Tempered stable distributions and processes in finance: numerical analysis

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Abstract. Most of the important models in finance rest on the assumption that randomness is explained through a normal random variable. However there is ample empirical evidence against the normality assumption, since stock returns are heavy-tailed, leptokurtic and skewed. Partly in response to those empirical inconsistencies relative to the properties of the normal distribution, a suitable alternative distribution is the family of tempered stable distributions. In general, the use of infinitely divisible distributions is obstructed by the difficulty to calibrate and simulate them. In this paper, we address some numerical issues resulting from tempered stable modelling, with a view toward the density approximation and simulation.

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1 Introduction

Since Mandelbrot introduced the α-stable distribution in modelling financial asset returns, numerous empirical studied have been done in both natural and economic sciences. The works of Rachev and Mittnik [19] and Rachev et al. [18], see also references therein, have focused attention on a general framework for market and credit risk management, option pricing, and portfolio selection based on the α-stable distribution. While the empirical evidence does not support the normal distribution, it is also not always consistent with the α-stable distributional hypothesis. Asset returns time series present heavier tails relative to the normal distribution and thinner tails than the α-stable distribution. Moreover, the stable scaling properties may cause problems in calibrating the model to real data. Anyway, there is a wide consensus to assume the presence of a leptokurtic and skewed pattern in stock returns, as showed by the α-stable modelling. Partly in response to the above empirical inconsistencies, and to maintain suitable properties of the stable model, a proper alternative to the α-stable distribution is the family of tempered stable distributions.

Tempered stable distributions may have all moments finite and exponential moments of some order. The latter property is essential in the construction of tempered stable option pricing models. The formal definition of tempered stable processes as been proposed in the seminal work of Rosiński [21]. The KoBol [4], the CGMY [5], the Inverse Gaussian (IG) and the tempered stable of Tweedie [23] are only some parametric examples in this class, that have an infinite dimensional parametrization by a family of measures [25]. Further extensions or limiting cases are also given by the fractional tempered stable framework [10], the bilateral gamma [15] and the generalized tempered stable distribution [7, 16]. The general formulation is difficult to use in practical applications, but it allows one to prove some interesting results regarding the calculus of the characteristic function and the random numbers generation. The infinite divisibility of this distribution allows one to construct the corresponding Lévy process and to analyze the change of measure problem and the process behavior as well.

The purpose of this paper is to show some numerical issues arising from the use of this class in applications to finance with a look to the density approximation and the random number generation for some particular cases, such as the CGMY and the KR case. The paper is related to some previous works of the authors [13, 14] where the exponential Lévy and the tempered stable GARCH models has been studied. The remainder of this paper is organized as follows. In Section 2 we review the definition of tempered stable distributions and focus our attention on the CGMY and KR distributions. An algorithm for the evaluation of the density function for the KR distribution is presented in Section 3. Finally, Section 4 presents a general random number generation method and an option pricing analysis via Monte Carlo simulation.

2 Basic definitions

The class of infinitely divisible distribution has a large spectrum of applications, and in recent years, particularly in mathematical finance and econometrics, non-normal infinitely divisible distributions have been widely studied. Before explaining the construction of tempered stable (TS_α) distributions, we want to recall the following well known result [22].

Theorem 1 (Lévy-Khintchine formula). *A real valued random variable X is infinitely divisible with characteristic exponent $\psi(z)$, i.e.*

$$E[e^{izX}] = e^{\psi(z)}$$

with $z \in \mathbb{R}$, if and only if there exists a triple (a_h, σ, ν) where $a_h \in \mathbb{R}$, $\sigma \geq 0$, h is a given truncation function, ν is a measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge x^2)\nu(dx) < \infty$$

and

$$\psi(z) = ia_h\theta - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}\backslash\{0\}} (e^{izx} - 1 - izh(x))\nu(dx) \tag{2.1}$$

for every $z \in \mathbb{R}$.

Let us now define the Lévy measure of a TS_α distribution.

Definition 2. *A real valued random variable X is TS_α if is infinitely divisible without Gaussian part and has Lévy measure ν that can be written in polar coordinated*

$$\nu(dr, du) = r^{-\alpha-1} q(r, u) dr\, \sigma(du), \tag{2.2}$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}. and

$$q : (0, \infty) \times S^{d-1} \mapsto (0, \infty)$$

is a Borel function such that $q(\cdot, u)$ is completely monotone with $q(\infty, u) = 0$ for each $u \in S^{d-1}$. A TS_α distribution is called a proper TS_α distribution if

$$\lim_{r \to 0^+} q(r, u) = 1$$

for each $u \in S^{d-1}$.

Furthermore, by [21, Theorem 2.3], the Lévy measure ν can be also rewritten in the form

$$\nu(A) = \int_{\mathbb{R}_0^d} \int_0^\infty I_A(tx)\alpha t^{-\alpha-1} e^{-t} dt R(dx), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{2.3}$$

where R is a unique measure on $\mathbb{R}^d$ such that $R(\{0\}) = 0$

$$\begin{cases} \int_{\mathbb{R}^d} (\|x\|^2 \wedge \|x\|^\alpha) R(dx) < \infty, & \alpha \in (0,2) \\ \int_{\mathbb{R}^d} (\log(1+2\|x\|^2) \wedge 1) R(dx) < \infty, & \alpha = 0 \end{cases} \tag{2.4}$$

Sometimes the only knowledge of the Lévy measure cannot be enough to obtain analytical properties of tempered stable distributions. Therefore, the definition of Rosiński measure R allows one to overcome this problem and to obtain explicit analytic formulas and more explicit calculations. For instance, the characteristic function can be rewritten by using directly the measure R instead of ν (see [21, Theorem 2.9]). Of course, given a measure R it is always possible to find the corresponding tempering function q; the converse is true as well. As consequence of this, the specification of a measure R satisfying conditions (2.4) or the specification of a completely monotone function q, defines uniquely a TS_α distribution.

Now, let us define two parametric examples. In the first example the measure R is the sum of two Dirac measures multiplied for opportune constants, while the spectral measure R of the second example has a nontrivial bounded support. If we set

$$q(r, \pm 1) = e^{-\lambda_\pm r}, \qquad \lambda > 0, \tag{2.5}$$

and the measure

$$\sigma(\{-1\}) = c_- \quad \text{and} \quad \sigma(\{1\}) = c_+, \tag{2.6}$$

we get

$$\nu(dr) = \frac{c_-}{|r|^{1+\alpha_-}} e^{-\lambda_- r} I_{\{x<0\}} + \frac{c_+}{|r|^{1+\alpha_+}} e^{-\lambda_+ r} I_{\{x>0\}}. \tag{2.7}$$

The measures Q and R are given by

$$Q = c_- \delta_{-\lambda_-} + c_+ \delta_{\lambda_+} \tag{2.8}$$

and

$$R = c_- \lambda_-^\alpha \delta_{-\frac{1}{\lambda_-}} + c_+ \lambda_+^\alpha \delta_{\frac{1}{\lambda_+}}, \tag{2.9}$$

where δ_λ is the Dirac measure at λ (see [21] for the definition of the measure Q).

Then the characteristic exponent has the form

$$\begin{aligned} \psi(u) =& iub + \Gamma(-\alpha)c_+((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + i\alpha\lambda_+^{\alpha-1}u) \\ &+ \Gamma(-\alpha)c_-((\lambda_- + iu)^\alpha - \lambda_-^\alpha - i\alpha\lambda_-^{\alpha-1}u), \end{aligned} \tag{2.10}$$

where we are considering the Lévy-Khinchin formula with truncation function $h(x) = x$. This distribution is usually referred to as the KoBoL or generalized tempered stable (GTS) distribution. If we take $\lambda_+ = M$, $\lambda_- = G$, $c_+ = c_- = C$, $\alpha = Y$ and $m = b$, we obtain that X is CGMY distributed with expected value m. The definition of the corresponding Lévy process follows.

Definition 3. *Let X_t the process such that $X_0 = 0$ and $E[e^{iuX_t}] = e^{t\psi(u)}$ where*

$$\begin{aligned}\psi(u) =& ium + \Gamma(-Y)C((M-iu)^Y - M^Y + iYM^{Y-1}u) \\ &+ \Gamma(-Y)C((G+iu)^Y - G^Y - iYG^{Y-1}u).\end{aligned}$$

We call this process the CGMY process with parameter (C, G, M, Y, m) where $m = E[X_1]$.

A further example is given by the KR distribution [13], with a Rosińsky measure of the following form

$$R(dx) = (k_+ r_+^{-p_+} I_{(0,r_+)}(x)|x|^{p_+-1} + k_- r_-^{-p_-} I_{(-r_-,0)}(x)|x|^{p_--1})\, dx. \quad (2.11)$$

where $\alpha \in (0,2)$, $k_+, k_-, r_+, r_- > 0$, $p_+, p_- \in (-\alpha, \infty) \setminus \{-1, 0\}$, and $m \in \mathbb{R}$. The characteristic function can be calculated by [21, Theorem 2.9] and it is given in the following result [13].

Definition 4. *Let X_t be a process with $X_0 = 0$ and corresponding to the spectral measure R defined in (2.11) with conditions $p \neq 0$, $p \neq -1$, $\alpha \neq 1$ and let $m = E[X_1]$. By considering the Lévy-Khinchin formula with truncation function $h(x) = x$, we have $E[e^{iuX_t}] = e^{t\psi(u)}$ with*

$$\begin{aligned}\psi(u) \ =& \frac{k_+\Gamma(-\alpha)}{p_+}\left({}_2F_1(p_+, -\alpha; 1+p_+; ir_+u) - 1 + \frac{i\alpha p_+ r_+ u}{p_+ + 1}\right) \\ & \frac{k_-\Gamma(-\alpha)}{p_-}\left({}_2F_1(p_-, -\alpha; 1+p_-; -ir_-u) - 1 - \frac{i\alpha p_- r_- u}{p_- + 1}\right) + ium,\end{aligned} \quad (2.12)$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function [1]. We call this process the KR process with parameter (k_+, k_-, r_+, r_-, p_+, p_-, α, m).

3 Evaluating the density function

In order to calibrate asset returns models through exponential Lévy process or tempered stable GARCH model [13, 14], one needs a correct evaluation of both the pdf and cdf functions. With the pdf function it is possible to construct a maximum likelihood estimator (MLE), while the cdf function allows one to assess the goodness of fit. Even if the MLE method may lead to local maximum rather than to a global one due to the multi dimensionality of the optimization problem, the results obtained seem to be satisfactory from the point of view of goodness of fit tests. Actually, an analysis on estimation methods for this kind of distributions would be interesting, but it is far from the purposes of this work.

Numerical methods are needed to evaluate the pdf function. By the definition of the characteristic function as the Fourier transform of the density function [8], we consider the inverse Fourier transform that is

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux} E[e^{iuX}] du \quad (3.1)$$

where $f(x)$ is the density function. If the density function has to be calculated for a large number of x values, the fast Fourier Transform (FFT) algorithm can be employed as described in [24]. The use of the FFT algorithm largely improves the speed of the numerical integration above and the function f is evaluated on a discrete and finite grid, consequently a numerical interpolation is necessary to x values out of the grid. Since a personal computer cannot deal with infinite numbers, the integral bounds $(-\infty,\infty)$ in equation (3.1) are replaced with $[-M,M]$, where M is large value. We take $M\sim 2^{16}$ or 2^{15} in our study and we have also noted that smaller values of M generate large errors in the density evaluation given by a wave effect in both density tails. We have to point out that the numerical integration as well as the interpolation may causes some numerical errors. The method above is a general method that can be used if the density function is not known in closed form.

While the calculus of the characteristic function in the CGMY case involves only elementary functions, more interesting is the evaluation of the characteristic function in the KR case that is connected with the Gaussian hypergeometric function. Equation (2.12) implies the evaluation of the hypergeometric ${}_2F_1(a,b;c;z)$ function only on the straight line represented by the subset $I=\{iy \mid y\in\mathbb{R}\}$ of the complex plane $\mathbb{C}$. We do not need a general algorithm to evaluate the function on the entire complex plane $\mathbb{C}$, but just on a subset of it. This can be done by means of the analytic continuation, without having recourse neither to numerical integration nor to numerical solution of a differential equation [17] (for a complete table of the analytic continuation formulas for arbitrary values of $z\in\mathbb{C}$ and of the parameters a, b, c, see [3] or [9]). The hypergeometric function belongs to the special function class and often occurs in many practical computational problems. It is defined by the power series

$$ {}_2F_1(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad |z|<1. \tag{3.2}$$

where $(a)_n:=\Gamma(a+n)/\Gamma(n)$ is the Ponchhammer symbol. By [1] the following relations are fulfilled

$$\begin{aligned}
{}_2F_1(a,b,c;z) &= (1-z)^{-b}{}_2F_1\left(b,c-a,c,\frac{z}{z-1}\right) \quad \text{if } \left|\frac{z}{z-1}\right|<1\\
{}_2F_1(a,b,c;z) &= (-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}{}_2F_1\left(a,a-c+1,a-b+1,\frac{1}{z}\right)\\
&\quad+(-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}{}_2F_1\left(b,b-c+1,b-a+1,\frac{1}{z}\right) \quad \text{if } \left|\frac{1}{z}\right|<1\\
{}_2F_1(a,b,c;-iy) &= \overline{{}_2F_1(a,b,c;iy)} \quad \text{if } y\in\mathbb{R}.
\end{aligned} \tag{3.3}$$

First by the last equality of (3.3), one can determine the values of ${}_2F_1(a,b,c;z)$ only for the subset $I_+=\{iy\mid y\in\mathbb{R}_+\}$ and then simply consider the conjugate for the set $I_-=\{iy\mid y\in\mathbb{R}_-\}$, remembering that ${}_2F_1(a,b,c;0)=1$. Second, we split the positive real line $\mathbb{R}_+$ in three subsets without intersection,

$$\begin{aligned}
I_+^1 &= \{iy \mid 0<y\le 0.5\}\\
I_+^2 &= \{iy \mid 0.5<y\le 1.5\}\\
I_+^3 &= \{iy \mid y>1.5\},
\end{aligned}$$

then we use (3.2) to evaluate ${}_2F_1(a,b,c;z)$ in I_+^1. Then, the first and the second equalities of (3.3) together with (3.2) are enough to evaluate ${}_2F_1(a,b,c;z)$ in I_+^2 and I_+^3 respectively. This subdivision allows one to truncate the series (3.2) to the integer $N = 500$ and obtain the same results as Mathematica. We point out that the value of y ranges in the interval $[-M, M]$ previously defined. This method together with the MATLAB vector calculus increase considerably the speed with respect to algorithms based on the numerical solution of differential equation [17]. Our method is grounded only on basic summations and multiplication. As a result the computational effort in the KR density evaluation is comparable to that of CGMY one. The KR characteristic function is necessary also to price options, not only for MLE estimation. Indeed, by using the approach of Carr and Madan [6] and the same analytic continuation as above, risk-neutral parameters may be directly estimated from option prices, without calibrate the underlining.

4 Simulation of TS_α processes

In order to generate random variate from TS_α processes, we will consider the general shot noise representation of proper TS_α laws given in [21]. There are different methods to simulate Lévy processes, but most of these methods are not suitable for the simulation of tempered stable processes due to the complicated structure of their Lévy measure. As emphasized in [21], the usual method of the inverse of Lévy measure [20] is difficult to implement, even if the spectral measure R has a simple form. We will apply [21, Theorem 5.1] to the previously considered parametric examples.

Proposition 5. *Let $\{U_j\}$ a i.i.d. sequence of uniform random variables in $(0,T)$, $\{E_j\}$ and $\{E'_j\}$ i.i.d. sequences of exponential variables of parameter 1 and $\{\Gamma_j\} = E'_1 + \ldots + E'_j$, $\{V_j\}$ a i.i.d. sequence of discrete random variables with distribution*

$$P(V_j = -G) = P(V_j = M) = \frac{1}{2},$$

a positive constant $0 < Y < 2$ and $\|\sigma\| = \sigma(S^{d-1}) = 2C$. Furthermore, $\{U_j\}$, $\{E_j\}$, $\{E'_j\}$ and $\{V_j\}$ are mutually independent. Then

$$X_t \stackrel{d}{=} \sum_{j=1}^{\infty} \left[\left(\frac{Y\Gamma_j}{2C} \right)^{-1/Y} \wedge E_j U_j^{1/Y} |V_j|^{-1} \right] \frac{V_j}{|V_j|} I_{\{U_j \le t\}} + t b_T \qquad t \in [0,T], \tag{4.1}$$

where

$$b_T = \begin{cases} -\Gamma(1-Y)C(M^{Y-1} - G^{Y-1}), & 0 < Y < 2 \text{ and } Y \neq 1 \\ (2\gamma + \log(2TC))C(M^{Y-1} - G^{Y-1}) & \\ \qquad -C(G^{Y-1}\log G - M^{Y-1}\log M), & Y = 1, \end{cases} \tag{4.2}$$

and γ is the Euler constant [1, 6.1.3], converges a.s. and uniformly in $t \in [0,T]$ to a CGMY process with parameters (C, G, M, Y, 0).

This series representation is not new in the literature, see [2, 12]. It is a slight modification of the series representation of the stable distribution [11], but here big jumps are removed. The shot noise representation for the KR distribution follows.

Proposition 6. *Let $\{U_j\}$ be a i.i.d. sequence of uniform random variables in $(0,T)$, $\{E_j\}$ and $\{E'_j\}$ i.i.d. sequences of exponential variables of parameter 1 and $\{\Gamma_j\} = E'_1 + \ldots + E'_j$, and constants $\alpha \in (0,2)$, $k_+, k_-, r_+, r_- > 0$ and, $p_+, p_- \in (-\alpha, \infty) \setminus \{-1, 0\}$. Let $\{V_j\}$ be a i.i.d. sequence of random variables with density*

$$f_V(r) = \frac{1}{\|\sigma\|}\left(k_+ r_+^{-p_+} I_{\{r > \frac{1}{r_+}\}} r^{-\alpha-p_+-1} + k_- r_-^{-p_+} I_{\{r < -\frac{1}{r_-}\}} |r|^{-\alpha-p_--1}\right)$$

where

$$\|\sigma\| = \frac{k_+ r_+^\alpha}{\alpha + p_+} + \frac{k_- r_-^\alpha}{\alpha + p_-}.$$

Furthermore, $\{U_j\}$, $\{E_j\}$, $\{E'_j\}$ and $\{V_j\}$ are mutually independent. If $\alpha \in (0,1)$, or if $\alpha \in [1,2)$ with $k_+ = k_-$, $r_+ = r_-$ and $p_+ = p_-$, then the series

$$X_t = \sum_{j=1}^{\infty} I_{\{U_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1} \right) \frac{V_j}{|V_j|} + tb \tag{4.3}$$

converges a.s. and uniformly in $t \in [0,T]$ to a KR tempered stable process with parameters $(k_+,\, k_+,\, r_+,\, r_+,\, p_+,\, p_+,\, \alpha,\, 0)$ with

$$b = -\Gamma(1-\alpha)\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1} \right).$$

If $\alpha \in [1,2)$ and $k_+ \neq k_-$ (or $r_+ \neq r_-$ or alternatively $p_+ \neq p_-$), then

$$X_t = \sum_{j=1}^{\infty} \left[I_{\{U_j \le t\}} \left(\left(\frac{\alpha \Gamma_j}{T\|\sigma\|} \right)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} |V_j|^{-1} \right) \frac{V_j}{|V_j|} - \frac{t}{T} \left(\frac{\alpha j}{T\|\sigma\|} \right)^{-1/\alpha} x_0 \right] + t b_T, \tag{4.4}$$

converges a.s. and uniformly in $t \in [0,T]$ to a KR tempered stable process with parameters $(k_+,\, k_-,\, r_+,\, r_-,\, p_+,\, p_-,\, \alpha,\, 0)$, where we set

$$b_T = \begin{cases} \alpha^{-1/\alpha} \zeta\left(\frac{1}{\alpha}\right) T^{-1} (T\|\sigma\|)^{1/\alpha} x_0 - \Gamma(1-\alpha) x_1, & 1 < \alpha < 2 \\ (2\gamma + \log(T\|\sigma\|)) x_1 - \left(\frac{k_+ r_+}{p_+ + 1} \left(\log r_+ - \frac{1}{p_+ + 1} \right) \right. & \\ \qquad \left. - \frac{k_- r_-}{p_- + 1} \left(\log r_- - \frac{1}{p_- + 1} \right) \right), & \alpha = 1. \end{cases}$$

with

$$x_0 = \|\sigma\|^{-1} \left(\frac{k_+ r_+^\alpha}{\alpha + p_+} - \frac{k_- r_-^\alpha}{\alpha + p_-} \right),$$

$$x_1 = \frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1},$$

ζ denotes the Riemann zeta function [1, 23.2], γ is the Euler constant [1, 6.1.3].

4.1 A Monte Carlo example

In this section, we assess the goodness of fit of random number generators proposed in the previous section. A brief Monte Carlo study is performed and prices of European put options with different strikes are calculated. We take into consideration a CGMY process with the same artificial parameters of the work [16] that is $C = 0.5$, $G = 2$, $M = 3.5$, $Y = 0.5$, interest rate $r = 0.04$, initial stock price $S_0 = 100$ and annualized maturity $T = 0.25$. Furthermore we consider also a GTS process defined by the characteristic exponent (2.10) and parameters $c_+ = 0.5$, $c_- = 1$, $\lambda_+ = 3.5$, $\lambda_- = 2$ and $\alpha = 0.5$, interest rate r, initial stock price S_0 and maturity T as in the CGMY case.

Monte Carlo prices are obtained through 50,000 simulations. The Esscher transform with $\theta = -1.5$ is considered to reduce the variance [12]. We want to emphasize that the Esscher transform is an exponential tilting [21], thus if applied to a CGMY or a GTS process, it modifies only parameters but not the form of the characteristic function.

In Table 1 simulated prices and prices obtained by using the Fourier transform method [6] are compared. Even if there is a competitive CGMY random number generator, where a time changed Brownian motion is considered [16], we prefer to use an algorithm based on series representation. Contrary to the CGMY case, in general there is not a constructive method to find the subordinator process that changes the time of the Brownian motion, that is we do not know the process T_t such that the TS_α process X_t can be rewritten as $W_{T(t)}$ [7]. The shot noise representation allows one to generate any TS_α process.

Table 1: European put option prices computed using the Fourier transform method (Price) and by Monte Carlo simulation (Monte Carlo).

CGMY			GTS		
Strike	Price	Monte Carlo	Strike	Price	Monte Carlo
80	1.7444	1.7472	80	3.2170	3.2144
85	2.3926	2.3955	85	4.2132	4.2179
90	3.2835	3.2844	90	5.4653	5.4766
95	4.5366	4.5383	95	7.0318	7.0444
100	6.3711	6.3724	100	8.9827	8.9968
105	9.1430	9.1532	105	11.3984	11.4175
110	12.7632	12.7737	110	14.3580	14.3895
115	16.8430	16.8551	115	17.8952	17.9394
120	21.1856	21.2064	120	21.9109	21.9688

Conclusions

In this work, we have focused our attention on the practical implementation of numerical methods involving the use of TS_α distributions and processes in the field of finance. Basic definitions are given and a possible algorithm to approximate the density function is proposed. Furthermore, a general Monte Carlo method is developed with a look to option pricing.

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