Construction of Lévy Drivers for Financial Models

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Abstract

We extend the Lévy-Khintchine representation for an infinitely divisible distribution to define a driving process in the context of the bond price framework developed earlier. We describe a methodology using subordination to construct such processes and we develop some examples in detail.

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1 Introduction

In our previous work [9] we have described the bond price process in terms of semimartingales where we used the characterization in terms of its set of characteristics. One advantage of this approach is that we can impose conditions needed for our results explicitly on the drift, diffusion, or jump components of the model. When the price dynamics is described by a diffusion with jumps driven by a Lévy process then the price itself is represented by a Lévy process. In this case the representation in terms of characteristics (for a fixed t) coincides with its Lévy-Khintchine representation.

In this paper, we first define a Lévy process to be used as driver for our financial model. To this end, we first construct an infinitely divisible distribution to describe the behavior of the increments. We then use a result that allows us to extend its Lévy-Khintchine representation to define the distribution of a Lévy process at each point in time. This extension is a special case of the set of characteristics which describes the process in terms of a semimartingale. Once this set is obtained then it may be used in our financial model since the price process (specified by its characteristics) was defined in terms of the characteristics of the driving process.

2 Summary of the General Model

This section is a summary of the framework for bond price dynamics in the context of a diffusion with jumps described in [9].

2.1 Introduction

We assume the canonical setting. Let $\mathsf{P}(t,T)$ be the price at time t of a bond which matures at time T. It is assumed that for each $T>0$, $(\{\mathsf{P}(t,T)\}_{0\le t\le T}$ is an optional, $\{\mathscr{F}_t\}$-adapted process, and for each t, $\mathsf{P}(t,T)$ is P-a.s. continuously differentiable in the T variable. Let $f(t,T)$ denote the T-*forward rate* at time t, defined by $f(t,T) = -\frac{\partial}{\partial T}\mathsf{P}(t,T)$. The *short rate* r is defined by $r_t = f(t,t)$, and the *money account process* B is defined by

$$\mathsf{B}_t = \exp\left(\int_0^t r_s ds\right).$$

In order to model the bond price dynamics we could start with a description of the forward rate or short rate dynamics. Alternatively, we could follow a direct approach, obtaining $\mathsf{P}(t,T)$ as the solution of a stochastic differential equation. Therefore, we are interested in studying dynamics of the following forms:

$$dr_t = a_t dt + b_t dW_t + \int_E q(t,x)\mu(dt,dx), \tag{1}$$

$$d\mathsf{P}(t,T) = \mathsf{P}(t-,T)\left\{m(t,T)dt + v(t,T)dW_t + \int_E n(t,x,T)\mu(dt,dx)\right\}, \tag{2}$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t + \int_E \delta(t,x,T)\mu(dt,dx). \tag{3}$$

The coefficients $b(t,T)$, $v(t,T)$, and $\sigma(t,T)$ are assumed to be m-dimensional row vector processes. The following technical assumptions will be needed:

Assumption

1. For any fixed $T > 0$, $n(t,x,T)$ and $\delta(t,x,T)$ are uniformly bounded. Furthermore, for each t,

$$\int_0^t \int_E h'(n(s,x,T))F(dx)ds < \infty,$$

where $h'(z) = |z|^2 \wedge |z|$ for $z \in \mathbb{R}$.

2. For each fixed ω, t, and (where appropriate) x, all the objects $m(t,T)$, $v(t,T)$, $n(t,x,T)$, $\alpha(t,T)$, $\sigma(t,T)$ and $\delta(t,x,T)$ are assumed to be continuously differentiable in the T-variable.

3. All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.

4. For any t the price curves $\mathsf{P}(\omega,t,T)$ are bounded functions for almost every ω.

Proposition 1. *If $f(t,T)$ satisfies (3), then $\mathsf{P}(t,T)$ satisfies*

$$d\mathsf{P}(t,T) = \mathsf{P}(t-,T)\Bigg[\left(r_t + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + S(t,T)dW_t + \int_E \left(e^{D(t,x,T)} - 1\right)\mu(dt,dx)\Bigg],$$

where

$$
\begin{aligned}
A(t,T) &= -\int_t^T \alpha(t,s)ds, \\
S(t,T) &= -\int_t^T \sigma(t,s)ds, \\
D(t,x,T) &= -\int_t^T \delta(t,x,s)ds.
\end{aligned}
\tag{4}
$$

2.2 Bond Markets, Arbitrage

We now present the framework (Björk, Kabanov and Runggaldier [4]) in which we will state results concerning the absence of arbitrage in a model of bond prices. It will be assumed throughout that the filtration $\mathbf{F}$ is the natural filtration generated by W and μ.

A *portfolio* in the bond market is a pair (g,h), where

1. g is a predictable process.
2. For each ω, t, $h_t(\omega,\cdot)$ is a signed finite Borel measure on $[t,\infty)$.
3. For each Borel set A the process $h_t(A)$ is predictable.

The *discounted bond prices* $\overline{\mathsf{P}}(t,T)$ are defined by

$$\overline{\mathsf{P}}(t,T) = \frac{\mathsf{P}(t,T)}{\mathsf{B}_t}.$$

A portfolio (g,h) is said to be *feasible* if the following conditions hold for every

t:

$$\int_0^t |g_s| ds < \infty, \qquad \int_0^t \int_s^\infty |m(s,T)||h_s(dT)| ds < \infty,$$

$$\int_0^t \int_s^\infty \int_E |n(s,x,T)||h_s(dT)|\nu(ds,dx) < \infty,$$

$$\text{and} \quad \int_0^t \left\{ \int_s^\infty |v(s,T)||h_s(dT)| \right\}^2 ds < \infty.$$

The *value process* corresponding to a feasible portfolio $\pi = (g, h)$ is defined by

$$V_t^\pi = g_t \mathsf{B}_t + \int_t^\infty \mathsf{P}(t,T) h_t(dT).$$

The *discounted value process* is

$$\overline{V}_t^\pi = \mathsf{B}_t^{-1} V_t^\pi.$$

A feasible portfolio is said to be *admissible* if there is a number $a \geq 0$ such that $V_t^\pi \geq -a$ P-a.s. for all t.

A feasible portfolio is said to be *self-financing* if the corresponding value process satisfies

$$\begin{aligned} V_t^\pi = V_0^\pi &+ \int_0^t g_s d\mathsf{B}_s + \int_0^t \int_s^\infty m(s,t)\mathsf{P}(s,t) h_s(dT) ds \\ &+ \int_0^t \int_s^\infty v(s,t)\mathsf{P}(s,t) h_s(dT) dW_s \\ &+ \int_0^t \int_s^\infty \int_E n(s,x,T)\mathsf{P}(s-,t) h_s(dT)\mu(ds,dx). \end{aligned}$$

The preceding relation can be interpreted formally as follows:

$$dV_t^\pi = g_t d\mathsf{B}_t + \int_t^\infty h_t(dT) d\mathsf{P}(t,T).$$

A *contingent T-claim* is a random variable $X \in L^0_+(\mathscr{F}_T, P)$. An *arbitrage portfolio* is an admissible self-financing portfolio $\pi = (g, h)$ such that the corresponding value process satisfies

1. $V_0^\pi = 0$

2. $V_T^\pi \in L^0_+(\mathscr{F}_T, P)$ with $P(V_T^\pi > 0) > 0$.

If no arbitrage portfolios exist for any $T > 0$ we say that the model is *arbitrage-free*.

Take the measure P as given. We say that a positive martingale $M = \{M_t\}_{t \geq 0}$ with $E^P(M_t) = 1$ for each t is a *martingale density* if for every $T > 0$ the process $\{\overline{\mathsf{P}}(t, T)M_t\}_{0 \leq t \leq T}$ ia a P-local martingale. If, moreover, $M_t > 0$ for all $t > 0$ we say that M is a *strict martingale density*.

We say that that a probability measure Q on $(\Omega, \mathscr{F})$ is a *martingale measure* if $Q_t \sim P_t$ and the process $\{\overline{\mathsf{P}}(t, T)\}_{0 \leq t \leq T}$ is a Q-local martingale for every $T > 0$. Here Q_t, P_t are the restrictions $Q_{|\mathscr{F}_t}$ and $P_{|\mathscr{F}_t}$, respectively.

Proposition 2. *Suppose that there exists a strict martingale density. Then the bond market model is arbitrage-free.*

We will make the following simplifying assumption:

Assumption For any positive martingale $N = \{N_t\}$ with $E^P(N_t) = 1$ there exists a probability measure Q on $\bigcup_{t\geq 0} \mathscr{F}_t$ such that $N_t = dQ_t/dP_t$.

The following results relate the coefficients in (2) and (3) with a model free of arbitrage.

Theorem 1. *Let the bond price dynamics be given by (2). There exists a martingale measure if and only if the following conditions hold:*

(i) There exists a predictable process ϕ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega, t, x)$ with $Y > 0$ satisfying

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx) ds < \infty.$$

and such that $E^P(\mathscr{E}(L)_t) = 1$ for all finite t, where the process L is defined by

$$L = \phi \cdot W + (Y - 1) * (\mu - \nu).$$

(ii) For all $T > 0$, and $t \in [0, T]$ we have

$$m(t,T) + \phi_t v(t,T)^T + \int_E Y(t,x) n(t,x,T) F(dx) = r_t. \tag{5}$$

The following theorem gives a similar result when we consider the forward rate dynamics.

Theorem 2. *Let the forward rate dynamics be given by (3). There exists a martingale measure if and only if the following conditions hold:*

(i) There exists a predictable process ϕ and a $\tilde{\mathscr{P}}$-measurable function $Y(\omega, t, x)$ with $Y > 0$ satisfying

$$\int_0^t \|\phi_s\|^2 ds < \infty, \qquad \int_0^t \int_E |Y(s,x) - 1| F(dx) ds < \infty.$$

and such that $E^P(\mathscr{E}(L)_t) = 1$ for all finite t, where the process L is defined by

$$L = \phi \cdot W + (Y - 1) * (\mu - \nu).$$

(ii) For all $T > 0$, and $t \in [0,T]$ we have

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 + \phi_t S(t,T)^T + \int_E Y(t,x)\left(e^{D(t,x,T)} - 1\right)F(dx) = 0,$$

where A, S and D are defined in (4).

3 Semimartingales with Independent Increments

In this short section we state a characterization of semimartingales with independent increments. These results will be used in the following section to establish the connection with Lévy processes.

Theorem 3. *Let X be a d-dimensional process with independent increments. Then X is also a semimartingale if and only if, for each $u \in \mathbb{R}^d$, the function $t \mapsto g(u)_t := E(\exp iu \cdot X_t)$ has finite variation over finite intervals.*

Theorem 4. *Let X be a d-dimensional semimartingale with $X_0 = 0$. Then it is a process with independent increments if and only if there is a version (B, C, ν) of its characteristics that is deterministic. Furthermore, in this case, with $J = \{t : \nu(\{t\} \times \mathbb{R}^d) > 0\}$ and for all $s \le t$, $u \in \mathbb{R}^d$ we have:*

$$\begin{aligned} E(e^{iu\cdot(X_t - X_s)}) = \exp\Big[& iu \cdot (B_t - B_s) - \frac{1}{2} u \cdot (C_t - C_s) \cdot u \\ & + \int_s^t \int_{\mathbb{R}^d} (e^{iu\cdot x} - 1 - iu \cdot h(x)) 1_{J^c}(r) \nu(dr, dx)\Big] \\ & \times \prod_{s<r\le t} \left\{ e^{-iu\cdot\Delta B_r} \left[1 + \int (e^{iu\cdot x} - 1)\nu(\{r\} \times dx)\right]\right\}. \end{aligned} \tag{6}$$

Corollary 1. *A d-dimensional semimartingale X is a process with stationary independent increments if and only if it is a semimartingale admitting a version (B, C, ν) of its characteristics that has the form*

$$B_t(\omega) = bt, \quad C_t(\omega) = ct, \quad \nu(\omega; dt, dx) = dt\, K(dx)$$

where $b \in \mathbb{R}^d$, c is a symmetric nonnegative $d \times d$ matrix, K is a positive measure on $\mathbb{R}^d$ that integrates $(|x|^2 \wedge 1)$ and satisfies $K(\{0\}) = 0$.

4 Construction of a Driving Process

In this section we develop a description of the type of processes we propose for financial applications. The approach is from specific to general. Infinitely divisible distributions extend quite naturally to additive and Lévy processes in law. Once a cádlág modification is chosen, this is seen to be a special case of our general approach in terms of semimartingales. We adopt the results and notation of K. Sato's beautiful book [18].

4.1 Infinitely Divisible Distributions

The class of infinitely divisible distributions arise naturally in a financial context. Below we define the class membership. Roughly speaking, a random variable follows an infinitely divisible distributions if it can be considered to be the sum of independent innovations. Asset returns, for example, are the accumulation of the returns accrued in non-overlapping time intervals. This class generalizes the Gaussian distribution to allow heavy tails and skewness (Shiryaev [21], Nolan [15]), and is the only class that contains the limit distributions of sums of iid random variables.

A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{n*}$, where μ^{n*} denotes the n-fold convolution of μ with itself.

We begin our discussion with the *Lévy-Khintchine representation* of the characteristic function $\widehat{\mu}(z) = \int e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$ of μ.

Theorem 5. *(i) Let $D = \{x \in \mathbb{R}^d : |x| \leq 1\}$. If μ is an infinitely divisible distribution on $\mathbb{R}^d$ then*

$$\begin{aligned}\widehat{\mu}(z) = \exp\Big[&-\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle \\ &+ \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_D(x))\nu(dx)\Big], \quad z \in \mathbb{R}^d\end{aligned} \tag{7}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty. \tag{8}$$

(ii) The representation of $\hat{\mu}(z)$ in (i) by $\gamma \in \mathbb{R}^d$, A and ν is unique.

(iii) Conversely, if $\gamma \in \mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a measure satisfying (8), then there is an infinitely divisible distribution μ whose characteristic function is given by (7).

As stated earlier, the motivation for describing a semimartingale in terms of characteristics was to generalize the *generating triplet* (γ, A, ν) for the infinitely divisible distribution μ. Here we start with an infinitely divisible distribution and develop a process and its characteristics in parallel, in order to adapt it to our general framework as a driving process.

The representation (7) can be rewritten in terms of another truncation function $c(x)$ in place of $1_D(x)$. Given a particular Lévy measure, we may be able to simplify the integrand in (7) by choosing an appropriate $c(x)$ while still ensuring that the integral is finite. In fact, if $c : \mathbb{R}^d \to \mathbb{R}$ is a measurable function such that

$$\int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x))\nu(dx) < \infty \tag{9}$$

for every $z \in \mathbb{R}^d$ then rearranging terms in (7) we obtain

$$\begin{aligned}\widehat{\mu}(z) = \exp\Big[&-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_c, z\rangle \\ &+ \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x))\nu(dx)\Big],\end{aligned} \tag{10}$$

with $\gamma_c \in \mathbb{R}^d$ defined by

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x(c(x) - 1_D(x))\nu(dx).$$

The representation (γ_c, A, ν) implied by (10) will also be called the generating triplet for μ. Note that the components A and ν are independent of the choice of c.

If $\int_{|x|\le 1} |x|\nu(dx) < \infty$ then (9) is satisfied with $c \equiv 0$ and we obtain the representation (γ_0, A, ν):

$$\widehat{\mu}(z) = \exp\Big[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\Big], \tag{11}$$

Likewise, if $\int_{|x|>1} |x|\nu(dx) < \infty$ we obtain the representation (γ_1, A, ν) from (10) with $c \equiv 1$.

4.2 Lévy Processes

An $\mathbb{R}^d$-valued stochastic process $\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is said to be an *additive process in law* if each of the following conditions hold.

1. X has the independent increments property.
2. $X_0 = 0$ a.s.
3. X is stochastically continuous.

An additive process in law with the stationary increments property is said to be a *Lévy process in law*. An additive (Lévy) process in law which is cádlág is called an *additive* (*Lévy*) *process*. An $\mathbb{R}$-valued increasing Lévy process is said to be a *subordinator*.

The following two results establish the correspondence between a family of infinitely divisible distributions and additive processes in law. Then the associated family of generating triplets offers a natural representation for the corresponding process. Later this will be seen to be a special case of the characteristics described in the context of semimartingales. However, we will need a restriction to ensure that an additive process is a semimartingale. In the case of a Lévy process, no restriction is needed.

Theorem 6. *(i) Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued additive process in law and, for $0 \leq s \leq t < \infty$, let $\mu_{s,t}$ be the distribution of $X_t - X_s$. Then $\mu_{s,t}$ is* *infinitely divisible and*

$$\mu_{s,t} * \mu_{t,u} = \mu_{s,u} \ \text{ for } 0 \le s \le t \le u < \infty,$$

$$\mu_{s,s} = \delta_0 \ \text{ for } 0 \le s < \infty,$$

$$\mu_{s,t} \to \delta_0 \ \text{ as } s \uparrow t,$$

$$\mu_{s,t} \to \delta_0 \ \text{ as } t \downarrow s.$$

(ii) Conversely, if $\{\mu_{s,t}\}_{0\le s\le t<\infty}$ *is a system of probability measures on* $\mathbb{R}^d$ *satisfying the properties in (i), then there is an additive process in law* $\{X_t\}_{t\ge 0}$ *such that for* $0 \le s \le t < \infty$, $X_t - X_s$ *has the distribution* $\mu_{s,t}$.

(iii) If $\{X_t\}$ *and* $\{X'_t\}$ *are* $\mathbb{R}^d$*-valued additive processes in law such that* $X_t \overset{\mathrm{d}}{=} X'_t$ *for any* $t \ge 0$*, then* $\{X_t\}$ *and* $\{X'_t\}$ *are identical in law.*

Theorem 7. *(i) Suppose that* $\{X_t\}_{t\ge 0}$ *is an* $\mathbb{R}^d$*-valued additive process in law. Let* $(\gamma(t), A_t, \nu_t)$ *be the generating triplet of the infinitely divisible distribution* $\mu_t = P_{X_t}$ *for* $t \ge 0$*. Then the following conditions are satisfied.*

(a) $\gamma(0) = 0$, $A_0 = 0$, $\nu_0 = 0$.

(b) If $0 \le s \le t < \infty$*, then* $\langle z, A_s z\rangle \le \langle z, A_t z\rangle$ *for* $z \in \mathbb{R}^d$ *and* $\nu_s(B) \le \nu_t(B)$ *for* $B \in \mathcal{B}(\mathbb{R}^d)$.

(c) As $s \to t$ *in* $[0,\infty)$, $\gamma(s) \to \gamma(t)$, $\langle z, A_s z\rangle \to \langle z, A_t z\rangle$ *for* $z \in \mathbb{R}^d$*, and* $\nu_s(B) \to \nu_t(B)$ *for* $B \in \mathcal{B}(\mathbb{R}^d)$ *with* $B \subset \{x : |x| > \epsilon\}$, $\epsilon > 0$.

(ii) Let $\{\mu_t\}_{t\ge 0}$ *be a system of infinitely divisible probability measures on* $\mathbb{R}^d$ *with generating triplets* $(\gamma(t), A_t, \nu_t)$ *satisfying (1)-(3) Then there exists, uniquely up to identity in law, an* $\mathbb{R}^d$*-valued additive process in law such that* $P_{X_t} = \mu_t$ *for* $t \ge 0$.

Let $\{X_t\}$ be an $\mathbb{R}^d$-valued additive process in law. Let (γ_t, A_t, ν_t) be its system of generating triplets. Construct the measure $\tilde{\nu}$ on $[0,\infty) \times \mathbb{R}^d$ such that

$$\tilde{\nu}([0,t] \times B) = \nu_t(B), \quad \text{for } t \ge 0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d) \tag{12}$$

by defining a set function as in (12) on the field of sets $[0,t] \times B$ with $t \ge 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$, and then extending to the σ-field which is equivalent to the Borel

σ-field of $[0,\infty) \times \mathbb{R}^d$. By Theorem 7*(i)* and (12) it follows that the following statements hold.

$$\tilde{\nu}(\{t\} \times \mathbb{R}^d) = 0 \quad \text{for } t \geq 0, \tag{13}$$

$$\int_{[0,t]\times\mathbb{R}^d} (1 \wedge |x|^2)\tilde{\nu}(ds, dx) < \infty \quad \text{for } t \geq 0. \tag{14}$$

Conversely, if a measure $\tilde{\nu}$ satisfies (13) and (14) then for each $t \geq 0$, the Lévy measure ν_t defined by (12) satisfies the conditions in Theorem 7*(i)*.

The following result implies that we can choose a modification $\{X_t'\}$ of $\{X_t\}$ that is an additive process.

Theorem 8. *Let* $\{X_t\}$ *be an an* $\mathbb{R}^d$*-valued additive or Lévy process in law. Then it has a cádlág modification.*

Since our interest is in semimartingales, by virtue of Theorem 3 we require $\{X_t'\}$ to be such that the function $t \mapsto \widehat{P}_{X_t}$ has finite variation over finite intervals. Hence by Theorems 3 and 4 with $\tilde{\nu}$ in (6), we identify $\{X_t'\}$ to be the semimartingale with characteristics $(\gamma_t, A_t, \tilde{\nu}(ds, dx))$. Since we have defined processes in this section to be stochastically continuous, then the last term in (6) is equal to 1 and the set $J = \emptyset$. The same conclusion also follows from (13).

If the additive process $\{X_t'\}$ has the stationary increments property (i.e. a Lévy process), then the condition in Theorem 3 is satisfied and it follows from Corollary 1 that its set of characteristics is $(t\gamma, tA, t\nu_1(dx))$. Conversely, given an infinitely divisible distribution μ on $\mathbb{R}^d$ with generating triplet (γ, A, ν), define the system of measures $\{\mu_{s,t}\}_{0\leq s\leq t<\infty}$ by the system of generating triplets $((s-t)\gamma, (s-t)A, (s-t)\nu)$. It follows easily from the representation

$$\widehat{\mu}_{s,t}(z) = \exp\Big[(t-s)\Big(-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_D(x))\nu(dx)\Big)\Big] \tag{15}$$

for $0 \leq s \leq t < \infty$ and $z \in \mathbb{R}^d$, that the conditions listed in Theorem 6 are satisfied. Then there is an additive process Y such that $Y_t - Y_s$ has distribution $\mu_{s,t}$ and which, in this case, has the stationary increments property. In the sequel we will construct Lévy processes by stipulating that its increments are described by a given infinitely divisible distribution.

4.3 Subordination of Lévy Processes

Now we shall construct a driving process by the method of subordination. This can be seen as a generalisation of our model by substituting the physical time, indexed by t, with an increasing non-negative Lévy process. The resulting process X_T is said to be *subordinated* to the *noise process* X by the *subordinator* T. In what follows we will specify a subordinator to be the Lévy process whose increments follow a given non-negative infinitely divisible distribution. Subordination can be interpreted as a transformation of the physical time to the "intrinsic time" of the underlying market. In other words, T will rescale the time axis to model periods of high or low business activity. $T(t)$ is interpreted as a measure of the cumulative trading volume up to the physical time t (Hurst, Platen, Rachev [10]).

Our goal is to construct the set of characteristics of the subordinated process in terms of the characteristics of the two component processes. Madan and Seneta [13] developed the *Variance Gamma* (VG) process by subordinating a Brownian motion with a gamma process for stock prices. Hurst, Platen and Rachev [10] used an $\alpha/2$-stable subordinator with a Brownian motion. These will be presented as examples of our methodology.

Rachev, Mittnik [16] studied the USD-CHF exchange rate using a subordinated model $Z_t = S(T_t)$. They gathered a data sample of $N = 128400$ spanning the period of 499 business days from 20 May 1985 to 20 May 1987. The average time between observations is 2 minutes, 6 seconds. Denote by $p_{\text{bid}}(t)$ and $p_{\text{ask}}(t)$, respectively, the bid and ask quote at time t for the exchange rate. For each $i = 1, \ldots, N$ the ith observation $x(t_i)$ is the logarithmic price at time t_i, defined by

$$x(t_i) = \frac{\log p_{\text{bid}}(t_i) + \log p_{\text{ask}}(t_i)}{2}.$$

Note that the set $\{x(t) : t \in \{t_i : 1 \leq i \leq N\}\}$ can be regarded as a sample path of the price process in *physical* time $\{(Z_t)\}$. On the other hand, $\{x(t_i) : 1 \leq i \leq N\}$ can be regarded as a sample path of the price process in *intrinsic* time $\{S(t)\}$.

Define the *return* $r(t_i; \Delta t)$ *at time* t_i over the period Δt by

$$r(t_i; \Delta t) = x(t_i) - x(t_i - \Delta t).$$

Note that the quantities $x(t_i) - x(t_{i-k})$, i.e. the returns at *k-quote* frequency, can be interpreted as price change in physical time or as price change in intrinsic (quote) time. The probabilistic structure of the process $S(t)$ was studied by estimating the pdf of the returns in intrinsic time. A stable model with an estimated $\alpha = 1.716$ provided an excellent fit for the returns at the 4-quote frequency. Given the average time elapsed between quotes, the relationship $t_i - t_k \approx 2(i - k)$ for $i > k$ was used to study the processes T and Z at the corresponding physical time scale. Define the *market time process* $\hat{T}$ by

$$\hat{T}(t) = \sum_{i=1}^{N} 1_{[t_i, \infty)}(t), \quad t \geq 0.$$

Then $\hat{T}(t)$ is the number of transactions up to time t, and $\hat{T}(t_i) = i$. The estimated pdf for the 8-minute time increments $\hat{T}(t) - \hat{T}(t-8)$ was studied to determine a model for the process T. The Weibull distribution provided the best fit. The Gamma distribution, which is infinitely divisible, also offered a good fit. In both cases the process $\{S(T_t)\}$ subordinated to the α-stable process S can be described in terms of stable distributions. On the other hand, the price process in physical time Z was similarly studied for 8-minute increments and obtained a stable fit with $\alpha = 1.3745$.

For any Lévy process X in this section it will be assumed that for *every* ω, $X(\omega)$ is cádlág and $X_0(\omega) = 0$. Let X and T be independent Lévy process defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$. We begin by specifying the characteristics of a subordinator (see Sato [18]).

Theorem 9. *Let $\{T_t\}_{t\geq 0}$ be a subordinator with Lévy measure ρ, drift β_0, and let $\lambda = P_{Z_1}$. Its second characteristic is zero and its Laplace transform is given by*

$$E[e^{-uZ_t}] = \int_{[0,\infty)} e^{-us}\lambda^t(ds) = e^{t\Psi(-u)}, \qquad u \geq 0,$$

where for any complex w with $\mathrm{Re}\ w \leq 0$,

$$\Psi(w) = \beta_0 w + \int_{(0,\infty)} (e^{ws} - 1)\rho(ds)$$

with

$$\beta_0 \geq 0 \qquad \text{and} \qquad \int_{(0,\infty)} (1 \wedge s)\rho(ds) < \infty.$$

Note that the theorem implies that a subordinator can only display jumps in the positive direction. This is obviously necessary, since we cannot go backwards in time. Moreover, the diffusion component has to be zero since otherwise there will be a negative change over any interval with positive probability.

The following result gives the characteristics of the subordinated process.

Theorem 10. *Let $\{T_t\}_{t\geq 0}$ be a subordinator with Lévy measure ρ, drift β_0, and $P_{T_1} = \lambda$. Let $\{X_t\}$ be an $\mathbb{R}^d$-valued Lévy process with generating triplet (γ, A, ν) and let $\mu = P_{X_1}$. Suppose that $\{X_t\}$ and $\{T_t\}$ are independent. Define*

$$Y(\omega) = X_{T_t(\omega)}(\omega), \qquad t \geq 0.$$

Then $\{Y_t\}$ *is a Lévy process and*

$$P[Y_t \in B] = \int_{[0,\infty)} \mu^s(B)\lambda^t(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

The generating triplet (γ', A', ν') *of* $\{Y_t\}$ *is as follows:*

$$\begin{aligned} \gamma' &= \beta_0\gamma + \int_{(0,\infty)} \rho(ds) \int_{|x|\le 1} x\mu^s(dx), \\ A' &= \beta_0 A, \\ \nu'(B) &= \beta_0\nu(B) + \int_{(0,\infty)} \mu^s(B)\rho(ds), \qquad B \in \mathcal{B}(\mathbb{R}^d \backslash \{0\}). \end{aligned} \tag{16}$$

4.3.1 Example: Variance-Gamma Process

We will now apply the previous result to obtain the characteristics for the Variance Gamma (VG) process (Madan, Seneta [13]). To this end, we first introduce the subordinator T which we define as the Lévy process such that

$$T_{t+s} - T_t \sim \Gamma\left(\frac{s}{\mu}, \frac{1}{\mu}\right) \tag{17}$$

where $\Gamma(c, \alpha)$ is the *gamma*-distribution with density

$$\frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x}, \ x > 0 \quad \text{ for } c > 0, \ \alpha > 0. \tag{18}$$

Lemma 1. *The generating triplet for the* $\Gamma(c, \alpha)$ *distribution is* $(0, 0, \rho)$*, where the Lévy measure* ρ *is given by*

$$\rho(dx) = cx^{-1}e^{-\alpha x}dx, \ x > 0. \tag{19}$$

It follows that the Γ-*subordinator* $\{T_t\}$ has characteristics $(0, 0, t\rho)$, with $c = 1/\mu$ and $\alpha = 1/\mu$.

Proof. Let μ be the probability measure with density (18). Denote its Laplace transform by $L_\mu(u)$. Then

$$L_\mu(u) = \left(1 + \frac{u}{\alpha}\right)^{-c}, \quad u \geq 0. \tag{20}$$

We will now see that

$$L_\mu(u) = \exp\left[c \int_0^\infty (e^{ux} - 1)\frac{e^{-\alpha x}}{x} dx\right]. \tag{21}$$

In fact,

$$\begin{aligned} \log(1 + \alpha^{-1}u) &= \int_0^u \frac{dy}{\alpha + y} = \int_0^u dy \int_0^\infty e^{-\alpha x - yx} dx \\ &= \int_0^\infty e^{-\alpha x} \left(\frac{e^{-ux} - 1}{-x}\right) dx, \end{aligned}$$

so that (21) now follows from (20).

For $w \in \mathbb{C}$, define $\Phi(w) = \int_0^\infty e^{wx} \mu(dx)$. Observe that Φ is analytic on $\{\mathrm{Re}\, w < 0\}$, continuous on $\{\mathrm{Re}\, w \leq 0\}$ and equal to $L_\mu(u)$ for $w = -u < 0$. Then Φ can be extended such that

$$\Phi(w) = \exp\left[c \int_0^\infty (e^{wx} - 1)\frac{e^{-\alpha x}}{x} dx\right], \quad \mathrm{Re}\, w \leq 0.$$

For $z \in \mathbb{R}$, it follows that

$$\hat{\mu}(z) = \Phi(iz) = \exp\left[c \int_0^\infty (e^{izx} - 1)\frac{e^{-\alpha x}}{x} dx\right],$$

and that the generating triplet of μ is $(0,0,\rho)$ with $\rho(dx)$ given by (19). $\square$

We state the following result for future reference. Let K_ν denote the modified Bessel function of the third kind with index ν (see, e.g., Watson [22]).

Lemma 2. *(Watson [22], p.80, 183)*

$$K_p(x) = \frac{1}{2}\left(\frac{x}{2}\right)^p \int_0^\infty e^{-t-x^2/(4t)} t^{-p-1} dt, \quad x>0,\, p \in \mathbb{R}, \tag{22}$$

$$K_{n+\frac{1}{2}}(x) = \sqrt{\pi/2}\, x^{-1/2} e^{-x} \left(1 + \sum_{i=1}^{n} \frac{(n+i)!}{(n-i)!i!}(2x)^{-i}\right), \quad x>0,\, n \in \mathbb{N}. \tag{23}$$

Let X be the process defined by $X_t = \sigma W_t + \theta t$ where W is a standard Brownian motion and $\sigma > 0$, $\theta \in \mathbb{R}$ are volatility and drift parameters, respectively. The *Variance Gamma process (VG)* is defined as the process Y subordinated to X by the Γ-subordinator T. Equivalently,

$$Y_t := X_{T(t)} = \sigma W_{T(t)} + \theta T(t).$$

By Theorem 10 the VG process has characteristics $(t\beta, 0, t\nu)$ for some $\beta \in \mathbb{R}$ and ν given by (16), which we compute as follows:

$$\begin{aligned}
\nu(dx) &= \int_0^\infty P^s_{\sigma W_1+\theta}(dx) c s^{-1} e^{-\alpha s} ds \\
&= \frac{c}{\sqrt{2\pi}\sigma} dx \int_0^\infty e^{-\frac{(s-\theta s)^2}{2\sigma^2 s}} s^{-3/2} e^{-\alpha s} ds \\
&= \frac{c}{\sqrt{2\pi}\sigma} e^{x\theta/\sigma^2} dx \int_0^\infty s^{-3/2} \exp\!\left[-\left(\alpha + \frac{\theta^2}{2\sigma^2}\right) s - \left(\frac{x^2}{2\sigma^2}\right)\frac{1}{s}\right] ds.
\end{aligned}$$

Using (22) and the change of variable $s' = \beta s$ with $\beta = \left(\alpha + \frac{\theta^2}{2\sigma^2}\right)$, the last integral is equal to

$$K_{\frac{1}{2}}\left(\sqrt{2x^2\beta/\sigma^2}\right)\left[\frac{1}{2}\left(\frac{\sqrt{2x^2\beta/\sigma^2}}{2}\right)^{1/2}\right]^{-1}.$$

Now using (23) with $n = 0$, it follows that

$$\nu(dx) = \frac{c}{|x|}e^{x\theta/\sigma^2}e^{-\frac{|x|}{\sigma}\sqrt{2\beta}}dx,$$

and substituting $c = 1/\mu$, $\alpha = 1/\mu$, we conclude that

$$\nu(dx) = \frac{1}{|x|\mu}\ \exp\left(\frac{x\theta}{\sigma^2} - \frac{|x|}{\sigma}\sqrt{\frac{2}{\mu} + \frac{\theta^2}{\sigma^2}}\right)dx, \quad -\infty < x < \infty.$$

4.3.2 Example: Subordination of Brownian Motion by $\alpha/2$-Stable

Using the same procedure, we now compute the characterization of the process subordinated to Brownian motion by the stable subordinator (Hurst, Platen, Rachev [10]). Define the subordinator T to be the Lévy process such that

$$T_{t+s} - T_t \sim S_{\alpha/2}(cs^{\alpha/2}, 1, 0), \quad c > 0,\ s, t \geq 0.$$

where $S_{\alpha/2}(cs^{\alpha/2}, 1, 0)$ is the $\alpha/2$-*stable distribution* (Samorodnitsky, Taqqu [17]) with characteristic function

$$\exp\left\{-sc^{\alpha/2}|z|^{\alpha/2}\left(1 - i\tan\left(\frac{\pi\alpha}{4}\right)\operatorname{sgn} z\right)\right\}, \quad z \in \mathbb{R}. \tag{24}$$

In order to obtain the set of characteristics for T, we will use the following results (Sato [18]).

Lemma 3. *Let μ be an infinite divisible distribution on $\mathbb{R}^d$ with characteristics (β, A, ν). Then μ is α-stable if and only if $A = 0$ and there is a finite measure λ on $S = \{x \in \mathbb{R}^d : |x| = 1\}$ such that*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{25}$$

Lemma 4. *Let μ be a non-trivial α-stable distribution on $\mathbb{R}^d$ with $0 < \alpha < 2$ and Lévy measure ν. Then $\int_{\{|x|\leq 1\}} |x|\nu(dx)$ is finite if and only if $\alpha < 1$. Also, $\int_{\{|x|>1\}} |x|\nu(dx)$ is finite if and only if $\alpha > 1$. The mass of ν is always infinite.*

Lemma 5. *The generating triplet of the $\alpha/2$-stable distribution defined in (24) is $(0, 0, \rho)$, where*

$$\rho(dr) = \frac{\lambda\, dr}{r^{1+\alpha/2}}, \quad r > 0 \tag{26}$$

with

$$\lambda = \frac{-c^{\alpha/2}}{\Gamma\left(-\frac{\alpha}{2}\right) \cos(\frac{\alpha\pi}{4})}. \tag{27}$$

It follows that the $\alpha/2$-*stable subordinator* $\{T_t\}$ has characteristics $(0, 0, t\rho)$.

Proof. In what follows, the Γ-function is extended from $(0, \infty)$ to any $s \in \mathbb{R}$ with $s \neq 0, -1, -2, \cdots$ by $\Gamma(s + 1) = s\Gamma(s)$. The following auxiliary result will be used:

$$\int_0^\infty \left(e^{wr} - 1\right) \frac{dr}{r^{1+\alpha'}} = \Gamma(-\alpha')(-w)^{\alpha'} \quad \text{for } \alpha' \in (0, 1), \tag{28}$$

which is valid for $w \neq 0$ complex such that $\operatorname{Re} w \leq 0$. Indeed, both sides of (28) are analytic on $\{w : \operatorname{Re} w < 0\}$ and continuous on $\{w : \operatorname{Re} w \le 0,\ w \ne 0\}$. Since

$$\begin{aligned}
\int_0^\infty (e^{-ur} - 1)\frac{dr}{r^{1+\alpha'}} &= -\int_0^\infty \int_0^r u\, e^{-uy} dy\, \frac{dr}{r^{1+\alpha'}} \\
&= -\frac{u}{\alpha'}\int_0^\infty e^{-uy} y^{-\alpha'} dy \\
&= \frac{\Gamma(1-\alpha')}{-\alpha'} u^{\alpha'} \\
&= \Gamma(-\alpha') u^{\alpha'} \quad \text{for } u > 0,
\end{aligned}$$

then (28) holds for real $w = -u < 0$. Hence it also holds on $\{w : \operatorname{Re} w \le 0,\ w \ne 0\}$. Since $d = 1$, observe that if $1_B(r\xi) > 0$ in (25) then $\xi \in \{-1, 1\}$. Then (25) reduces to

$$\begin{aligned}
\rho(B) = \lambda_{-1} &\int_0^\infty 1_B(-r)\frac{dr}{r^{1+\alpha/2}} \\
&+ \lambda_1 \int_0^\infty 1_B(r)\frac{dr}{r^{1+\alpha/2}} \quad \text{for } B \in \mathcal{B}(\mathbb{R}),
\end{aligned} \tag{29}$$

where $\lambda_j := \lambda(\{j\}) \ge 0$ and $j \in \{-1, 1\}$ such that $\lambda_{-1} + \lambda_1 > 0$. It follows from Lemma 3, Lemma 4, and (11) that the characteristic function of μ is of the form

$$\log \widehat{\mu}(z) = \int_{\mathbb{R}} \left(e^{izx} - 1\right) \rho(dx) + i\gamma_0 z, \quad z \in \mathbb{R}. \tag{30}$$

We shall now compute the integral in (30) with ρ defined by (29). Let $\alpha' = \alpha/2$. Choose the branch $(-w)^{\alpha'} = |w|^{\alpha'} e^{i\alpha' \arg(-w)}$ with $\arg(-w) \in (-\pi, \pi]$ in (28), implying that

$$\int_0^\infty \left(e^{izr}-1\right)\frac{dr}{r^{1+\alpha'}} = \Gamma(-\alpha')|z|^{\alpha'}\exp\!\left(-i\frac{\pi\alpha'}{2}\mathrm{sgn}(z)\right)$$
$$= \Gamma(-\alpha')|z|^{\alpha'}\cos\!\left(\frac{\pi\alpha'}{2}\right)\left[1-i\tan\!\left(\frac{\pi\alpha'}{2}\right)\mathrm{sgn}(z)\right].$$

Hence the integral in (30) with (29) is equal to

$$\Gamma(-\alpha')|z|^{\alpha'}\cos\!\left(\frac{\pi\alpha'}{2}\right)$$
$$\times\left\{\lambda_{-1}\left[1-i\tan\!\left(\frac{\pi\alpha'}{2}\right)\mathrm{sgn}(-z)\right]+\lambda_1\left[1-i\tan\!\left(\frac{\pi\alpha'}{2}\right)\mathrm{sgn}(z)\right]\right\}$$
$$=\Gamma(-\alpha')|z|^{\alpha'}\cos\!\left(\frac{\pi\alpha'}{2}\right)$$
$$\times\ (\lambda_1+\lambda_{-1})\left\{1-i\left(\frac{\lambda_1-\lambda_{-1}}{\lambda_1+\lambda_{-1}}\right)\tan\!\left(\frac{\pi\alpha'}{2}\right)\mathrm{sgn}(z)\right\}.$$

From the uniqueness in the Lévy-Khintchine representation it now follows from (24) and (30) that $\lambda_{-1}=0$, $\lambda_1=\lambda$ as defined in (27), and $\gamma_0=0$. Therefore (30) simplifies to

$$\log\widehat{\mu}(z)=\lambda\int_0^\infty\left(e^{izr}-1\right)\frac{dr}{r^{1+\alpha/2}},\quad z\in\mathbb{R},$$

from which (26) immediately follows. □

From Theorem 10 the process $\{W_{T(t)}\}$ subordinated to Brownian motion has characteristics $(0,0,t\nu)$ with ν given by (16). Therefore we conclude that

$$\begin{aligned}\nu(dx) &= \int_0^\infty P^s_{W_1}(dx)\frac{\lambda\, ds}{s^{1+\alpha/2}}\\ &= \frac{\lambda}{\sqrt{2\pi}}\int_0^\infty s^{-\frac{3+\alpha}{2}}e^{-x^2/2s}ds\, dx\\ &= \frac{\lambda\, 2^{\alpha/2}}{\sqrt{\pi}}\Gamma\!\left(\frac{\alpha+1}{2}\right)\frac{dx}{|x|^{1+\alpha}}, \quad -\infty < x < \infty.\end{aligned}$$

4.3.3 Example: Subordination of α-Stable by Gamma

Motivated by the results in Hurst, Platen, Rachev [10] cited in Section 4.3, we provide an expression for the characteristics of the subordination of the α-stable Lévy process with $1 < \alpha < 2$ by the Γ subordinator. Although the stable distribution is absolutely continuous with respect to Lebesgue measure, there is no known closed-form expression for the pdf valid for a range of values of α. We will then leave the Lévy measure expressed in terms of the series representation of the pdf (see [6]). To this end, we begin with the following representation for the characteristic function of the α-stable distribution on $\mathbb{R}$.

Theorem 11. *Let $0 < \alpha \leq 2$. If μ is an α-stable distribution on $\mathbb{R}$, then*

$$\widehat{\mu}(z) = \exp(-c_1|z|^\alpha e^{-i(\pi/2)\theta\alpha\,\mathrm{sgn}\, z}), \tag{31}$$

where $c_1 > 0$ and $\theta \in \mathbb{R}$ with $|\theta| \leq (\frac{2-\alpha}{\alpha}\wedge 1)$. The parameters c_1 and θ are uniquely determined by μ. Conversely, for any c_1 and θ, there is an α-stable distribution μ satisfying (31).

Denote the parameters in (31) by $(\alpha,\theta,c_1)_Z$ and denote the density of μ by $p(x;(\alpha,\theta,c_1)_Z)$.

Theorem 12. *The density for the distribution μ on $\mathbb{R}$ defined in (31) with $1 < \alpha < 2$ is given by*

$$p(x;(\alpha,\theta,c_1)_Z) = c_1^{-1/\alpha} p(c_1^{-1/\alpha} x;(\alpha,\theta,1)_Z) \quad \textit{for } x > 0$$

$$\textit{and} \quad p(x;(\alpha,\theta,c_1)_Z) = p(-x;(\alpha,-\theta,c_1)_Z) \quad \textit{for } x < 0,$$

where

$$p(x;(\alpha,\theta,1)_Z) = \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{\Gamma(1+k/\alpha)}{k!}(-x)^k \sin\!\left(\frac{k\pi}{2}(\theta-1)\right), \; x > 0.$$

Let T be the $\Gamma(\gamma,\beta)$-subordinator (19). Let X be the Lévy process such that X_1 is α-stable with parameters c_1, θ in the representation (31). Then the Lévy measure ν of the subordinated process X_T is

$$\begin{aligned}\nu(dx) &= \int_0^\infty P^s_{X_1}(dx)\gamma s^{-1} e^{-\beta s} ds \\ &= \gamma c_1^{-1/\alpha} dx \int_0^\infty p'(s,x) e^{-\beta s}\frac{ds}{s^{1+1/\alpha}},\end{aligned}$$

where

$$\begin{aligned}p'(s,x) = \Big[&p\big(-(sc_1)^{-1/\alpha}x;(\alpha,-\theta,1)_Z\big)1_{(-\infty,0)}(x) \\ &+ p\big((sc_1)^{-1/\alpha}x;(\alpha,\theta,1)_Z\big)1_{(0,\infty)}(x)\Big].\end{aligned}$$

5 Concluding Remarks

We have presented a summary of our earlier work regarding term structure models, where we expressed the results in terms of the characteristics of the driving process. Here we have described a methodology for constructing Lévy processes as potential drivers for our model. To illustrate, we derived the characteristics of some processes from the literature with infinite Lévy measure.

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References

[1] O.E. Barndorff-Nielsen, "Exponentially Decreasing Distributions for the Logarithm of Particle Size", *Proc. R. Soc. Lond. A.* 353, 401-419 (1977).

[2] O.E. Barndorff-Nielsen, "Hyperbolic Distributions and Distributions on Hyperbolae", *Scand. J. Statist.* 5, 151-157, (1978).

[3] O.E. Barndorff-Nielsen, N, Shephard, "Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics", *J. R. Statist. Soc. B* 63, Part 2, 167-241, (2001).

[4] T. Björk, Y. Kabanov, W. Runggaldier, "Bond Market Structure in the Presence of Marked Point Processes", *Math. Finance* 7, 211-239, (1997).

[5] J.Y. Campbell, A.W. Lo, A.C. MacKinlay, *The Econometrics of Financial Markets.* Princeton University Press, Princeton, N.J. (1997).

[6] W. Feller, *An Introduction of Probability Theory and its Applications.* Vol.2, Second Ed., Wiley, New York, (1971).

[7] P. Carr, D.B. Madan, E.C. Chang, "The Variance Gamma Process and Option Pricing",*European Finance Review* 2, 79-105, (1998).

[8] I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series, and Products*, corrected and enlarged ed., Academic Press, San Diego, (1980).

[9] J. Hernandez, “A General Framework for Term Structure Models Driven by Lévy Processes”, *Journal of Concrete and Applicable Mathematics*, vol 2, no.4 (2004).

[10] S.R. Hurst, E. Platen, S.T. Rachev, “Option Pricing for a Logstable Asset Price Model”, *Mathematical and Computer Modelling* 29, 105-119, (1999).

[11] R. Jarrow, D. Madan, “Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns”, *Math. Finance* 5, 311-336, (1995).

[12] D.B. Madan, “Purely Discontinuous Asset Price Processes”, preprint, University of Maryland, (1999).

[13] D.B. Madan, E. Seneta, “The Variance Gamma (V.G.) Model for Share Market Returns”, *Journal of Business*, 63, 511-524, (1990).

[14] C. Marinelli, S.T. Rachev, R. Roll, “Subordinated Exchange Rate Models: Evidence for Heavy Tailed Distributions and Long-Range Dependence”, in *Stable Models in Finance*, Pergamon Press, (1999).

[15] J.P. Nolan, “Modeling Financial Data with Stable Distributions”, in *Handbook of Heavy Tailed Distributions in Finance*, (ed. S.T. Rachev). Elsevier-North Holland, Amsterdam, (2003).

[16] S.T. Rachev, S. Mittnik, *Stable Paretian Models in Finance*, Wiley, West Sussex, (2000).

[17] G. Samorodnitsky, M.S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York, (1994).

[18] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, (2000).

[19] P.J. Schönbucher, *Credit Risk Modelling and Credit Derivatives*, Ph.D. dissertation, University of Bonn, (2000).

[20] A.N. Shiryaev, J. Jacod, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, (1987).

[21] A.N. Shiryaev, *Essentials of Stochastic Finance: Facts, Models, Theory*, World Scientific, Singapore, (1999).

[22] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, (1944).