alpha)}$$
$$\text{the function } |s-x|^{\alpha-1} \text{ is } d\tau(s) \times dF_X^{-1}(x) \text{ integrable in } R^2 \Big\}.$$

5) If $X \underset{-\alpha}{\geq} Y$ *, then for any integer* $k \geq \alpha - 1$ *the inequality* $E\left(\min_{1 \leq i \leq k} X_i \right) \geq E\left(\min_{1 \leq i \leq k} Y_i \right)$ *holds, where* X_i*,* Y_i *i=1,…,k are i.i.d. copies respectively of X and Y.*

From the above discussion it follows that there exist many different ways to discriminate the choices available to investors. We distinguish between orders and their dual/survival orders, bounded and unbounded orders, and risk and uncertainty orders. Moreover, there exists a strong connection among orderings and risk/uncertainty measures that will be more thoroughly treated in the next section.

3. New Measures for Orderings, Probability Functionals

Most of portfolio theory is based on minimizing a distance from a benchmark or minimizing potential possible losses while maintaining constant some portfolio characteristics. As observed by Rachev et al (2005), these problems can be reformulated from the point of view of the theory of probability metrics. In particular, we are generally interested in *probability functionals* $\mu:\Lambda\times\Lambda\to R$ (where Λ is a space of real valued random variables defined on $(\Omega,\Im,P)$) satisfying the following property for any pair of random variables X,Y:

Identity property: $f(X)=f(Y)\Leftrightarrow\mu(X,Y)=0$; where $f(X)$ identifies some characteristics of the random variable *X*.

From this property we can distinguish among three main groups of probability functionals (namely, primary, simple, and compound) depending on certain modifications of the identity property (see Rachev (1991)). *Compound probability functionals* identify the random variable almost surely (i.e., for any pair of random variables $X,Y:\mu(X,Y)=0\Leftrightarrow P(X=Y)=1$). *Simple probability functionals* identify the distribution (i.e., for any pair of random variables X,Y: $\mu(X,Y)=0\Leftrightarrow F_X=F_Y$). *Primary probability functionals* determine only some random variable characteristics.

Typically, with respect to the portfolio selection problem, the two probability functionals μ studied are those that identify:

1) the uncertainty of the random variable in a given absolute moment. Thus, we can say that some portfolios are "equivalent in uncertainty" if they present the same dispersion that can be measured in different ways (see Ortobelli et al (2005)). For example, we can consider "equivalent in uncertainty" portfolios with:

- the same absolute central moment $\mu(X,Y)=0\Leftrightarrow E(|X-E(X)|^p)=E(|Y-E(Y)|^p)$;
- the same distance by a given benchmark Z $\mu(X,Y)=0\Leftrightarrow d(X,Z)=d(Y,Z)$, where d measure a distance between the random variable and the benchmark Z;
- the same level of concentration valued with an opportune moment p i.e. $\mu(X,Y)=0\Leftrightarrow E(|X_1-X|^p)=E(|Y_1-Y|^p)$, and where X_1 is an independent copy of *X* and Y_1 is an independent copy of *Y*.

2) the losses in the distributional tail behavior. Thus, for example we can assume "equivalent in losses (risk)" two investments that present

- the same distributional tail $\mu(X,Y)=0\Leftrightarrow F_X(x)=F_Y(x)\;\;\forall x\in(-\infty,t]$ for a given *t*.
- the same Expected Shortfall or Conditional Value at Risk (CVaR) $\mu(X,Y)=0\Leftrightarrow CVaR_p(X)=CVaR_p(Y)$ for a given probability $p\in(0,1)$;
- the same power of the tail valued on the left tails with an opportune moment $\mu(X,Y)=0\Leftrightarrow E\left((t-X)_+^p\right)=E\left((t-Y)_+^p\right)$ for a given threshold $t\in R$;

- the same distance valued on the tail respect to a given benchmark Z we get $\mu(X,Y)=0 \Leftrightarrow d_{tail}(X,Z)=d_{tail}(Y,Z)$ where d_{tail} measure a distance on the tails between the random variable and the benchmark.

Further extensions that describe primary, simple, and compound probability metrics as tracking error measures can be found in Stoyanov et al (2006) and Ortobelli et al (2006).

3.1 FORS Orderings

One of the principal problems in economics is the ordering of choices in the face of uncertainty. Basically, any observer can deduce the decision makers' preferences from their behavior in the market. Starting from this logical deduction, utility theory classifies the optimal choices of different categories of market agents (for example, risk averse, non-satiable, non-satiable risk averse) under ideal market conditions. In particular, the fundamentals of utility theory under uncertainty conditions have been developed by von Neumann and Morgenstern (1953). Several improvements and further advancements of the theory have been proposed, even in recent years (see, among others, Machina (1982), Yaari (1987), Gilboa and Schmeidler (1989), and Maccheroni et al (2005)). Roughly speaking, in utility theory the ordering of uncertain choices begins with the selection of a finite number of axioms characterizing the preferences of a given category of market agents. The second step of the theory involves representing the preferences of market agents preferences employing "utility functionals" that summarize the decision makers' behavior. Clearly, there exist a correspondence among the orderings of utility functional, the orderings of preferences, and the orderings of random goods. Thus, when the utility functionals are characterized, it is possible to identify the different categories of market agents. Consequently, we can also identify the optimal choices of a given category of market agents when we order some utility functionals. In particular, we define *efficient for a given category of market operators* all the admissible choices that cannot be preferred (dominated) by all the agents of the same category. Moreover, any utility functional under uncertainty conditions is strictly linked to a probability functional. Therefore, in order to capture the agents' behavior, next we propose to study orderings among probability functionals which are induced by orderings among preferences.

According to the definition of probability functionals (see Rachev (1991)), we want to discuss the main relevant properties of a probability functional with respect to the portfolio selection problem. It is well known that the most important property that characterizes any probability functional μ associated with a portfolio choice problem is the consistency with a stochastic order (see Ortobelli et al (2005)). In terms of probability functional, the consistency is defined as: X

dominates Y *with* respect to a given order of preferences $\succ$ implies $\mu(X,Z) \le \mu(Y,Z)$ for a fixed arbitrary benchmark Z.

We define a *FORS measure induced by order* $\succ$ any probability functional $\mu : \Lambda \times \Lambda \to R$ that is consistent with a given order of preferences $\succ$. The order of preferences $\succ$ could be characterized:

a) either with some axioms that identify the decision makers' preferences (as in utility theory);

b) or with an order that identifies the preferences of a particular category of investors (such as orders $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$, $\underset{-\alpha}{>}$, unbounded $\underset{-\alpha}{>}$ and (dual) α-(bounded) R-S order). In this case, we define an α*-FORS measure induced by order* $\succ$, when the order of preference $\succ$ refers to a given category of investors characterized by the parameter α.

Observe that in the definition of consistency, no rule relative to the benchmark Z is described. Therefore, we can consider as a subclass of probability functionals consistent with an order of preference, all the risk measures $\mu : \Lambda \to R$. In particular, the recent literature in financial economics has highlighted the importance of some particular properties of risk measures (see, among others, Artzner et al (1999), Frittelli and Rosazza Gianin (2002), Föllmer and Sheid (2002), and Ortobelli et al (2005)). We recall that a convex measure $\mu(X)$ valued on a family of random variables $X \in \Lambda$, satisfies the properties:

1. Monotone for every $X, Y \in \Lambda$ $X \ge Y \Rightarrow \mu(X) \le \mu(Y)$
2. Translation invariant, $\forall X \in \Lambda$ and $m \in R$ such that $\mu(X+m) = \mu(X) - m$
3. Convex $\forall X, Y \in \Lambda$, $\forall a \in [0,1]$,

$$\mu(aX + (1-a)Y) \le a\mu(X) + (1-a)\mu(Y)$$

If additionally we even consider positive homogeneity,

4. positive homogeneous $\forall \alpha \ge 0$ $\forall X \in \Lambda$, $\mu(\alpha X) = \alpha\mu(X)$

then, we have a *coherent static risk measure*.

Thus any coherent risk measure is a FORS measure $\mu : \Lambda \to R$ induced by the monotonic order, i.e. $\forall X, Y \in \Lambda$ such that X>Y *P*-almost surely implies $\mu(X) \le \mu(Y)$.

Definition 4 *We call convex* α*-FORS measure induced by order* $\succ$ *any probability functional* $\mu : \Lambda \to R$ *that is consistent with a translation invariant, convex* α*-FORS order. We call coherent*

α-FORS measure induced by order $\succ$ any probability functional $\mu:\Lambda \to R$ that is consistent with a translation invariant, convex and positive homogeneous α-FORS order.

Although in many cases convex/coherent risk measures are convex/coherent FORS measures, this definition better specifies the consistency. For example, for every $\alpha \geq 1$ and for every $\beta \in (0,1)$ the measure

$$\frac{-\Gamma(\alpha+1)}{\beta^{\alpha}} F_X^{(-(\alpha+1))}(\beta)$$

is a coherent (α+1)- FORS measure consistent with $\underset{-(\alpha+1)}{\geq}$ order (see Ortobelli et al (2006)). However, this measure is not necessarily consistent with $\underset{-(\gamma+1)}{\geq}$ when $\gamma > \alpha$ (i.e. it is not a coherent $\gamma+1$- FORS measure). Among the typical FORS functionals we can consider the following ones:

1. $-F_X^{(-\alpha)}(p)$, for a fixed benchmark $p \in (0,1)$, (is induced by $\underset{-\alpha}{>}$ order);
2. $F_X^{(\alpha)}(t)$, for a fixed benchmark $t \in R$, (is induced by $\underset{\alpha}{>}$ order);
3. $\tilde{\rho}_{t,\alpha}(X) = E\left(\left|t - X\right|^{\alpha-1}\right)$ for a fixed benchmark $t \in R$, (is induced by α - *R-S* order);
4. $\tilde{\rho}_{\alpha}(X) = E\left(\left|X - X_1\right|^{\alpha-1}\right)$, for the benchmark X_1 that is an independent copy of *X* (is induced by α bounded - *R-S* order).

As for the previous ordering analysis, we deduce that there exist two types of FORS *measures*:

- Measures of *risk* (tails, losses) which are induced by orderings of tails, such as $\underset{\alpha}{>}$, $\overset{b}{\underset{\alpha}{>}}$, $\underset{-\alpha}{>}$, that we call *FORS risk measures* (that are measures of reward if we multiply the functions for (-1));
- Measures of *uncertainty* (concentration, dispersion) which are induced by orderings of uncertainty such as (dual) α-(bounded) *R-S* orders, that we call *FORS uncertainty measures*.

Similarly we can extend the previous definition to reward measures isotonic to orderings. Thus any probability functional μ associated to a portfolio choice problem that satisfies the property:

Isotonicity: *X* dominates *Y* respect to a *risk* ordering $\succ$ implies $\mu(X,Z) \geq \mu(Y,Z)$ for a fixed arbitrary benchmark Z.

is called *FORS reward measure* induced by risk order $\succ$.

Clearly any consideration done for FORS risk measures can be easily extended to FORS reward measures. Moreover, all the above examples of FORS functionals induced from a given ordering of preference $\succ$ are parametric. However, under the opportune hypotheses, we can also say the converse. As a matter of fact, one can develop many other kinds of orderings using the fractional integral in the following way.

Definition 5 *Let* $\rho_X : [a,b] \to \bar{R}$ *(with* $-\infty \le a < b \le +\infty$*) be a bounded variation function, for every random variable X belonging to a given class* Λ*. Furthermore, assume that* ρ_X *is a simple probability functional over the class* Λ *(i.e.* $\forall X, Y \in \Lambda$, $\rho_X = \rho_Y \Leftrightarrow F_X = F_Y$*). If, for any fixed* $\lambda \in [a,b]$, $\rho_X(\lambda)$ *is a FORS risk measure induced by a risk ordering* $\succ$*, then, we call FORS risk orderings induced by* $\succ$ *the following new class of orderings defined for every* $\alpha > 0$, $\forall X, Y \in \Lambda_{(\alpha)} = \left\{ X \in \Lambda \,\middle|\, \left| \int_a^b |t|^{\alpha-1} \, d\rho_X(t) \right| < \infty \right\}$

$$X \underset{\succ,\alpha}{FORS}\, Y \;\; iff \;\; \rho_{X,\alpha}(u) \le \rho_{Y,\alpha}(u) \;\; \forall u \in [a,b]$$

where $\rho_{X,\alpha}(u) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^{u^-} (u-t)^{\alpha-1} d\rho_X(t) & if\ \alpha > 0,\ \alpha \ne 1 \\ \rho_X(u) & if\ \alpha = 1 \end{cases}$. *We call* ρ_X *FORS measure associated to the FORS ordering of random variables belonging to* Λ.

Similarly we can define *FORS uncertainty orderings.*

Definition 6 *We say that X dominates Y in the sense of* α *FORS uncertainty ordering induced by* $\succ$ *(we simple write* $X \underset{\succ,unc\ \alpha}{FORS} Y$*) if and only if* $\int_a^x (x-s)_+^{\alpha-1} d\rho_{\pm X}(s) \le \int_a^x (x-s)_+^{\alpha-1} d\rho_{\pm Y}(s)$ $\forall x \in [a,b]$ *(i.e. when* $X \underset{\succ,\alpha}{FORS} Y$ *and* $-X \underset{\succ,\alpha}{FORS} -Y$*).*

Given a FORS ordering then it is possible to define a survival ordering considering

$$\bar{\rho}_{X,\alpha}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t^+}^b (u-t)^{\alpha-1} d\rho_X(u) & if\ \alpha > 0,\ \alpha \ne 1 \\ -\rho_X(t) & if\ \alpha = 1 \end{cases} \qquad \bar{\rho}_{X,\alpha}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha-v)} \displaystyle\int_{t^+}^b (u-t)^{\alpha-v-1} \bar{\rho}_{X,v}(u) du & \forall \alpha > v \ge 1 \vee \forall 1 > \alpha > v > 0 \\ \lim\limits_{t_n \nearrow t} \dfrac{1}{\Gamma(\alpha-v)} \displaystyle\int_{t_n^+}^b (u-t_n)^{\alpha-v-1} \bar{\rho}_{X,v}(u) du & \forall \alpha > 1 > v > 0 \end{cases}$$

and we say $X \underset{\succ,sur\ \alpha}{FORS} Y$ iff $\bar{\rho}_{X,\alpha}(t) \le \bar{\rho}_{Y,\alpha}(t)$ for every *t* belonging to[a,b]. However, in this case, we cannot generally say that the results obtained for survival orders are equivalent to those obtained for orders applied to the opposite of the random variables. Thus survival FORS ordering is an alternative to the original one.

Note that if we assume in Definition 5 that ρ_X is a primary (instead of simple) probability functional induced by $\succ$, then the probability functionals $\rho_{X,\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_a^{u^-}(u-t)^{\alpha-1}d\rho_X(t)$ defined for $\alpha>1$ are again FORS measures induced by $\succ$. In addition, if σ_X is a FORS probability functional induced by a given FORS ordering $\underset{\succ,v}{FORS}$, then σ_X is again a FORS measure induced by order $\succ$.

For any FORS risk ordering induced by $\succ$, we can easily define an inverse (dual) ordering, if the FORS measure ρ_X is monotone. In this case, we consider the left inverse of ρ_X (i.e., $\rho_X^{-1}(x)=\inf\left\{u\in[a,b]:\rho_X(u)\geq x\right\}$ for any positive x belonging to the imagine of ρ_X). However, many of the extensions we have observed for stochastic dominance orders and its dual are still valid for FORS orderings as underlined in the following remark.

Remark 3 *The following implications hold for a FORS ordering of a random variables class* Λ.

1) *For every* $\alpha>v>0$ $X\ \underset{\succ,v}{FORS}\,Y$ *implies* $X\ \underset{\succ,\alpha}{FORS}\,Y$ *and we can write*

$$\rho_{X,\alpha}(t)=\begin{cases}\frac{1}{\Gamma(\alpha-v)}\int_a^{t^-}(t-u)^{\alpha-v-1}\rho_{X,v}(u)du & \forall\alpha>v\geq1\vee\forall1>\alpha>v>0\\ \lim_{t_n\searrow t}\frac{1}{\Gamma(\alpha-v)}\int_a^{t_n}(t_n-u)^{\alpha-v-1}\rho_{X,v}(u)du & \forall\alpha>1>v>0\end{cases}$$

2) *For any monotone increasing FORS measure* ρ_X *associated to a FORS ordering, the left inverse* ρ_X^{-1} *is a FORS reward measure and* $-\rho_X^{-1}$ *is itself a FORS ordering induced by* $\succ$.

3) *Suppose* $|\rho_X(b)|<\infty$, $|\rho_X(a)|<\infty$ *for every X belonging to* Λ. *Then we can extend* ρ_X *on all the real line R assuming* $\rho_X(u)=\rho_X(a)\ \forall u\leq a$ *and* $\rho_X(u)=\rho_X(b)\ \forall u\geq b$. *Moreover, we say X unbounded* $\underset{\succ,\alpha}{FORS}$ *dominates Y iff* $\rho_{X,\alpha}(u)\leq\rho_{Y,\alpha}(u)$ *for every* $u\in R$ *where we define*

$$\rho_{X,\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{u^-}(u-t)^{\alpha-1}d\rho_X(t)\ \forall u\in R.$$

If ρ_X *is monotone unbounded* $\underset{\succ,\alpha}{FORS}$ *order implies* $\underset{\succ,\alpha}{FORS}$ *order.*

In addition, an equivalent formulation of FORS orderings is given by the following corollary that generalizes the representation of orderings using utility functionals.

Corollary 3 *Suppose* ρ_X *is FORS measure associated with a FORS ordering* $\succ$ *on a given class of random variables X belonging to* Λ. *Then, given* $X, Y \in \Lambda_{(\alpha)}$, $X \underset{\succ,\alpha}{FORS} Y$ *if and only if* $\int_a^b \phi(u) d\rho_{X,1}(u) \geq \int_a^b \phi(u) d\rho_{Y,1}(u)$ *for every* ϕ *belonging to*

$$W^{\alpha} = \Big\{ \phi(x) = -\int_{x^+}^{b} (s-x)^{\alpha-1} d\tau(s) - k(b-x)^{\alpha-1} \Big| k \geq 0,\ k=0 \text{ if } b=\infty;\ \tau \text{ is a } \sigma-\text{finite positive measure s.t. } \forall X \in \Lambda_{(\alpha)} \text{ the function } |s-x|^{\alpha-1} \text{ is } d\tau(s) \times d\rho_X(x) \text{ integrable in } [a,b]\times[a,b] \Big\}.$$

Moreover for every $1 \leq \alpha < v$, $\phi_v \in W^v$ *if and only if there exists a function* $\phi_\alpha \in W^\alpha$ *such that* $\phi_v(x) = \int_{x^+}^{b} (s-x)^{v-\alpha-1} \phi_\alpha(s) ds$.

Moreover, even some of the moments properties we have verified for the stochastic dominance orders can be replaced in some sense for FORS orderings as follows from the proposition below.

Proposition 4 *Suppose* $\rho_X : [a,b] \to \overline{R}$ *is FORS measure associated to a FORS ordering* $\succ$ *on a given class of random variables X belonging to* Λ. *Then the following implications hold for any opportune pair of random variables X and Y belonging to* Λ.

a) $X \underset{\succ,\alpha}{FORS} Y$, $(\alpha > 1)$ *implies for any increasing and invertible function* $H : supp(X,Y) \to [a,b]$ *such that* $|H(z)-x|^{\alpha-1}$ *is* $dF_Z(z) \times d\rho_X(x)$ *integrable for Z equal either to X or to Y :*

$$E\left(\rho_{X,\alpha}(H(X))\right) = \int_a^b E\left(\left(H(X)-s\right)_+^{\alpha-1}\right) d\rho_X(s) \leq \int_a^b E\left(\left(H(X)-s\right)_+^{\alpha-1}\right) d\rho_Y(s) = E\left(\rho_{Y,\alpha}(H(X))\right)$$

$$E\left(\rho_{X,\alpha}(H(Y))\right) = \int_a^b E\left(\left(H(Y)-s\right)_+^{\alpha-1}\right) d\rho_X(s) \leq \int_a^b E\left(\left(H(Y)-s\right)_+^{\alpha-1}\right) d\rho_Y(s) = E\left(\rho_{Y,\alpha}(H(Y))\right)$$

In particular, when $supp(X,Y) = [c,d]$ *we can take* $H(x) = \frac{x-c}{d-c}(b-a) + a$.

b) *If* $X \underset{\succ,sur\ \alpha}{FORS} Y$ *and* $X \underset{\succ,\alpha}{FORS} Y$ *(i.e.* $\overline{\rho}_{X,\alpha}(u) \leq \overline{\rho}_{Y,\alpha}(u)$ *and* $\rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u)$ *for every real* $u \in [a,b]$*) and* $\int_a^b |s|^r d\rho_X(s) = \int_a^b |s|^r d\rho_Y(s)$ *for a given* $r > \alpha - 1$, *then* $F_X = F_Y$, *otherwise it implies* $\int_a^b |s|^r d\rho_X(s) < \int_a^b |s|^r d\rho_Y(s)$ *for every* $r > \alpha - 1$.

The fact that a FORS measure ρ_X (associated with a FORS ordering) is a simple probability functional over a given class of random variables, qualifies the FORS ordering itself. Next we propose a further characterization of FORS orderings.

Suppose $|t|<+\infty$ and let $\rho_{X,\alpha+is}(t)=\frac{1}{\Gamma(\alpha+is)}\int_a^t (t-x)^{\alpha+is-1}d\rho_X(x)$ be the complex extension of the FORS measure $\rho_{X,\alpha}(t)$ ($\alpha>1$) associated to a FORS ordering. Then, as a consequence of Lemma 1, for every real $\alpha>v\geq 1$, $\forall X\in\Lambda_{(\alpha)}$ and $s,k\in R$, we get:

$$\rho_{X,\alpha+is}(t)=\frac{1}{\Gamma(\alpha+is)}\int_{t-a}^{0}(u)^{\alpha+is-1}d\rho_X(t-u)=\frac{1}{\Gamma(\alpha-v+i(s-k))}\int_a^t (t-u)^{\alpha-v+i(s-k)-1}\rho_{X,v+ik}(u)du.$$

That is, $\forall v\in[1,\alpha)$ the functions $\Im_{X,v}(p+is)=\Gamma(p+is)\rho_{X,v+p+is}(t)=\int_0^\infty f_v(x)x^{p+is-1}dx$ are the Mellin transforms of the functions $f_v(x):=\rho_{X,v}(t-x)I_{[0,t-a]}(x)$, defined $\forall p\in(0,\alpha-v]$ and $\forall s\in R$. Thus, for the properties of the Mellin transform, we get the following inversion formula $\forall v\in[1,\alpha)$, $\forall X\in\Lambda_{(\alpha)}$, $\forall p\in(0,\alpha-v]$:

$$\rho_{X,v}(t-x)I_{[0,t-a]}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Im_{X,v}(p+im)x^{-p-im}dm$$

and in particular $\rho_X(t-x)I_{[0,t-a]}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Im_{X,1}(p+im)x^{-p-im}dm \quad \forall p\in(0,\alpha-1]$ (see, among others, Titchmarsh (1937), Szmydt, and Zieman (1992), and Ortobelli (1999)). Observe that the Mellin transform is an analytical function. Then, if we know the values that the transform $\Im_{X,v}(s_n)$ assumes on a countable complex sequence $\{s_n\}_{n\in N}$ ($s_n\in C$) and even in its accumulation point s (i.e. $s_n\xrightarrow{n\to\infty}s$) we univocally determines $\rho_{X,v}(x)$ for every $x\in[a,t]$. That is, the α fractional integral valued on a given point t and for every $\alpha\in(1,p]$ represents itself a transform because $h_X(u)=\Gamma(u-1)\rho_{X,u}(t)$ $\forall u>1$ is the Mellin transform of $\rho_X(t-x)I_{[0,t-a]}(x)$ valued on the real line. From this simple observation, we get a systematic way to generate FORS orderings based on the following theorem.

Theorem 1 *Suppose $|b|<+\infty$ and $\rho_X^{(1)}:[a,b]\to R$ is a FORS1 measure associated with a FORS1 ordering $\succ$ defined on a class of random variables $X\in\Lambda$. If $\rho_X^{(1)}$ is a monotone function, then the* *probability functional* $\rho_X^{(2)} : [1, p] \to R$ *with* $\rho_X^{(2)}(u) = \rho_{X,u}^{(1)}(b)$ *points out a FORS*2 *measure* (*induced by* $\succ$) *on the class of random variables* $\Lambda_{p_1} = \left\{ X \in \Lambda / p_1 > 1 : \left| \rho_{X,p_1}^{(1)}(b) \right| < +\infty \right\}$ *and it is associated to the following new FORS*2 *ordering induced by the previous one* $\succ$ *defined for every* $\alpha > 0$, $\forall X, Y \in \Lambda_{p_1,(\alpha)} = \left\{ Z \in \Lambda_{p_1} \,\middle|\, \left| \int_1^{p_1} u^{\alpha-1} d\rho_X^{(2)}(u) \right| < \infty \right\}$:

$$X \underset{\succ,\alpha}{FORS2}\, Y \;\; iff\;\; \rho_{X,\alpha}^{(2)}(u) \le \rho_{Y,\alpha}^{(2)}(u) \;\; \forall u \in [1, p].$$

Thus, given a FORS1 ordering, we can define a second level of ordering FORS2 and the definition can be extended recursively. As a matter of fact, we can easily get a k-level of FORSk ordering $\rho_X^{(k)} : [1, p_k] \to R$ with $\rho_X^{(k)}(u) = \rho_{X,u}^{(k-1)}(p_{k-1})$ on the class of random variables $\Lambda_{p_k} = \left\{ X \in \Lambda_{p_{k-1}} / p_k > 1 : \left| \rho_{X,p_k}^{(k-1)}(p_{k-1}) \right| < +\infty \right\}$ where $p_0 = b$. An immediate consequence of the proposed analysis is given by the following corollary.

Corollary 4 *Under the assumption of the previous theorem if* $X \underset{\succ,1}{FORSk}\, Y$, *then* $X \underset{\succ,\alpha}{FORSm}\, Y$ *for every m>k and* $\alpha \ge 1$. *In particular, if* σ_X *is a FORSk probability functional induced by the k-th level of a FORS ordering* $\underset{\succ,v}{FORSk}$ ($v \ge 1$), *then* σ_X *is also a FORS measure induced by order* $\succ$.

Thus, it follows from the above corollary that the new orders are finer than the generating one. This could permit to better characterize the investors' choices under uncertainty. However, several new questions arise by the introduction of k-level orderings. For example, it could be interesting analyze the relations/differences existing among functionals $\rho_{X,\alpha}^{(k)}$ and $\rho_{X,\beta}^{(s)}$ for $s \ne k$ and/or $\alpha, \beta > 1$, $\alpha \ne \beta$ in order to understand their impact on investors' preferences. We also believe that some of the "moments" properties verified by Fishburn (1980b) and O'Brien (1984) can be extended to FORS type orderings. However, because of space constraints, we cannot be exhaustive in our analysis and further analysis of these issues will be the object of future research.

3.1.1 Examples of FORS measures and orderings

Typical examples of FORS ordering are the classical stochastic orders and their dual which are induced by the first stochastic dominance order. Consider the following examples of FORS measures and orderings.

Moment FORS measures: For any fixed real t, $\rho_X(\lambda)=\Gamma(\lambda+1)F_X^{(\lambda+1)}(t)=E\left(\left(t-X\right)_+^{\lambda}\right)$ is a primary probability functional over the class of p-integrable random variables $\Lambda=L^p=\left\{X/E\left(|X|^p\right)<+\infty\right\}$. In addition, $\rho_X(\lambda)$ defined for every $\lambda\geq m$ and a given m<p is a FORS measure induced by $\underset{m+1}{\overset{b}{>}}$. Then for every $\alpha\geq 1$, the measure ${}_m\rho_{X,\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_m^u(u-s)^{\alpha-1}d\rho_X(s)$ $\forall u\geq m$ with m<p is a FORS measure induced by $\underset{m+1}{\overset{b}{>}}$ that identifies the distribution of the tail (i.e., the measure ${}_m\rho_{X,\alpha}(u)={}_m\rho_{Y,\alpha}(u)$ $\forall u\geq m$ for a given $\alpha\geq 1\Leftrightarrow F_X(x)=F_Y(x)$ $\forall x\leq t$). This is a logical consequence of the inverse Mellin transform applied to the moment curve of the positive random variable $\left(t-X\right)_+$ that univocally determines the distribution of the tail.

Weak Moment FORS orderings: Let us consider the class of random variables above bounded *p*-integrable $\Lambda=\left\{Z\in L^p/Z\leq b<+\infty\right\}$. Then for every m<p, we can consider ${}_m\rho_X(\lambda)=E\left(\left(b-X\right)^{\lambda}\right)$ $\forall\lambda\geq m$ is a FORS measure induced by the m+1 stochastic dominance order $\underset{m+1}{\overset{b}{>}}$ that is also a simple probability functional over the class Λ. Thus for every $m\geq 0$ the following probability functional ${}_m\rho_{X,\alpha}(u)=\frac{1}{\Gamma(\alpha)}\int_m^u(u-s)^{\alpha-1}d_m\rho_X(s)$ $\forall u\geq m$ identifies a FORS ordering induced by the order $\succ\equiv\underset{m+1}{\overset{b}{>}}$. That is, for every pair of random variables *X* and *Y* in the class Λ:

$$X\underset{\succ,\alpha}{FORS}Y \text{ if and only if } {}_m\rho_{X,\alpha}(u)\leq {}_m\rho_{Y,\alpha}(u)\quad\forall u\geq m.$$

Similar analysis can be done with below bounded random variables. Thus for below and above bounded random variables $\tilde{\Lambda}=\left\{Z/-\infty<a\leq Z\leq b<+\infty\right\}$, we can express moment FORS orderings induced by the order $\succ\equiv\underset{m+1}{>}$ considering that $\forall Z\in\tilde{\Lambda}$ we have $Z\leq b$ and $-Z\leq -a$. Thus if $\forall\lambda\geq m$ ${}_m\rho_X(\lambda)=E\left(\left(b-X\right)^{\lambda}\right)$ then ${}_m\rho_{-X}(\lambda)=E\left(\left(X-a\right)^{\lambda}\right)$. Consequently, for every pair of random variables *X* and *Y* belonging to $\tilde{\Lambda}$, we can say that *X* dominates *Y* in the sense of α moment FORS uncertainty ordering induced by the risk ordering $\succ\equiv\underset{m+1}{>}$, when $X\underset{\succ,\alpha}{FORS}Y$ and $-X\underset{\succ,\alpha}{FORS}-Y$.

***Gini-FORS measure*:** For any fixed $\beta \in (0,1)$ $h_X(v) = \frac{-\Gamma(v)}{\beta^v} F_X^{(-v)}(\beta)$ is a primary probability functional over the class of random variables $\Lambda = \left\{ X / p > 1 : \left| F_X^{(-p)}(\beta) \right| < +\infty \right\}$. As explained in Ortobelli et al (2006), the following measures are strictly linked to the extended Gini measure. This justifies the name we gave it. In addition, $h_X(v)$ defined for every $v \geq m$ and a given m<p is a FORS measure induced by $\underset{-m}{>}$ that is coherent when $p > v \geq 2$. Then for every $\alpha \geq 1$, the measure ${}_m h_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_m^u (u-s)^{\alpha-1} dh_X(s)$ $\forall u \geq m$ with m<p is a FORS measure induced by $\underset{-m}{>}$ that identify the distribution of the tail (i.e. the measure ${}_m h_{X,\alpha}(u) = {}_m h_{Y,\alpha}(u)$ $\forall u \geq m$ for a given $\alpha \geq 1 \Leftrightarrow F_X^{-1}(q) = F_Y^{-1}(q)$ $\forall q \leq \beta$).

Gini- FORS orderings: Let us consider for every $\lambda \geq m > 1$ ${}_m\rho_X(\lambda) = -\Gamma(\lambda) F_X^{(-\lambda)}(1)$ is a FORS measure induced by the *m* dual stochastic dominance order $\underset{-m}{>}$ that is also a simple probability functional over the class $\Lambda = \left\{ X / \lambda > m : \left| F_X^{(-\lambda)}(1) \right| < +\infty \right\}$. Thus for every $m > 1$, the following probability functional ${}_m\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_m^u (u-s)^{\alpha-1} d_m\rho_X(s)$ $\forall u \geq m$ identifies a FORS ordering induced by the order $\underset{-m}{>}$. That is, for every pair of random variables *X* and *Y* in the class Λ:

$$X \underset{\succ,\alpha}{FORS} Y \text{ if and only if } {}_m\rho_{X,\alpha}(u) \leq {}_m\rho_{Y,\alpha}(u) \ \ \forall u \in [m, \lambda].$$

Moreover for $m \geq 2$, at less of a multiplicative factor ${}_m\rho_X(u) = -\Gamma(u) F_X^{(-u)}(1)$ is a coherent FORS functional. As for the weak moment FORS ordering, we can define a Gini-FORS uncertainty order. Observe that several portfolio selection problems can be defined extending the classic order considerations to FORS orderings (see Ortobelli et al (2006)).

4. Dynamic Probability Functionals and Orderings

The previous discussion deals with the problem of ordering and quantifying today the riskiness of future financial positions with maturity *T*. For this reason, the previous approach can be considered a "static" approach. Next we extend this approach considering two classes of dynamic probability functionals: for random variables and for stochastic processes. In all the cases we assume a filtered probability space $\left(\Omega, \Im, (\Im_t)_{0 \leq t \leq T}, P\right)$ with $T \in (0, \infty)$, that satisfies the usual conditions; that is, the probability space $(\Omega, \Im, P)$ is complete, $(\Im_t)$ is right continuous and $\Im_0$ contains all the null sets of $\Im$.

4.1 Dynamic probability functionals for random variables

For every *t* belonging to $[0,T]$ we denote by $L^0(\Im_t)$ the space of all real valued $\Im_t$ measurable random variables defined on $(\Omega, \Im_t, P)$. Intuitively, any dynamic probability functional is a map $(\mu_t)_{t\in[0,T]}$ indexed by the interval of time$[0,T]$ where at any instant *t* the random variable μ_t measures a distance by a benchmark or the possible losses "conditionally" to the information available at time *t.* Thus we call *dynamic probability functional* for a random wealth defined on $L^p(\Im_T)$ $p \geq 0$ any map $(\mu_t)_{t\in[0,T]}$ such that $\mu_t : L^p(\Im_T) \times L^p(\Im_T) \to L^0(\Im_t)$ for every $t \in [0,T]$ and μ_0 is a static probability functional in the sense previously described. In this case at any time $t \in (0,T]$ and for any $X, Y \in L^p(\Im_T)$, $\mu_t(X,Y)$ is a $\Im_t$ measurable random variable and it is a conditional probability functional. According to this, $\forall t \in (0,T]$ any equality and/or inequality of μ_t should be considered valid P almost surely. As for static probability functionals, next we discuss the main properties of a dynamic probability functional. In particular, we define

Dynamic identity property (for random variables): $f(X) = f(Y) \Leftrightarrow \mu_t(X,Y) = 0$ P-a.s. *for any* $t \in [0,T]$*; where f(X) identifies some characteristics of random variable X.*

Even in this case we can identify *primary*, (weak, strong) *simple* and *compound* dynamic probability functionals. In particular, weak simple dynamic probability functional implies that for every $t \in [0,T]$ $\mu_t(X,Y) = 0$ $P-a.s.$ if and only if $F_{E(X/\Im_t)} = F_{E(Y/\Im_t)}$ $\forall t \in [0,T]$. Strong simple dynamic probability functional implies that for every $t \in [0,T]$ $\mu_t(X,Y) = 0$ $P-a.s.$ if and only if the processes $\{E(X/\Im_t)\}_{0\leq t\leq T}$ and $\{E(Y/\Im_t)\}_{0\leq t\leq T}$ have the same finite dimensional distributions (i.e., for any integer $n \geq 1$, real numbers $0 \leq t_1 < ... < t_n \leq T$ and $A \in B_{R^n}$ Borel sigma algebra of R^n we have $P\left(\left(E(X/\Im_{t_1}),...,E(X/\Im_{t_n})\right) \in A\right) = P\left(\left(E(Y/\Im_{t_1}),...,E(Y/\Im_{t_n})\right) \in A\right)$).

Similarly *compound* dynamic probability functional implies that for every $t \in [0,T]$ $\mu_t(X,Y) = 0$ $P-a.s. \Leftrightarrow P(X=Y) = 1$ that is the process $\{E(X/\Im_t)\}_{0\leq t\leq T}$ is a version of the process $\{E(Y/\Im_t)\}_{0\leq t\leq T}$. In addition, as a consequence of the right continuity of the filtration, the conditional processes have right continuous sample paths and $P(X = Y) = 1$ implies the processes

$\{E(X/\Im_t)\}_{0\le t\le T}$ and $\{E(Y/\Im_t)\}_{0\le t\le T}$ are indistinguishable. Instead *primary* dynamic probability functional determines only some random variable characteristics at some time *t* belonging to the interval $[0,T]$. Therefore, with respect to the portfolio selection problem, we could have dynamic probability functionals μ_t that identify the uncertainty or the losses of the random variable at a given time *t*. For example, we can say that two portfolios $X,Y\in L^p(\Im_T)$ (for an opportune $p\ge 1$) for any $t\in[0,T]$ are "equivalent in uncertainty" conditionally to the information at time *t* if their conditional expectation present:

- the same conditional absolute central moment; that is, :

$$\mu_t(X,Y)=0 \Leftrightarrow E(|X-E(X)|^p/\Im_t)=E(|Y-E(Y)|^p/\Im_t) \text{ P a.s.}$$

- the same conditional level of concentration valued with an opportune moment *p;* that is,

$$\mu_t(X,Y)=0 \Leftrightarrow E(|X_1-X|^p/\Im_t)=E(|Y_1-Y|^p/\Im_t), \text{ P a.s.}$$

  where X_1 is an independent $\Im_T$ measurable copy of X and Y_1 is an independent $\Im_T$ measurable copy of *Y*.

Similarly, we can assume for any $t\in[0,T]$ "equivalent in losses (risk)" conditional on the information at time *t* if their conditional expectation present:

- the same distributional tail $\mu_t(X,Y)=0 \Leftrightarrow F_{E(X/\Im_t)}(x)=F_{E(Y/\Im_t)}(x)\ \forall x\in(-\infty,a_t]$ for a given threshold $a_t\in R$;
- the same power of the tail valued on the left tails with an opportune moment $\mu_t(X,Y)=0 \Leftrightarrow E\left((a_t-E(X/\Im_t))_+^p\right)=E\left((a_t-E(Y/\Im_t))_+^p\right)$ for a given threshold $a_t\in L^p(\Im_t)$.

According to the definition of dynamic probability functionals for random variables $(\mu_t)_{t\in[0,T]}$ we can define the consistency as follows:

Inter-temporal risk consistency (for random variables): *for any pair of random variables* $X,Y\in L^p(\Im_T)$ *X dominates Y respect to a given order of preferences* $\succ$ *implies that for a given benchmark* $Z\in L^p(\Im_T)$ $\mu_0(X,Z)\le\mu_0(Y,Z)$, *and for any* $t\in(0,T]$ *the order of preferences* $\succ$ *induce an order* $\succ_t$ *among the* $\Im_t$ *measurable random functionals* μ_t *i.e.* $\mu_t(Y,Z)\succ_t\mu_t(X,Z)$.

We call *FORS dynamic measures induced by order* $\succ$ all the dynamic probability functionals $(\mu_t)_{t\in[0,T]}$ that satisfy the inter-temporal risk consistency with respect to a given order of preferences. Most of the real cases studied in the literature require that the risk (uncertainty) of financial position X is strictly lower than the risk (uncertainty) of position Y (i.e., $\mu_t(Y,Z) \geq \mu_t(X,Z)$ P a.s.). However, it would be that this risk (uncertainty) at a given time $t \in (0,T]$ is lower only for a given category of investors. For this reason in, the previous definition we require that the order of preference induce an "order of preferences" even for the risk $\mu_t(Y,Z) \succ_t \mu_t(X,Z)$. Moreover, in some cases we could have the equivalency between the relations $\mu_T(Y,Z) \succ_T \mu_T(X,Z)$ and $X \succ Y$ considering that the map $\mu_T(\bullet,Z)$ is $\Im_T$ measurable.

Clearly, many new questions are born from the above definitions. As a matter of fact, we can distinguish several different types of human behavior when we introduce the new variable "time". For example, it is logic to think that there exists an investor who reserves his preferences to X with respect to Y, if there exist at least a time $t \in [0,T]$ such that $E(X/\Im_t)$ is preferred to $E(Y/\Im_t)$ with respect to a given FORS ordering. In addition, as observed by Detlefsen and Scandolo (2005), we could expect that different measurements of the same payoff at different dates should be related in some way. We think this way has to take into account some macroeconomic investors' expectation of the future evolution of the market. Thus, we could consider investors with:

- *Increasing expectative during the time* $[0,T]$. That is, with the passing of time, there is aN increasing interest in random wealth valued at time T and the class of investors interested In these random wealths increases. For example, suppose for any pair of random wealths $X,Y \in L^p(\Im_T)$ the relation $X \underset{\succ,(\alpha-1)}{FORS} Y$ implies for every t belonging to $(0,T]$ $E(X/\Im_t) \underset{\succ,\beta(t)}{FORS} E(Y/\Im_t)$ where $\beta(t)$ is a decreasing positive function that forecasts the evolution of market interest during the period $[0,T]$ (say $\beta(t) = \alpha - \frac{t}{T}$ for a given $\alpha \geq 2$). Therefore $\forall t \in [0,T]$ we get $\mu_0(E(X/\Im_t),Z) \leq \mu_0(E(Y/\Im_t),Z)$ and $\mu_t(Y,Z) \succ_t \mu_t(X,Z)$.
- *Decreasing expectative during the time* $[0,T]$. That is, with the passing of time, there is a decreasing interest in random wealths valued at time T and the class of investors interested In these random wealths decreases. For example, suppose for any pair of random wealths $X,Y \in L^p(\Im_T)$ the relation $X \underset{\succ,\alpha+1}{FORS} Y$ implies for every t belonging to $(0,T]$ $E(X/\Im_t) \underset{\succ,\beta(t)}{FORS} E(Y/\Im_t)$ where $\beta(t)$ is an increasing positive function on $[0,T]$ (say

$\beta(t)=\alpha+\frac{t}{T}$ for a given $\alpha\geq 1$). Therefore $\mu_0(X,Z)\leq\mu_0(Y,Z)$ and $\mu_t(Y,Z)\succ_t\mu_t(X,Z)$, but it is not necessarily verified that $\mu_0(E(X/\Im_t),Z)\leq\mu_0(E(Y/\Im_t),Z)$ $\forall t\in[0,T)$.

- *Constant expectative during the time* $[0,T]$. For example, suppose for any pair of random wealths $X,Y\in L^p(\Im_T)$ the relation $X\underset{\succ,\alpha}{FORS}Y$ implies for every t belonging to $(0,T]$ $E(X/\Im_t)\underset{\succ,\alpha}{FORS}E(Y/\Im_t)$ Therefore $\forall t\in[0,T]$, we get $\mu_0(E(X/\Im_t),Z)\leq\mu_0(E(Y/\Im_t),Z)$ and $\mu_t(Y,Z)\succ_t\mu_t(X,Z)$.
- *Mixed expectative during the time* $[0,T]$. That is, we do not have a definite relation with the passing of time.

Therefore when the investors' choices are made in a dynamic framework, we can essentially identify two forecasting steps: (1) forecasting the evolution of the market and (2) forecasting the best choices consistently with the evolution of the market. The first forecasting step is fundamental to understand the evolution of investors' expectations. The second forecasting step has been the subject of many discussions in the recent literature on risk measures. Clearly, the choice of opportune FORS dynamic measures is strictly dependent on the decision made in the first step.

Typical examples of FORS dynamic measures are given by the following $\Im_t$ measurable $\mu_t(X,Z)$ functionals:

1. $\mu_0(X,p)=\frac{-\Gamma(\beta(0))}{(p)^{\beta(t)-1}}F_X^{(-\beta(0))}(p)$ for a given constant benchmark p and for every $t\in(0,T]$ $\mu_t(X,a_t(w))=\frac{-\Gamma(\beta(t))}{(a_t(w))^{\beta(t)-1}}F_{E(X/\Im_t)}^{(-\beta(t))}(a_t(w))$, $\forall w\in\Omega$, where a_t is a $\Im_t$ measurable benchmark that assume values on (0,1]. Thus we assume there is a $\underset{-\beta(t)}{>}$ order between the conditional values at each time t (with $\beta(t)\geq 1$, $\forall t\in[0,T]$) and $X\succ Y$ implies that $\mu_t(X,a_t)\leq\mu_t(Y,a_t)$ P-a.s. $\forall t\in[0,T]$.
2. $\mu_0(X,k)=F_X^{(\beta(0))}(k)$ for some given constant benchmark k and for every $t\in(0,T]$ $\mu_t(X,a_t(w))=F_{E(X/\Im_t)}^{(\beta(t))}(a_t(w))$, $\forall w\in\Omega$, where a_t is a $\Im_t$ measurable benchmark. Thus we assume there exist a $\underset{\beta(t)}{>}$ order between the conditional values at each time t (with $\beta(t)\geq 1$, $\forall t\in[0,T]$) and $X\succ Y$ implies that $\mu_t(X,a_t)\leq\mu_t(Y,a_t)$ P-a.s. $\forall t\in[0,T]$.

3. $\mu_0(X,k)=E\left(\left(k-X\right)_+\right)$ for a fixed benchmark k and for every $t\in(0,T]$ $\mu_t(X,k)=E\left(\left(k-X\right)_+/\Im_t\right)$. Thus we could assume that $X\succ Y$ implies $\mu_t(Y,Z)\underset{\beta(t)}{>}\mu_t(X,Z)$ $\forall t\in[0,T]$ where $\beta(t)$ is an opportune function (say $\beta(t)\geq 1$, $\forall t\in[0,T]$).

4. $\mu_0(X,k)=E\left(\left|k-X\right|^{\beta(0)-1}\right)$ for some given constant benchmark k and for every $t\in(0,T]$ $\mu_t(X,a_t(w))=E\left(\left|a_t(w)-E(X/\Im_t)\right|^{\beta(t)-1}\right)$ $\forall w\in\Omega$, where a_t is a $\Im_t$ measurable benchmark. Thus we could assume there exist a $\beta(t)$- *R-S* order (with $\beta(t)\geq 2$, $\forall t\in[0,T]$) between the conditional values at each time *t* and $X\succ Y$ implies that $\mu_t(X,a_t)\leq\mu_t(Y,a_t)$ P-a.s. $\forall t\in[0,T]$.

5. $\mu_0(X,k)=E\left(\left|k-X\right|^{\beta(0)-1}\right)$ for some given constant benchmark k and for every $t\in(0,T]$ $\mu_t(X,k)=E\left(\left|k-X\right|^{\beta(t)-1}/\Im_t\right)$ (with an opportune $\beta(t)\geq 2$, $\forall t\in[0,T]$). Thus we could assume that $X\succ Y$ implies $\mu_t(Y,k)$ *FSD* $\mu_t(X,k)$ $\forall t\in[0,T]$.

As for static FORS measures, we distinguish *FORS dynamic risk measures* (the first three examples) from *FORS dynamic uncertainty measures* (the last two examples). In the most simple case we could require that the inter-temporal preferences corresponds to the monotony P almost surely everywhere (i.e., if $X\geq Y$, $\mu_t(X,Z)\leq\mu_t(Y,Z)$ for any $t\in[0,T]$ P a.s.). This is a case of constant expectations during the time $[0,T]$. As a matter of fact, as a consequence of the $\Im_T$ measurability of $\mu_t(\bullet,Z)$ and $E(Y/\Im_t)$, we get $X\geq Y$ P *a.s.* implies $E(X/\Im_t)\geq E(Y/\Im_t)$ P a.s., $\mu_t(E(X/\Im_s),Z)\leq\mu_t(E(Y/\Im_s),Z)$. Thus, when functionals of functionals are finite, $X\geq Y$ P *a.s.* also implies $\mu_t(\mu_s(Y,Z),Z)\leq\mu_t(\mu_s(X,Z),Z)$ P *a.s.* for every $s,t\in[0,T]$.

In addition to these relations, we could require further types of time relations: time consistency, recursivity, super-martingale property (see typical examples of dynamic risk measures defined by Frittelli and Rosazza Gianin (2004), Detlefsen and Scandolo (2005), and Artzner et al. (2002)). As for the static probability functional, no rule relative to the benchmark Z is described in the above definition. Thus, as a sub-case of the previous definition, we can consider a FORS dynamic measure $(\mu_t)_{t\in[0,T]}$ induced by an order preference $\succ$ such that $X\succ Y$ implies $\mu_t:L^p(\Im_T)\to L^0(\Im_t)$, $\mu_0(X)\leq\mu_0(Y)$ and $\mu_t(Y)$ is preferred to $\mu_t(X)$ with respect to a given inter-temporal order of preferences $\succ_t$.

Dynamic risk measures for random variables defined in the literature are a particular sub-case of our previous analysis. So, even convex and coherent dynamic risk measures are particular FORS dynamic risk measures consistent with the monotonic order. Recall that a coherent dynamic risk measure is a map $(\mu_t)_{t\in[0,T]}$ such that $\mu_t : L^p(\Im_T) \to L^0(\Im_t)$ with $\mu_T(X) = -X$ for every $X \in L^p(\Im_T)$ and that is:

1. Monotone: $\forall X, Y \in L^p(\Im_T)$ ($p \geq 1$) $X \geq Y \Rightarrow \forall t \in [0,T]$, $\mu_t(X) \leq \mu_t(Y)$ P-a.s.

2. Translation invariant: $\forall t \in [0,T]$ $\forall X, Z \in L^p(\Im_T)$ such that Z is $\Im_t$ measurable $\mu_t(X + Z) = \mu_t(X) - Z$ P-a.s.

3. Convex: $\forall X, Y \in L^p(\Im_T)$ $\forall t \in [0,T]$, $\forall a \in [0,1]$,

$$\mu_t(aX + (1-a)Y) \leq a\mu_t(X) + (1-a)\mu_t(Y) \text{ P-a.s.}$$

If additionally we consider positive homogeneity, then

4. Positive homogeneous: $\forall \alpha \geq 0$ $\forall X \in L^p(\Im_T)$ $\forall t \in [0,T]$, $\mu_t(\alpha X) = \alpha\mu_t(X)$ P-a.s.,

we have a coherent dynamic risk measure. Examples of convex and coherent dynamic measures can be found in Frittelli and Rosazza Gianin (2004), Detlefsen and Scandolo (2005), and Riedel (2004). However even if this definition serves to price correctly the risk, it appears too strong because it does not take into account the different perception of risk of the several operators in the market. For this reason we introduce a weak definition in terms of FORS dynamic measures. When a FORS dynamic risk measure is convex and translation invariant at least with respect to constant variables, then we say that it is a *dynamic convex* FORS *measure*. If it additionally is also positive homogeneous, then we say that it is a *dynamic coherent FORS measure*. Typically, examples of dynamic coherent FORS risk measures are given by the first of the previous examples for $\beta(t) \geq 2$, $\forall t \in [0,T]$. Similarly we can extend the previous definition to dynamic reward measures isotonic to orderings. Thus any dynamic probability functional for random variables $(\mu_t)_{t\in[0,T]}$ associated with a portfolio choice problem that satisfies the property:

Inter-temporal reward isotonicity (for random variables): *for any pair of random variables* $X, Y \in L^p(\Im_T)$ *X dominates Y respect to a risk ordering implies that for a given benchmark* $Z \in L^p(\Im_T)$ $\mu_0(X,Z) \leq \mu_0(Y,Z)$, *and for any* $t \in (0,T]$ *the risk ordering induce an order* $\succ_t$ *among the* $\Im_t$ *measurable random functionals i.e.* $\mu_t(X,Z) \succ_t \mu_t(Y,Z)$.

is called a *FORS dynamic reward measure* induced by *risk* order $\succ$. Any consideration done for FORS dynamic risk measures can be easily extended to FORS dynamic reward measures.

4.2 Static and Dynamic Probability Functionals for Stochastic Processes

Typically on a filtered probability space $\left(\Omega,\Im,(\Im_t)_{0\le t\le T},P\right)$ that satisfies the usual assumptions, we consider indistinguishable classes of RCLL processes belonging to the Banach space:

$$\Re^p=\left\{X:[0,T]\times\Omega\to R/X\ RCLL;\ \Im_t\ adapted;\ \|X\|_{\Re^p}<\infty\right\}$$

with the norm $\|X\|_{\Re^p}:=\left\|\sup_{0\le t\le T}|X_t|\right\|_p$, where $\|.\|_p$ is the norm in L^p. As for random variables we can propose static and dynamic measures for stochastic processes.

4.2.1 Static measures for processes

Intuitively, any static probability functional for stochastic processes is a map $\mu:\Re^p\times\Re^p\to R$ that satisfies some properties. A typical example is:

The identity property (for stochastic processes): $f(X)=f(Y)\Leftrightarrow\mu(X,Y)=0$ *where f(X) identifies some characteristics of the stochastic process X.*

Even in this case we can identify *primary*, (weak strong) *simple,* and *compound* static probability functionals for processes. In particular, weak simple dynamic probability functional implies that for every $t\in[0,T]$ $\mu_t(X,Y)=0$ $P-a.s.$ if and only if $F_{X_t}=F_{Y_t}$ $\forall t\in[0,T]$. Strong simple probability functional implies that $\mu(X,Y)=0$ if and only if the processes $X=\{X_t\}_{0\le t\le T}$ and $Y=\{Y_t\}_{0\le t\le T}$ have the same finite dimensional distributions. Similarly, weak (strong) *compound* probability functional implies that for every $t\in[0,T]$, $\mu(X,Y)=0\Leftrightarrow P(X_t=Y_t)=1$; that is, the process $X=\{X_t\}_{0\le t\le T}$ is a version of (indistinguishable by) the process $Y=\{Y_t\}_{0\le t\le T}$.

When we consider RCLL processes, there is no distinction between weak and strong compound probability functionals because we have right continuous sample paths of the processes. Instead, *primary* probability functional identifies only some stochastic processes characteristics. For example, we can say that two processes are "equivalent in uncertainty" if they present the same absolute central moment at each time t (i.e., $\mu(X,Y)=0\Leftrightarrow E(|X_t-E(X_t)|^p)=E(|Y_t-E(Y_t)|^p)$ $\forall t\in[0,T]$) or we can say that the processes are "equivalent in losses (risk)" if they present the same distributional tail at each time *t* (i.e., $\mu(X,Y)=0\Leftrightarrow F_{X_t}(x)=F_{Y_t}(x)$ $\forall x\in(-\infty,a_t]$ $\forall t\in[0,T]$ for a given threshold $a_t\in R$).

According to the definition of static probability functionals for stochastic processes $\mu : \Re^p \times \Re^p \to R$, we can define the consistency and isotonicity as follows:

Risk consistency-isotonicity (for stochastic processes): *we say that that the measure* $\mu(.)$ *is consistent* (*isotonic*) *respect to a given* (*risk*) *order of preferences* $\succ$, *if for any pair of stochastic processes* $X, Y \in \Re^p$ *such that* $X \succ Y$ *implies that for a given benchmark* $Z \in \Re^p$ $\mu(X,Z) \le \mu(Y,Z)$ ($\mu(X,Z) \ge \mu(Y,Z)$).

We call *FORS* (*reward*) *measures induced by order* $\succ$ on $\Re^p$ all the probability functionals μ that satisfy the risk consistency (isotonic) with respect to a given (risk) order of preferences among stochastic processes in $\Re^p$. Clearly first of all we need to define an order of preferences among stochastic processes. Typically we can think that the order of preferences induces for any $t \in [0,T]$ an order $\succ_t$ among the $\Im_t$ measurable random variables X_t and Y_t of the two processes (i.e., $X_t \succ_t Y_t \quad \forall t \in [0,T]$). Therefore, typical examples of static risk (the first three examples) or uncertainty (the last two examples) of FORS measures are the following functionals between processes:

1. $\mu(X,k) = -\frac{1}{T}\int_0^T \frac{\Gamma(\beta(t))}{k^{\beta(t)-1}} F_{X_t}^{(-\beta(t))}(k)dt$ for a given constant benchmark *k*. Thus everytime $X \succ Y$ implies $X_t \underset{-\beta(t)}{>} Y_t$ order $\forall t \in [0,T]$, then $\mu(X,k) \le \mu(Y,k)$ (where we assume that the function $g(t) = \frac{\Gamma(\beta(t))}{k^{\beta(t)-1}} F_{X_t}^{(-\beta(t))}(k) \quad \forall t \in [0,T]$ is Lebesgue integrable).

2. $\mu(X,k) = \int_0^T F_{X_t}^{(\beta(t))}(k)dt$ for some given constant benchmark *k*. Thus every time $X \succ Y$ implies $X_t \underset{\beta(t)}{>} Y_t$ order $\forall t \in [0,T]$, then $\mu(X,k) \le \mu(Y,k)$ (where we assume that the function $g(t) = F_{X_t}^{\beta(t)}(k) \quad \forall t \in [0,T]$ is Lebesgue integrable).

3. $\mu(X,k) = F^{(\beta)}_{\int_0^T X_t dt}(k)$ for some given constant benchmark *k*. Thus every time $X \succ Y$ implies $\int_0^T X_t dt \underset{\beta}{>} \int_0^T Y_t dt$, then $\mu(X,k) \le \mu(Y,k)$.

4. $\mu(X,k) = \int_0^T E\left(\left|k - X_t\right|^{\beta(t)-1}\right)dt$ for some given constant benchmark *k*. Thus every time $X \succ Y$ implies X_t $\beta(t)$ *R-S dominates* Y_t $\forall t \in [0,T]$, then $\mu(X,k) \le \mu(Y,k)$ (where we assume that the function $g(t) = E\left(\left|k - X_t\right|^{\beta(t)-1}\right) \quad \forall t \in [0,T]$ is Lebesgue integrable).

5. $\mu(X,k) = E\left(\left|k - \frac{1}{T}\int_0^T X_t dt\right|^{\beta-1}\right)$ for some given constant benchmark *k.* Thus every time $X \succ Y$ implies $\int_0^T X_t dt$ β *R-S dominates* $\int_0^T Y_t dt$, then $\mu(X,k) \le \mu(Y,k)$.

Observe that no rule relative to the benchmark Z is described in the above definition. Thus, as sub-case of the previous definition, we can consider a static FORS measure μ induced by an order preference $\succ$ such that $X \succ Y$ implies $\mu : \Re^p \to R$, $\mu(X) \le \mu(Y)$. Typical examples are the convex and coherent static FORS measures defined by Cheiridito et al (2004, 2005). A convex measure for stochastic processes satisfies the properties:

1. Monotone for every $X, Y \in \Re^\infty$ $X \ge Y \Rightarrow \mu(X) \le \mu(Y)$
2. Translation invariant, $\forall X \in \Re^\infty$ and $m \in R$ such that $\mu(X+m) = \mu(X) - m$
3. Convex $\forall X, Y \in \Re^\infty$, $\forall a \in [0,1]$,

$$\mu(aX + (1-a)Y) \le a\mu(X) + (1-a)\mu(Y)$$

If additionally we consider even positive homogeneity

4. Positive homogeneous $\forall \alpha \ge 0$ $\forall X \in \Re^p$, $\mu(\alpha X) = \alpha\mu(X)$

then we have a coherent static risk measure for processes. When a *FORS* risk measure for stochastic processes is convex and translation invariant, then we say that is a *convex FORS measure* for stochastic processes. If it additionally is also positive homogeneous, then we say that it is a *coherent FORS measure* for stochastic processes. Typical examples of coherent FORS measure for stochastic processes is given by the first example earlier when $\beta(t) \ge 2$ $\forall t \in [0,T]$ or by $\mu(X,k) = \frac{\Gamma(\beta)}{k^{\beta-1}} F^{(-\beta)}_{\frac{1}{T}\int_0^T X_t dt}(k)$ with $\beta \ge 2$. Other examples can be found in Cheridito et al. (2004, 2005). Analogously, we can extend the previous definition to dynamic risk (reward) measures for stochastic processes.

4.2.2 Dynamic measures for processes

Mimicking the definition of dynamic measures for random wealth, a dynamic probability functional for a stochastic process is a map $(\mu_t)_{t\in[0,T]}$ indexed by the interval of time $[0,T]$ where at any instant *t* the random variable μ_t measures the possible losses or a distance by a benchmark process "conditional" on the information available at time *t.* Thus we call *dynamic probability functional* for a stochastic process to be defined on $\Re^p$ any map $(\mu_t)_{t\in[0,T]}$ such that $\mu_t : \Re^p \times \Re^p \to L^0(\Im_t)$ for every $t\in[0,T]$ and μ_0 is a static probability functional in the sense previously described. As for dynamic measures for random wealth, at any time $t\in(0,T]$ and for any $X,Y\in\Re^p$, $\mu_t(X,Y)$ is a $\Im_t$ measurable random variable and it is a conditional probability functional. As for static probability functionals of processes/random variables, next we discuss the main properties of a dynamic probability functional. Thus we generally consider the

Dynamic identity property (for stochastic processes): $f(X)=f(Y)\Leftrightarrow \mu_t(X,Y)=0$ P-a.s. *for any* $t\in[0,T]$*; where f(X) identifies some characteristics of stochastic process X.*

In this case, weak simple dynamic probability functional implies that for every $t\in[0,T]$ $\mu_t(X,Y)=0\ P-a.s.$ if and only if $F_{X_t}=F_{Y_t}$ $\forall t\in[0,T]$. Strong simple probability functional implies that for every $t\in[0,T]$ $\mu_t(X,Y)=0\ P-a.s.$ if and only if the processes $\{X_t\}_{0\le t\le T}$ and $\{Y_t\}_{0\le t\le T}$ have the same finite dimensional distributions. Similarly *compound* dynamic probability functional implies that for every $t\in[0,T]$ $\mu_t(X,Y)=0\ P-a.s.\Leftrightarrow P(X_t=Y_t)=1$; that is, X is a modification of Y and if $X,Y\in\Re^p$ they are indistinguishable. Instead *primary* dynamic probability functional determines only some stochastic processes characteristics at some time *t* belonging to the interval $[0,T]$.

According to the definition of dynamic probability functionals for stochastic processes $(\mu_t)_{t\in[0,T]}$, we can define the consistency as follows:

Inter-temporal risk consistency (for stochastic processes): *for any pair of stochastic processes* $X,Y\in\Re^p$ *X dominates Y respect to a given order of preferences* $\succ$ *implies that for a given benchmark* $Z\in\Re^p$ $\mu_0(X,Z)\le\mu_0(Y,Z)$*, and for any* $t\in(0,T]$ *the order of preferences* $\succ$ *induce an order* $\succ_t$ *among the* $\Im_t$ *measurable random functionals i.e.* $\mu_t(Y,Z)\succ_t \mu_t(X,Z)$ *respect to the processes* X,Y *valued only on the period* $[t,T]$ *(i.e.* $\{X_s\}_{t\le s\le T}$ *and* $\{Y_s\}_{t\le s\le T}$*).*

Similarly, we can extend this definition to dynamic reward measures isotonic to orderings of preferences. Analogously, we call *FORS dynamic measures induced by order* $\succ$ among stochastic processes all the dynamic probability functionals $(\mu_t)_{t\in[0,T]}$ that satisfy the inter-temporal consistency with respect to a given order of preferences for stochastic processes. Therefore, typical examples of FORS dynamic measure are given by the following $\Im_t$ measurable $\mu_t(X,Z)$ functionals defined for $X,Y\in\Re^p$ and an opportune p :

1. $\mu_0(X,q) = -\frac{1}{T}\int_0^T \frac{\Gamma(\beta(s))}{q^{\beta(s)-1}} F_{X_s}^{(-\beta(s))}(q)ds$ for a given constant benchmark q and for every $t \in (0,T]$ the $\Im_t$ measurable function $\mu_t(X,a_t(w)) = -\frac{1}{T-t}\int_t^T \frac{\Gamma(\beta(s))}{(a_t(w))^{\beta(s)-1}} F_{X_s}^{(-\beta(s))}(a_t(w))ds$, defined $\forall w \in \Omega$, where a_t is $\Im_t$ measurable benchmark that assumes values on (0,1] and the function $g(s) = \frac{\Gamma(\beta(s))}{(a_t(w))^{\beta(s)-1}} F_{X_s}^{\beta(s)}(a_t(w))$ is Riemann integrable on $[0,T]$ $\forall w \in \Omega$. Thus when $X \succ Y$ implies $X_t \underset{-\beta(t)}{>} Y_t$ at each time *t*, then $\mu_t(X,a_t) \le \mu_t(Y,a_t)$ P-a.s. $\forall t \in [0,T]$.

2. $\mu_0(X,k) = \frac{\Gamma(\beta(0))}{k^{\beta-1}} F_{\frac{1}{T}\int_0^T X_t dt}^{(-\beta(0))}(k)$ for a fixed benchmark k and for every $t \in (0,T]$ the $\Im_t$ measurable function $\mu_t(X,a(w)) = \frac{\Gamma(\beta(t))}{(a_t(w))^{\beta(t)-1}} F_{\frac{1}{T-t}\int_t^T X_s ds}^{(-\beta(t))}(a_t(w))$ defined $\forall w \in \Omega$, where a_t is $\Im_t$ measurable benchmark that assumes values on (0,1]. Thus when $X \succ Y$ implies $\frac{1}{T-t}\int_t^T X_s ds \underset{-\beta(t)}{>} \frac{1}{T-t}\int_t^T Y_s ds$ at each time *t* then $\mu_t(X,a_t) \le \mu_t(Y,a_t)$ P-a.s. $\forall t \in [0,T]$.

3. $\mu_0(X,k) = \int_0^T E\left((k-X_t)_+^{\beta(t)}\right)dt$ and for every $t \in (0,T]$ $\mu_t(X,k) = \int_t^T E\left((k-X_s)_+^{\beta(s)} / \Im_t\right)ds$ for a fixed benchmark *k* and a continuous function $\beta(t) \ge 1$. Thus we could require that $X \succ Y$ implies $\mu_t(Y,k)$ *FSD* $\mu_t(X,k)$ $\forall t \in [0,T]$.

4. $\mu_t(X,k) = E\left(\left(k - \frac{1}{T-t}\int_t^T X_s ds\right)_+ / \Im_t\right)$ for a fixed benchmark *k* and for every $t \in (0,T]$. Thus we could assume that $X \succ Y$ implies $\mu_t(Y,k) \underset{\beta(t)}{>} \mu_t(X,k)$ $\forall t \in [0,T]$ where $\beta(t)$ is an opportune function (say $\beta(t) \ge 1$, $\forall t \in [0,T]$).

5. $\mu_0(X,k) = E\left(\left|k - \frac{1}{T}\int_0^T X_s ds\right|^{\beta(0)-1}\right)$ for some given constant benchmark k and for every $t \in (0,T]$ the $\Im_t$ measurable function $\mu_t(X,a_t(w)) = E\left(\left|a_t(w) - \frac{1}{T-t}\int_t^T X_s ds\right|^{\beta(t)-1}\right)$ $\forall w \in \Omega$, where a_t is a $\Im_t$ measurable benchmark. Thus when $X \succ Y$ implies $\frac{1}{T-t}\int_t^T X_s ds$ $\beta(t)$- *R-S*

$\frac{1}{T-t}\int_t^T Y_s ds$ at each time t (with $\beta(t) \geq 2, \forall t \in [0,T]$), then $\mu_t(X, a_t) \leq \mu_t(Y, a_t)$ P-a.s. $\forall t \in [0,T]$.

Even in this case we distinguish *FORS dynamic risk measures* (the first four examples) from *FORS dynamic uncertainty measures* (the last example) for stochastic processes.

Observe that no rule relative to the benchmark Z is described in the above definition. Thus, we can consider a dynamic FORS measure $(\mu_t)_{t\in[0,T]}$ induced by an order preference $\succ$ such that $X \succ Y$ implies $\mu_t : \Re^p \to L^0(\Im_t)$, $\mu_t(X) \leq \mu_t(Y)$. Typical examples are the convex and coherent dynamic FORS measures. A convex measure for stochastic processes is a map $(\mu_t)_{t\in[0,T]}$ such that $\mu_t : \Re^p \to L^0(\Im_t)$ with $\mu_T(X) = -X_T$ for every $X \in \Re^p$ and it satisfies the properties:

1. Monotone: for every $X, Y \in \Re^p$ ($p \geq 1$) $X \geq Y \Rightarrow \forall t \in [0,T], \mu_t(X) \leq \mu_t(Y)$ P-a.s.
2. Translation invariant: $\forall X \in \Re^p$, $\forall t \in [0,T]$ and $\Im_t$ measurable function Z, then $\mu_t(X+Z) = \mu_t(X) - Z$ P-a.s.
3. Convex $\forall X, Y \in \Re^p$, $\forall t \in [0,T]$, $\forall a \in [0,1]$,

$$\mu_t(aX + (1-a)Y) \leq a\mu_t(X) + (1-a)\mu_t(Y) \text{ P-a.s.}$$

If additionally we consider even the positive homogeneity

4. Positive homogeneous $\forall \alpha \geq 0\ \forall X \in \Re^p\ \forall t \in [0,T], \mu_t(\alpha X) = \alpha\mu_t(X)$ P-a.s.

then we have a coherent dynamic risk measure for processes. Examples can be found using the conditional version of the static ones proposed by Cheridito et al. (2004, 2005). When a *FORS* risk measure for stochastic processes is convex and translation invariant at least with respect to a constant, then we say that it is a *convex* FORS measure for stochastic processes. If it additionally is also positive homogeneous, then we say that it is a *coherent dynamic FORS measure* for stochastic processes. Typical examples of coherent dynamic FORS measure for stochastic processes are given by the first two examples given earlier when $\beta(t) \geq 2\ \forall t \in [0,T]$.

4.3 Dynamic orderings for stochastic processes

From the previous definition of FORS dynamic measures for wealth and processes, we easily deduce that there exist infinite possible dynamic orderings among processes $\{X_t\}_{0\leq t\leq T}$ that could be even conditional processes of a future wealth X when we assume $X_t = E(X/\Im_t)$. In particular, we can define dynamic orderings as follows.

Definition 7 *Assume* $\rho_X : [a,b]\times[0,T] \to R$ *with* $-\infty \le a < b \le +\infty$ *defined for a given class of stochastic process X belonging to* Λ. *Suppose that for any* $X \in \Lambda$, *for every fixed* $t \in [0,T]$ $\rho_X(u,t)$ *is a bounded variation function respect to the first component u. Furthermore, assume that* ρ_X *identifies the distribution functions of X=*$\{X_t\}_{0\le t\le T} \in \Lambda$ *(i.e.,* $\forall X, Y \in \Lambda$, $\rho_X = \rho_Y \Leftrightarrow F_{X_t} = F_{Y_t}$ $\forall t \in [0,T]$*). If, for any fixed* $(\lambda,t) \in [a,b]\times[0,T]$, $\rho_X(\lambda,t)$ *is a FORS risk measure induced by a risk ordering* $\succ$ *among processes* $X, Y \in \Lambda$ *such that* $X \succ Y$ *implies* $\rho_X(\lambda,t) \le \rho_Y(\lambda,t)$. *Then, we call FORS risk dynamic ordering induced by* $\succ$ *the following new class of orderings defined for every* $\alpha > 0$ *over the class,* $\Lambda_{(\alpha)} = \left\{ Z \in \Lambda \middle| \sup_{0\le t\le T} \left| \int_a^b |u|^{\alpha-1} d\rho_Z(u,t) \right| < \infty \right\}$:

$$X \underset{dyn\succ,\alpha}{FORS} Y \text{ iff } \rho_{X,\alpha}(u,t) \le \rho_{Y,\alpha}(u,t) \ \ \forall (u,t) \in [a,b]\times[0,T]$$

where $\rho_{X,\alpha}(u,t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^{u^-} (u-s)^{\alpha-1} d\rho_X(s,t) & \text{if } \alpha > 0, \alpha \ne 1 \\ \rho_X(u,t) & \text{if } \alpha = 1 \end{cases}$.

We call ρ_X *FORS dynamic measure associated with the FORS dynamic ordering. In addition, we say that X dominates Y in the sense of* α *dynamic FORS uncertainty ordering induced by* $\succ$ *(we simply write* $X \underset{dyn\succ,unc\ \alpha}{FORS} Y$*) if and only if* $\forall (u,t) \in [a,b]\times[0,T]$ $\int_a^u (u-s)_+^{\alpha-1} d\rho_{\pm X}(s,t) \le \int_a^u (u-s)_+^{\alpha-1} d\rho_{\pm Y}(s,t)$ *(i.e. when* $X \underset{dyn\succ,\alpha}{FORS} Y$ *and* $-X \underset{dyn\succ,\alpha}{FORS} -Y$*).*

Even in this case we can propose a class of utility function representative of dynamic FORS orderings.

Corollary 5 *Suppose* ρ_X *is FORS dynamic measure associated with a FORS dynamic ordering* $dyn\succ$ *on a given class of stochastic process X belonging to* Λ. *Then, given* $X, Y \in \Lambda_{(\alpha)}$, $X \underset{dyn\succ,\alpha}{FORS} Y$ *if and only if* $\int_a^b \phi(u) d\rho_X(u,t) \ge \int_a^b \phi(u) d\rho_Y(u,t)$ *for every* $t \in [0,T]$ *and for every* ϕ *belonging to*

$$W^\alpha = \left\{ \phi(x) = -\int_{x^+}^b (s-x)^{\alpha-1} d\tau(s) - k(b-x)^{\alpha-1} \middle| k \ge 0,\ k = 0 \text{ if } b = \infty; \tau \text{ is a } \sigma\text{-finite positive measure s.t. } \forall X \in \Lambda_{(\alpha)} \text{ the function } |s-x|^{\alpha-1} \text{ is } d\tau(s)\times d\rho_X(x,t) \text{ integrable in } [a,b]\times[a,b]\ \forall t \in [0,T] \right\}.$$

Moreover for every $1 \le \alpha < v$, $\phi_v \in W^v$ *if and only if there exists a function* $\phi_\alpha \in W^\alpha$ *such that* $\phi_v(x) = \int_{x^+}^{b} (s-x)^{v-\alpha-1} \phi_\alpha(s)ds$.

Table 1

This table summarizes the main classification of FORS orderings and measures. We distinguish between primary and compound probability functionals; between static and dynamic orderings/measures; between uncertainty and risk orderings/measures; between orderings and survival/dual orderings; between bounded and unbounded orderings and among different level of orderings.

			Identity Property		Consistency/Isotonicity Property	
			STATIC	DYNAMIC	STATIC	DYNAMIC
FORS MEASURES (RISK and REWARD)	Random Variables	Primary				
		Simple		weak strong		
		Compound		weak strong		
	Stochastic Processes	Primary				
		Simple	weak strong	weak strong		
		Compound	weak strong	weak strong		
	Alpha-FORS Risk Measures				Alpha-FORS Uncertainty Measures	

	Static FORS ordering	Dynamic FORS ordering
FORS ORDERINGS	FORS ordering	Dual FORS ordering
	FORS ordering	Survival FORS ordering
	Bounded FORS ordering	Unbounded FORS ordering
	Risk FORS ordering	Uncertainty FORS ordering
	FORSk ordering (k-th level of FORS ordering)	

Similar to the static case, we get a simple way to produce dynamic FORS ordering.

Proposition 5 *Suppose* $|b| < +\infty$ *and* $\rho_X^{(1)} : [a,b] \times [0,T] \to R$ *is a* $FORS1$ *dynamic measure associated with a* $FORS1$ *dynamic ordering* $dyn \succ$ *defined on a class of stochastic processes* $X \in \Lambda$. *If* $\rho_X^{(1)}(u,t)$ *is a monotone function in u, then the probability functional* $\rho_X^{(2)} : [1,p] \times [0,T] \to R$ *with* $\rho_X^{(2)}(u,t) = \rho_{X,u}^{(1)}(b,t)$ *defines a* $FORS2$ *dynamic measure (induced by* $dyn \succ$*) on the class of stochastic processes*

$$\Lambda_p = \left\{ X \in \Lambda \,/\, p > 1 \ \sup_{0 \le t \le T} \left| \rho_{X,p}^{(1)}(b,t) \right| < +\infty \right\}$$

and it is associated with the following new FORS2 ordering induced by the previous one $dyn \succ$ *defined for every* $\alpha > 0$, $\forall X, Y \in \Lambda_{p,(\alpha)} = \left\{ Z \in \Lambda_p \;\middle|\; \sup_{0 \le t \le T} \left| \int_1^p u^{\alpha-1} d\rho_X^{(2)}(u,t) \right| < \infty \right\}$

$$X \underset{dyn\succ,\alpha}{FORS2} Y \;\; iff \;\; \rho_{X,\alpha}^{(2)}(u,t) \le \rho_{Y,\alpha}^{(2)}(u,t) \;\; \forall (u,t) \in [1,p] \times [0,T].$$

Iterating the procedure we can create different levels of dynamic orderings. On the other hand, most of the previous properties and results proved for static ordering can be opportunely extended to a dynamic context. In order to summarize the main orderings and uncertainty/risk measures identified in this paper, we describe in Table 1 the classification of FORS measures/orderings in static and dynamic context.

5. Conclusions

This paper unifies the classical theory of stochastic dominance and investor preferences with the recent literature of risk measures applied to the investors' choice problem. Thus, after distinguishing primary, simple, and compound probability functionals, we propose new orderings, risk and reward measures.

In the first part of this paper, we use the well known stochastic dominance orderings to classify some of the main aspects of orderings and probability functionals. Then we describe a new class of orderings based on probability functionals that satisfy the consistency property with respect to a given order of preferences. Finally we extend to a dynamic context the results obtained with static probability functionals and orderings in the second part of this paper. Clearly this analysis could not be exhaustive and several theoretical and empirical studies are still necessary in order to valuate the impact for investors' choices.

APPENDIX: Proofs

Proof of Proposition 1: Consider $\alpha \in (0,1)$. Let *X* and *Y* be bounded random variables continuous on the extremes of the support, then *X, Y*$< \sup\{x / F_X(x) + F_Y(x) < 2\} = b < +\infty$. Suppose $X \underset{1}{\overset{b}{>}} Y$, then we obtain

$$\Gamma(\alpha) F_X^{(\alpha)}(b) = \int_{-\infty}^{b} (b-x)^{\alpha-1} dF_X(x) \ge \int_{-\infty}^{b} (b-x)^{\alpha-1} dF_Y(x) = F_Y^{(\alpha)}(b)\Gamma(\alpha)$$

because *X* and *Y* are continuous on the extreme b and the function $u(x) = (b-x)^{\alpha-1} I_{[x<b]}$ is an increasing function for $x \le b$ and $\Gamma(\alpha) F_Y^{(\alpha)}(b) = E(u(Y));\; \Gamma(\alpha) F_X^{(\alpha)}(b) = E(u(X))$. Furthermore,

because the function $g(\alpha)=F_X^{(\alpha)}(b)=E\left((b-X)^{\alpha-1}\right)$ is analytic, then there exists $r\in(\alpha,1)$ such that $F_X^{(r)}(b)>F_Y^{(r)}(b)$, otherwise $F_X=F_Y$ because using the Mellin transform we can univocally determine the distribution functions. In the case where *X* and *Y* are bounded random variables not necessarily continuous on the extremes $X,\ Y<b<+\infty$, then for every increasing function $v(x)=(t-x)^{\alpha-1}I_{[x\le b]}$ and $t>b$ it follows

$$E(v(X))=\int_{-\infty}^{b}(t-x)^{\alpha-1}dF_X(x)\ge\int_{-\infty}^{b}(t-x)^{\alpha-1}dF_Y(x)=E(v(Y))$$

and similarly the inequality is strict for some *t*. There are analogous considerations when $X,Y>\inf\{x/F_X(x)+F_Y(x)>0\}=a>-\infty$ because *-X, -Y<-a* and $X\underset{1}{\overset{b}{>}}Y$ iff $-Y\underset{1}{\overset{-a}{>}}-X$. Next, suppose *X* and *Y* are a pair of random variables such that $X\underset{1}{\overset{b}{>}}Y$. Observe that $X\underset{1}{\overset{b}{>}}Y$ iff $X_-^{(M)}\underset{1}{\overset{M}{\ge}}Y_-^{(M)}$, $X_+^{(M)}\underset{1}{\overset{b}{\ge}}Y_+^{(M)}$ and at least one dominance is strict, where $X=X_+^{(M)}+X_-^{(M)}$ and $X_+^{(M)}=XI_{[X\ge M]};X_-^{(M)}=XI_{[X<M]}$ for every $M\in supp\{X,Y\}$. Thus the thesis follows. □

Proof of Remark 1 Points 1 and 3 are a logical consequence of point 4 and of $F_{cX+t}(x)=F_X\left(\frac{x-t}{c}\right),\ \forall t,x\in R,\ c>0.$ Thus, at less of this affine transformation, the thesis follows for any $\alpha>0$. Point 2 is a consequence of the distribution function $F_{X_+^{(M)}}(t)=0,\ \forall t<M$, $F_{X_+^{(M)}}(t)=F_X(t)$ $\forall t\ge M$ while $F_{X_-^{(M)}}(t)=F_X(t),\ \forall t<M$, and $F_{X_-^{(M)}}(t)=1,\forall t\ge M.$ Thus, for every $M\in R,$ the order $X\underset{\alpha}{\overset{b}{\ge}}Y$ implies $X_-^{(M)}\underset{\alpha}{\overset{M}{\ge}}Y_-^{(M)}$. Observe that generally we cannot say that $X\underset{\alpha}{\ge}Y$ implies $X_-^{(M)}\underset{\alpha}{\ge}Y_-^{(M)}$. □

Proof of Proposition 2: The first part of the proposition summarizes one of Fishburn's (1980b) and O'Brien's (1984) results. In order to prove the inequalities given by (5), recall that if $X\underset{\alpha}{\overset{b}{\ge}}Y$, then $F_X^{(\alpha)}(X)\le F_Y^{(\alpha)}(X)$; $F_X^{(\alpha)}(Y)\le F_Y^{(\alpha)}(Y)$. If we apply the Fubini theorem to the expected value of these random variables, we get $\forall\alpha\ge1$:

$$\Gamma(\alpha)E\left(F_X^{(\alpha)}(X)\right)=\frac{1}{2}E\left(\left|X_1-X_2\right|^{\alpha-1}\right)\le\Gamma(\alpha)E\left(F_Y^{(\alpha)}(X)\right)=E\left(\left(X_1-Y_1\right)_+^{\alpha-1}\right).$$

Similarly we obtain the other inequality. □

Proof of Corollary 1: Points a and c of this corollary are consequences of the previous proposition and of the definition of α-*R-S* order (α bounded *R-S* order). Point b is a consequence of the Fishburn (1980b) and O'Brien (1984) necessary condition of moments expressed in the previous proposition; that is generally true only if we consider unbounded dominance orders. The implication point d is a consequence of point c and of point b of Proposition 2.□

Proof of Proposition 3: In order to prove a, suppose *X* and *Y* are bounded random variables, then *X, Y*> $a=\inf\{x/F_X(x)+F_Y(x)>0\}>-\infty$. Moreover, we can suppose *X* dominates strictly *Y* in the sense of 2-R-S order because *X* and *Y* admit a finite first moment. Thus, $E\left((X-a)_+^{\alpha-1}\right)=\int_a^b(x-a)^{\alpha-1}dF_X(x)\geq E\left((Y-a)_+^{\alpha-1}\right)=\int_a^b(x-t)^{\alpha-1}dF_Y(x)$ for every $\alpha\in[1,2)$ because $f(x)=(x-a)^{\alpha-1}$ is a strictly concave increasing function on the support of the random variables *X* and *Y*. In addition, there exists $r\in(\alpha,2)$ such that $g_X(r)=E\left((X-a)^{r-1}\right)=E\left((X-a)_+^{r-1}\right)>g_Y(r)=E\left((Y-a)^{r-1}\right)$, otherwise $F_X=F_Y$ because the function $g_X(r)$ is analytic. Similar considerations can be done when *X, Y*<*M* because *–X, -Y*>*-M.* Thus point a follows and $\alpha\in[1,2)$ such that *X* α*-R-S Y* (*X* α bounded *R-S Y*) cannot exist when *X* and *Y* are bounded and they admit finite the first moment. Point b generalizes Theorem 2.6 of Li and Zhu (1994). By the previous Lemma, we know that $|x|^r=\frac{1}{B(\alpha,r-\alpha+1)}\int_0^{|x|}(|x|-y)^{\alpha-1}y^{r-\alpha}dy$ for every $r>\alpha-1$. Then, as a consequence of Fubini theorem we get:

$$B(\alpha,r-\alpha+1)E\left(|X|^r\right)=\int_a^b\left(\int_0^{|x|}(|x|-y)^{\alpha-1}y^{r-\alpha}dy\right)dF_X(x)=\int_0^b y^{r-\alpha}\left(\int_y^b(x-y)^{\alpha-1}dF_X(x)\right)dy+$$
$$+\int_a^0(-y)^{r-\alpha}\left(\int_a^y(y-x)^{\alpha-1}dF_X(x)\right)dy=\Gamma(\alpha)\int_0^b y^{r-\alpha}\bar{F}_X^{(\alpha)}(y)dy+\Gamma(\alpha)\int_a^0(-y)^{r-\alpha}F_X^{(\alpha)}(y)dy.$$

Thus $B(\alpha,r-\alpha+1)E\left(|Y|^r-|X|^r\right)=\int_0^b y^{r-\alpha}\left(\bar{F}_Y^{(\alpha)}(y)-\bar{F}_X^{(\alpha)}(y)\right)dy+\Gamma(\alpha)\int_a^0(-y)^{r-\alpha}\left(F_Y^{(\alpha)}(y)-F_X^{(\alpha)}(y)\right)dy$. If *X* α-*R-S Y* and $E\left(|X|^r\right)=E\left(|Y|^r\right)$ with $r>\alpha-1$, it follows $F_Y^{(\alpha)}=F_X^{(\alpha)}$ (i.e., $F_X=F_Y$), otherwise $E\left(|X|^r\right)<E\left(|Y|^r\right)$, for every $r>\alpha-1$ for which there exist finite the *r*-th moment. If *X* and *Y* are symmetric with null mean *X*=*-X* and *Y*=*-Y,* thus it follows point c.□

Proof of Remark 2: While the first three points follows by the previous discussion, points 4 and 5 are a logical consequence of the analysis proposed by Muliere and Scarsini (1989). □

Proof of Remark 3: It follows by the previous definitions and discussions.

Proof of Corollary 3: The proof of this corollary is analogous to the proof given by Muliere and Scarsini (1989), Fishburn (1976, 1980a) and Muller (1997) with some little differences. In particular observe that if $X\ \underset{\succ,\alpha}{FORS}\ Y$ then $\rho_{X,\alpha}(b)\leq\rho_{Y,\alpha}(b)$ and

$$\int_a^b \int_a^{u^-} (u-s)^{\alpha-1} d\rho_X(s) d\tau(u) \le \int_a^b \int_a^{u^-} (u-s)^{\alpha-1} d\rho_Y(s) d\tau(u).$$

By the Fubini-Tonelli theorem this is equivalent to $\int_a^b \phi(u) d\rho_X(u) \ge \int_a^b \phi(u) d\rho_Y(u)$. Conversely, let us consider $\rho_{X,\alpha}(u) = \frac{1}{\Gamma(\alpha)} \int_a^{u^-} (u-s)^{\alpha-1} d\rho_X(s) = \int_a^b \phi_{(u)}(s) d\rho_X(s)$ where $\tau_{(u)}(y) = \frac{I_{(u,b]}(y)}{\Gamma(\alpha)}$ and $\phi_{(u)}(s) = \frac{(u-s)^{\alpha-1} I_{[a,u)}(s)}{\Gamma(\alpha)} = \int_{s^+}^b (z-s)^{\alpha-1} d\tau_{(u)}(z)$. Clearly, for every $u \in [a,b]$, $-\phi_{(u)}(s) \in W^\alpha$ and for every $u \in [a,b]$ $\int_a^b -\phi_{(u)}(s) d\rho_X(s) \ge \int_a^b -\phi_{(u)}(s) d\rho_Y(s)$ implies that $\forall u \in [a,b]$ $\rho_{X,\alpha}(u) \le \rho_{Y,\alpha}(u)$, iff $X \underset{\succ,\alpha}{FORS} Y$. Moreover, as a consequence of Lemma 1, for every $\alpha < v$, we have that $\phi_v \in W^v$ if and only if

$$\phi_v(x) = -\int_{x^+}^b (s-x)^{v-1} d\tau_v(s) - k(b-x)^{v-1} = -\int_{x^+}^b \frac{(s-x)^{v-\alpha-1}}{B(v-\alpha,\alpha)} \left(\int_{s^+}^b (y-s)^{\alpha-1} d\tau_v(y) - k(b-s)^{\alpha-1} \right) ds = \int_{x^+}^b (s-x)^{v-\alpha-1} \phi_\alpha(s) ds$$

where $\phi_\alpha \in W^\alpha$, $\phi_\alpha(s) = -\int_{s^+}^b (y-s)^{\alpha-1} d\tau_\alpha(y) - \frac{k(b-s)^{\alpha-1}}{B(v-\alpha,\alpha)}$ and $\tau_\alpha(y) = \frac{\tau_v(y)}{B(\alpha, v-\alpha)}$. □

Proof of Proposition 4: Considering that $P(X \le H^{-1}(t)) = P(H(X) \le t)$ $\forall t \in [a,b]$, if we apply the Fubini Tonelli theorem we get point a. The proof of point b is practically the same as for point b of Proposition 3.

Proof of Theorem 1: By Holder inequality we know that for every $\alpha \in (1,p]$ and $X \in \Lambda_p$

$$\left(\left|\Gamma(\alpha)\rho_{X,\alpha}(b)\right|\right)^{1/(\alpha-1)} = \left(\left|\int_a^b (b-t)^{\alpha-1} d\rho_X(t)\right|\right)^{1/(\alpha-1)} \le \left(\left|\int_a^b (b-t)^{p-1} d\rho_X(t)\right|\right)^{1/(p-1)} = \left(\Gamma(p)\left|\rho_{X,p}(b)\right|\right)^{1/(p-1)} < +\infty.$$

Thus $h_X(u) = \rho_{X,u}(b)$ is defined for every $X \in \Lambda_p$ and for every $u \in (1,p]$. Moreover, if $X \underset{1}{FORS1} Y$ implies $X \underset{u}{FORS1} Y$ for any $u \in (1,p]$ and thus $h_X(u) = \rho_{X,u}(b) \le \rho_{Y,u}(b) = h_Y(u)$. Therefore $h_X(u) = \rho_{X,u}(b)$ is a FORS2 measure induced by $\succ$ on the class Λ_p for any fixed $u \in (1,p]$. In addition, if for any $u \in (1,p]$ we have $h_X(u) = \rho_{X,u}(b) = \rho_{Y,u}(b) = h_Y(u)$, then by applying the inverse Mellin transform we get $\rho_X(t) = \rho_Y(t)$ for every $t \in [a,b]$. This implies $F_X = F_Y$ and h_X is a simple probability functional. Thus using Definition 4 we prove the theorem. □

Proof of Corollary 5: The proof is practically the same given in Corollary 3.

Proof of Proposition 5: The proof is practically the same given in Theorem 1.

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