Distributional Analysis of the Stocks Comprising the DAX 30

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Abstract

In this paper, we analyze the returns of stocks comprising the German stock index DAX with respect to the α-stable distribution. We apply nonparametric estimation methods such as the Hill estimator as well as parametric estimation methods conditional on the α-stable distribution. We find for both the nonparametric and parametric estimation methods that the α-stable hypothesis cannot be rejected for the return distribution. We then employ the GARCH model; the fit of innovations modeled with an underlying α-stable distribution is compared to the fit obtained from modeling the innovations with the skew-t distribution. The α-stable distribution is found to outperform the skew-t distribution.

Keywords Stable distributions, heavy-tails, tail estimation, ARMA-GARCH, DAX 30

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1 Introduction

As shown by Mandelbrot (1963b) and Fama (1965a), stock returns have an underlying heavy-tailed distribution. In other words, they are leptokurtic. This can also be found in Clark (1973) and Blattberg and Gonedes (1974). What followed these initial findings was a vast amount of monographs and articles covering the stock price behavior with emphasis on the U.S. capital market. An exhaustive account of these studies is provided in Rachev (2003). Research with respect to this issue for the German equity market is not as extensive. In the appendix to this paper, we provide a table that summarizes the findings of studies for the German and Austrian equity markets.

In this paper, we investigate the distribution behavior of daily logarithmic stock returns for German blue chip companies. While the distribution that is assumed in major theories in finance and risk management is the Gaussian distribution, we show that the α-stable distribution offers a reasonable improvement if not the best choice among the alternative distributions that have been proposed in the literature over the past four decades.

The most important equity index in Germany is the DAX® index which contains the 30 most liquid German blue chip stocks. Prices used to compute the return were obtained from the Frankfurt Stock Exchange.[1] The return for

[1] In addition, the automated quotations for the same stocks from the Xetra® were analyzed to determine whether there are deviations in the results caused by the slightly different regulatory procedures offered by the two exchanges. However, because significant differences between the prices were not observed, results for the automated quotation are omitted. The

each stock includes cash dividends and is adjusted for stock splits and capital adjustments. The period investigated is January 1, 1988 through September 30, 2002.

Inclusion in the DAX depends on requirements such as market capitalization and trading volume. As a result, some of the 30 constituent stocks are periodically replaced by others. During the period of investigation, there were 55 stocks that had been included in the DAX. To assure that the statistics estimated were generated from sufficient data, we restricted the sample to stocks with a minimum of 1,000 observations. This reduces the original number of candidate stocks from 55 to 35.

The problems related to the correct assessment of the empirical distribution of the returns are with respect to the overall shape, tail estimation, and determination of existing moments. Particularly in the context of finite sample observations, the last can easily lead one to mistakenly conclude in favor of distributions with lighter tails. To exemplify, the moments of a Gaussian distribution exist to all orders. This is not the case, for example, with the Pareto or students-t distributions even though sample moments of those distributions exist since data samples are finite. It can be shown that even these can grow quickly with increasing order which is usually the case with financial data.

The paper is organized as follows. In the next section, the basic notion of α-stable random variables is reviewed. In Section 3, we present the results based on non-parametric estimation methods for the return distribution. Section 4 provides methods and results of the parametric estimation techniques conditional on the α-stable class of distributions. Section 5 models volatility clustering based on different error distributions and reports the results of the alternative GARCH models. A summary of our findings is presented in the final

stock prices from both sources were provided by the capital market database Karlsruher Kapitalmarkt Datenbank (KKMDB) at the University of Karlsruhe.

section.

2 Definition of α-Stable Distributions

For an exhaustive treatment of the topic of α-stable randomness, Samorodnitsky and Taqqu (1994) should be consulted which has become a standard in this field, over the years. Here, a brief idea is given as to what the meaning of α-stable distributions implies, in the definition below.

Definition 2.1. If for any $a, b > 0$ and independent copies X_1, X_2 of X, there exist $c > 0$ and $d \in \mathbb{R}$ such that

$$aX_1 + bX_2 \stackrel{d}{=} cX + d \tag{1}$$

where $\stackrel{d}{=}$ denotes equality in distribution, then X is a stable random variable.

Generally, α-stable random quantities are described by the quadruple $(\alpha, \beta, \sigma, \mu)$ or with the notation of Samorodnitsky and Taqqu (1994), $S_\alpha(\sigma, \beta, \mu)$, where the index of stability, α, is the characteristic parameter of the tail as well as the peak at the median. Scaling is described by σ, β indicates the degree of skewness whereas μ is the location parameter which is not necessarily the mean.

An important property of the α-stable random variables is that they can be looked upon as the distributional limit of a standardized sum of an increasing number of *i.i.d.* random variables. They are said to have a domain of attraction (DA). This is a generalization of the central limit theorem known for the Gaussian distribution. Note that the normal distribution is a special case of the α-stable distributions. In that case, $\alpha = 2$, β is meaningless, μ is the mean, and the variance is $2\sigma^2$.

Even though an analytical form of the probability density function (*pdf*) does not exist for most combinations of the four parameters, the distribution can be identified by the unique characteristic functions which are given to be as in

Definition 2.2. X is said to be stable if there exist $0 < \alpha \leq 2, \sigma \geq 0, \beta \in [-1, 1]$, and $\mu \in \mathbb{R}$ such that

$$\Phi(\theta) \equiv \begin{cases} \exp\left\{-\sigma^\alpha|\theta|^\alpha(1 - i\beta(\operatorname{sign}\theta)\tan(\frac{\pi\alpha}{2})) + i\mu\theta\right\}, & \alpha \neq 1 \\ \exp\left\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(\operatorname{sign}\theta)\ln|\theta|) + i\mu\theta\right\}, & \alpha = 1. \end{cases} \quad (2)$$

In general, α-stable distributions are favorable for modelling financial returns because of their ability to display skewness often observed in reality. The possibly more important feature, however, is that they can capture the leptokurtosis of financial returns. In the tails, α-stable distributions decay like a Pareto distribution, hence, they are also referred to as Pareto-stable. As is often the case, large price movements are more frequent than indicated by the normal distribution which can be particularly harmful if price changes are negative.

3 Nonparametric Estimation of Return Distribution

In this section, we report the results of three nonparametric tests for the return distribution: kurtosis, Kolmogorov-Smirnov, and Hill tail.

3.1 Kurtosis

An initial statistic of interest to reveal information as to whether a sample can be considered normal or heavy-tailed is the kurtosis defined as

$$\mathfrak{K} = \frac{\mathbb{E}(X - \mu)^4}{\sigma^4}.$$

In the normal case, this statistic takes on the value 3 whereas in the case of heavy-tails, the values are higher.

As can be seen from column (2) of Table 1, for the stocks in this study kurtosis is significantly greater than 3, indicating leptokurtosis for all 35 returns series.[2] This finding agrees with the findings of other researchers who have investigated the German equity market. See, for example, Kaiser (1997) and Schmitt (1994). For financial data, kurtosis is usually greater than 3 as stated in Franke, Härdle and Hafner (2004).

3.2 Kolmogorov-Smirnov test

As a test for Gaussianity, we apply the two-sided Kolmogorov-Smirnov test with its well-known test statistic

$$K_n = \sup_{x \in \mathbb{R}} |F_0(x) - F_n(x)|$$

where F_0 is the theoretical cumulative distribution function *(cdf)* tested for and F_n is the sample distribution. For all but one stock in our study, the Gaussian distribution could be safely rejected at the 95% confidence level. The values for the Kolmogorov-Smirnov test are given in columns (3) through (6) of Table 1.

3.3 Hill tail-estimator

The following approach uses the semi-parametric Hill estimation of the tail index as a proxy for the extreme Pareto part of the tail if it should exist. The tail estimator was first introduced by Hill (1975) to infer the Pareto-type behavior for the sample data. The estimator applies if the tails of the underlying *cdf* follow the Pareto law with tail index α_P. The Pareto *cdf* is in the DA of the

[2]In Table 1, WKN is the abbreviation of the German word "Wertpapierkennummer" which means security code number.

α-stable Paretian law for $0 < \alpha < 2$[3] with tail probability in the limit

$$P(Y \geq y) = 1 - F(y) \approx Ly^{-\alpha}, \; y \to \infty.$$

with slowly varying L. With $X_{(n)} < X_{(n-1)} < \ldots$, the estimator is defined as[4]

$$\hat{\alpha}_{Hill} = \frac{r}{\sum_{i=1}^{r} \ln X_{(i)} - r \cdot \ln X_{(r+1)}} \tag{3}$$

which under certain conditions is consistent.[5]

A problem arises with respect to the determination of the proper threshold index r indicating the beginning of the Pareto tail of the underlying *cdf*. This may suffice to hint at the questionable quality of the estimator.[6] Annaert, De Ceuster and Hodgson (2005) investigated the reliability of the Hill estimator. Based on Monte Carlo simulation, they find that the Hill estimator retrieves the heavy-tailed characteristic or tail parameter with sufficient exactness whenever the true underlying Pareto-stable distribution is in the realm of non-Gaussianity. However, the parameter space in the simulation of Annaert, De Ceuster and Hodgson (2005) was very limited in that β and μ were set to 0 and γ was restricted to 0.01. We, on the other hand, conducted a different Monte Carlo simulation with a more flexible parameter space. As a result, we cannot confirm their support for the Hill estimator. Instead, our findings cast serious doubt on the Hill estimator's reliability because it systematically overestimates the tail parameter. Even for fairly low α, we find that the estimator trespasses the border-line value 2 with a high probability.

As just mentioned, a problem arises with respect to the determination of

[3] For different values of α, the characteristic exponent of the α-stable parametrization and the Pareto tail parameter do not correspond.

[4] Indices in parentheses denote the ordered sample.

[5] See Rachev and Mittnik (2000).

[6] Admittedly, there have been attempts to find methodologies for assessing the appropriate tail sizes. See, for example, Lux (2001).

the proper threshold index r indicating the beginning of the Pareto tail of the underlying *cdf* when computing the Hill estimator. As r increases, α_H gradually descends to cross the conditional value of the estimated α-stable parameter. Beyond certain values of r, α_H falls to approach the value of 1. The Hill estimator estimates the α-stable characteristic parameter correctly, in some instances, at tail lengths of between 10% and 15%. But no common threshold value can be determined for all the stocks analyzed in this study.[7]

With this ambiguity existing as to where the tail of the underlying sample distribution begins, Lux (1996a) still rejects the hypothesis of tails stemming from an α-stable distribution for German blue chip stocks as a result of Hill estimation based on varying tail lengths of 2.5%, 5%, 10%, and 15%. Covering an earlier period, Akgiray, Booth and Loistl (1989) performed a test for the tail indices of the most liquid German stocks based on maximum likelihood estimation of the generalized Pareto distribution, $1-(1+\gamma\frac{x}{\omega})^{1/\gamma}$, rather than the Hill estimator. They also rejected the α-stable hypothesis for the tails even though they cannot deny the overall good fit this class of distributions provides, and suggest a universal 10% tail area optimal.

Results of the Hill estimation of the tail index for our sample stocks are reported in Table 2 with standard errors and 95% confidence bounds, respectively. The instability of the estimator for varying tail lengths becomes strikingly obvious. The plots (not displayed here) reveal that the tail corresponds to the characteristic stable parameter for tail sizes roughly within 10% and 15%. As can be seen by the lower bounds, when the respective tail lengths represent the extreme 15% of the returns, in 31 out of 35 cases, we cannot reject a stable distribution at the 95% confidence level. Still, we find that the Hill estimator is inappropriate to serve as a reliable estimator for the tail index.

[7]Problems of this sort are also mentioned in Rachev and Mittnik (2000).

4 Parametric Estimation Conditional on the α-Stable Distribution

So far, we have rejected the hypothesis of Gaussian returns. Additionally, we concluded that the Hill estimator does not suffice to determine the tail index. Hence, the hypothesis of Pareto-type tails in the realm of α-stability could not be rejected. Now, conditional on the assumption that the α-stable distribution is correct, we set about to estimate the four stable parameters based on three different techniques: maximum likelihood estimation (MLE), quantile estimation, and characteristic function based estimation. All estimation results can be found in Table 3.

4.1 Maximum likelihood estimation

In the following, parameter estimates are obtained conditional on the α-stable distribution function. For conducting MLE of the parameters with the likelihood $f(x|\alpha, \gamma, \beta, \mu_1)$, two methods have been suggested. The first method, suggested by Nolan (1999),[8] minimizes the information matrix which is known to be the negative inverse of the Hessian matrix of the likelihood function. This is done by some numerically efficient gradient search. The second method is based on a computationally efficient Fast Fourier Transformation (FFT) introduced by Mittnik, Doganoglu and Chenyao (1999). We will refer to the first and second methods as the Nolan method and FFT method, respectively.

The FORTRAN program code of the Nolan method used in this study is incorporated in an executable program offered on Nolan's internet web page. Applying some constraints concerning the boundaries, etc., values obtained for α for our sample of stocks are between 1.4605 and 1.9117. The values of β are significantly different from no skewness, i.e. $\beta = 0$, with a majority indicating

[8] The reader can find a vast resource of α-stable MLE on Nolan's web site at American University including his executable program codes.

slight positive skewness.[9] The FFT method[10] applies an FFT approximation of the *pdf* to conduct the computation of the likelihood. The benefit of the FFT method is the reduction in computation time.[11] The estimates for α corresponded to those obtained from the Nolan method, ranging from 1.44617 to 1.8168. For the FFT method, too, the values for β generally suggest skewness for most stocks.

The minimum value obtained for α was identical for both, the Nolan and the FFT method. It was also found at the same stock. This is in contrast to the maxima which were, additionally, obtained at different stocks. Interestingly, though, the maximum value estimated by the Nolan method matched the α value estimated for that very same stock by the FFT method.

4.2 Quantile estimation

While Fama and Roll (1971) provided the foundation for the quantile estimator, it was McCulloch (1986) who modified the estimator, providing estimation of parameters for skewed α-stable *pdf*s. The estimator matches sample quantiles and theoretical quantiles tabulated for different values of the parameter tuple.[12]

The values for α for our sample of stocks range from 1.3975 to 1.8019. It is somewhat striking that the values seem to be slightly lower than those obtained from the MLE using both the Nolan and FFT methods.[13]

[9] For further complications inherent in the program code as to the computational results, the reader is referred to the manual given by the program's author.

[10] Because estimates from the FFT method do not significantly deviate from the Nolan method, they are not listed here.

[11] The code in MATLAB was provided by Stoyan Stoyanov, FinAnalytica Inc.

[12] The implementation of the McCulloch estimator in MATLAB was enabled through the translation of the original FORTRAN code by Stoyan Stoyanov, FinAnalytica Inc.

[13] This type of downward bias was found, for example, as a result of Monte Carlo studies by Blattberg and Gonedes (1974) using the quantile estimator by Fama and Roll (1971) and should be less likely when applying the estimator by McCulloch (1986) due to the fact that it is a consistent estimator.

4.3 Characteristic function based estimation

The last of the three estimators we used in this study is the characteristic function based estimator. Its existence is not surprising since the theoretical characteristic functions of the α-stable distribution are known. Hence, one only needs to fit the sample characteristic function (SCF) and retrieve the parameters. Generally, this approach is based on Koutrouvelis (1981). Let the SCF be

$$\hat{\Phi}(\theta) = \frac{1}{n} \sum_{t=1}^{n} \exp\{i\theta\tilde{y}_t\}. \tag{4}$$

Ordinary Least Squares (OLS) estimates for the stable parameters are obtained from the natural logarithm of equation (4).

The problem with the numerical method as proposed by Koutrouvelis (1981) is that the frequencies θ_k most suitable for the respective regression must be looked up in tables indexed by sample size and initial parameter estimates. This leads to a large computational effort. Kogon and Williams (1998) remedied this shortcoming by using a common, finite interval for the θ_k with fixed grid size for all parameters and samples. This procedure, called the Fixed-Interval (FI) estimator, results in a substantial computational improvement. They suggested that the best interval would be $[.1, 1]$ with up to 50 equally spaced grid points. While with respect to precision the FI estimator is slightly inferior to the original one by Koutrouvelis (1981) for some parameter tuples, this is more than offset by its speed.

For the characteristic function based estimator, an implementation in MATLAB of the FI estimator has been used.[14] The fixed interval was set as suggested by Kogon and Williams (1998) with 10 scalar frequency points and step size .1. Estimation results are reported in Table 3. The values obtained for α for the

[14]This has been implemented by Stoyan Stoyanov, FinAnalytica Inc.

stocks in our sample are between 1.5377 and 1.8828. Values for $|\beta| > 0.1$ can be observed in two cases. The majority of the values indicates slight positive skewness. Computation time was significantly reduced compared to the previous alternatives.

It is evident from all estimation results that the parameters indicate non-Gaussian distributions of the returns, i.e. values α are well below 2. Results are reasonably close throughout the different methods despite theoretical descrepancies of the three estimators.

5 Modeling the Returns as GARCH

Our last set of empirical results, and possibly the most interesting, are those obtained from an analysis of the autoregressive moving average (ARMA) innovations with respect to generalized autoregressive conditional heteroscedasticity (GARCH). The ARMA-GARCH model used in this study is

$$r_t = \sum_{i=1}^{p} \phi_i r_{t-1} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}$$
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-1}^2 + \sum_{i=j}^{p} \beta_j h_{t-j}$$

where $\epsilon_t | \mathfrak{F}_t \sim N(0, h_t)$ and $\mathfrak{F}_t$ is the filtration at time t. Empirically, it has been observed by Bollerslev (1986) that a simple GARCH(1,1) performs at least as well as a long-lagged ARCH(8) process. An attribute of the special GARCH(1,1) process for modeling financial data series is its capability to capture leptokurtosis. While the results from fitting the returns series to ARMA structures are not displayed here,[15] in most cases the preferred model was an MA(1). In some cases, the AR(1) model was found to provide the best fit. The selection cri-

[15]Results are available upon request.

terion considered first-differencing appropriate for one stock, only, but in that instance it was found that the suggested model was not invertible. Consequently, we imposed a no-first-differencing restriction and obtained a MA(1) model with similar value for the selection criterion. In all cases, diagnostic and significance checks suggested that the returns are best modeled as white noise.

Francq and Zakoïan (2004) prove that under quite general conditions for pure GARCH as well as ARMA-GARCH Quasi-MLE such as Berndt, Hall, Hall, and Hausmann (BHHH) produces asymptotically normal estimators when innovations satisfy second moment conditions. In the case of infinite variance processes such as α-stable innovations, convergence occurs even faster.

The introduction of a students-t distribution permitting skewness or, simply, skew-t distribution, serves as a reasonable competitor to the α-stable distribution in this context. First introduced as a multivariate version by Hansen (1994), Fernandez and Steel (1998) present the univariate *pdf* about mean or location zero as

$$p(\epsilon_t|\tau,\nu,\lambda) = 2\frac{\Gamma(\frac{\nu+1}{2})\tau}{\Gamma(\frac{\nu}{2})(\pi\nu)^{1/2}(\gamma+\frac{1}{\gamma})}\times$$
$$\left[1+\frac{\tau^2}{\nu}\epsilon_t^2\left\{\frac{1}{\gamma^2}I_{[0,\infty)}(\epsilon_t)+\gamma^2 I_{(-\infty,0)}(\epsilon_t)\right\}\right]^{-\frac{\nu+1}{2}}. \tag{5}$$

Parameter ν indicates the degrees of freedom as with the t-distribution and the parameter γ corresponds to skewness, with $\gamma = 1$ indicating symmetry. Any other value for γ indicates skewness of some degree. The parameter τ^2 is interpreted as precision. It is inversely proportional to the scaling parameter, σ^2, which, in turn, is a real multiple of the variance if it exists. In applications in the literature, τ is very often set equal to 1.[16] Equation (5) reduces to the regular students-t *pdf* when $\beta = 0$, $\lambda = 1$, and $\tau = 1$.[17]

[16]See, for example, Garcia, Renault, and Veredas (2004).

[17]Alternative representations of the skew-t *pdf* can be found, for example, in Jones and

The model we suggest is a GARCH(1,1) structure of the generalized form

$$c_t^\delta = \alpha_0 + \alpha|\epsilon_{t-1} - \mu|^\delta + \beta c_{t-1}^\delta$$

where we know $\delta = 2$ from the original set-up with Gaussian innovations. In the Gaussian case, $c_t = h_T^{1/2}$. This is impossible, however, if the distribution under consideration does not have finite moments of order $> \delta$ for some $\delta < 2$. Since first absolute moments exist for the theoretical distributions fitted to the log return series as well as innovations, $\delta = 1$ is chosen as in Rachev and Mittnik (2000).[18] The GARCH(1,1) structure is parsimonious regarding parameter use and still enjoys popularity for its great flexibility in financial applications as noted by Nelson (1991) and several of his later articles, as well as others.

In our paper, the skew-t and the α-stable distributions were tested against each other as alternative distributions for the ARMA residuals, $\{\epsilon_t\}$, virtually being the log returns in many cases. In fact, the normal distribution was also analyzed; however, because it performed poorly, we did not consider it any further.

Depending on the distribution, notation $S_{\alpha,\beta}^\delta$GARCH(r,s) and $t_{\nu,\lambda}^\delta$GARCH(r,s) can be used to indicate α-stable or skew-t innovations, respectively. Preference is based on the maximized logarithmic likelihood[19] functions of the *iid* $\{r_t\}$. In the α-stable case, the likelihood equals

$$\prod_{t=1}^{n} \frac{1}{c_t} S_{\alpha,\beta}\left(\frac{\epsilon_t - \mu}{c_t}\right)$$

which in contrast to the normal and skew-t distributions is known not to have

Faddy (2003).

[18] Mittnik and Paolella (2003) leave more room to play in the sense that δ enters as a variable with respect to which can be optimized for each distribution, respectively.

[19] Conditioning starting values are set equal to their expected values. However, as argued in Mittnik and Paolella (2003), these values have little to no impact on the outcome of the estimation.

an analytical solution. Consequently, it has to be approximated numerically.

For a numerical approximation of the α-stable likelihoods, MATLAB encoded numerical FFT approximations were performed. The skew-t likelihoods are analytically solvable.[20] Results show that for some stocks, the skew-t and α-stable alternatives behave alike according to the log-likelihood values.

The fit was also compared using the Anderson-Darling (AD) goodness-of-fit test,

$$AD = \sup_{x \in \mathbb{R}} \frac{|F_s(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)\left(1 - \hat{F}(x)\right)}}$$

with $\hat{F}(\cdot)$ denoting the estimated parametric *pdf* and $F_s(\cdot)$ the empirical sample *pdf* computed as

$$F_s(x) = \frac{1}{n} \sum_{t=1}^{n} I_{(-\infty,x]} \left(\frac{\epsilon_t - \hat{\mu}}{\hat{h}_t^{1/2}} \right).$$

$I(\cdot)$ is the indicator function.

The AD-statistic is well suited for detecting poorness of fit, particularly in the tails of the *cdf*. As can be seen in Table 4, the α-stable outperforms the skew-t alternative in most instances.[21] Even though, we tested lag structures of up to $(r = 5, s = 5)$,[22] our preference was with a lag structure of (1,1) justifying the GARCH(1,1) model for the reasons commonly cited in literature.

[20]Basic GARCH estimation programs in MATLAB provided by Kevin Sheppard from the University of California at San Diego were altered by us to allow for the α-stable distribution. The current internet location is `http://www.kevinsheppard.com/research/ucsd_garch/ucsd_garch.aspx`.

[21]Analyzing foreign exchange data of US Dollar versus several important international currencies, Mittnik and Paolella (2003) found comparable results. But one has to keep in mind that their counterpart distribution is the student-t with less flexibility than the skew-t we use in this study. So, our results might be considered even more striking, in this context.

[22]Tabulated results of lags up to five are available upon request.

6 Conclusion

All of the tests performed in this study reject the Gaussian hypothesis for the logarithmic returns of the German blue chip stocks we analyzed. The nonparametric estimation results indicate that the rejection of the stable hypothesis by other researchers is not based on a reliable empirical test. The modeling of returns using α-stable distributions we report seems promising in spite of the lack of an analytic form of the probability distribution function. This is due to the tight fit of the approximated α-stable *cdf* to the empirical *cdf* combined with dependable estimation of the stable parameters.

As a negative aspect mentioned by several researchers, for example, Lux (1996a), the α-stable alternative sometimes slightly overemphasizes the mass in the extreme parts of the tails compared to finite empirical data vectors. This is in contrast to our findings. We discovered that the tail shape of the α-stable class is extremely suitable for the returns we considered, particularly in the context of GARCH modeling. The alternatives in our study provided by the normal and the skew-t distributions could not systematically outperform the α-stable distribution. Instead, they produced equivalent results, at best. Particularly with respect to fitting the empirical tails, they performed poorly.

Theoretically, using the α-stable distribution is reasonable because it is the distributional limit of series of standardized random variables in the domain of attraction. Thus, the α-stable class is a natural candidate for modeling the return distribution. Practically, when protecting portfolios against extreme losses, it becomes particularly important to asses the extreme parts of the lower tails adequately. Hence, the stable Paretian distribution ought to be favored due to its very good overall fit of the distribution function in addition to the superior tail fit.

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Appendix: Summary of research for the German stock market

Study (German)	Period covered	Frequency	Equity instrument	Methodology used	Conclusion
Akgiray, Booth and Loistl (1989)	Jan 1974 - Dec 1982	daily	50 stocks	MLE of γ of generalized Pareto comparison with stable α, $2 \times \sigma_\gamma$ confidence bounds	reject α-stable assumption on the account of all α outside $2 \times \sigma_\gamma$
Hecker (1974)[d]	Jan 1968 - Apr 1971	daily	54 stocks	chi-square	rejection of normal distribution[b]
Hanssen (1976)	1961 - 1972[c]	daily	50 stocks	binomial sign test of the stability of the variance	constant variance rejected at $p = .05$
Hecker (1974)[d]	Jan 1958 - Dec 1962	daily	37 stocks	chi-square	rejection of normal distribution[b]
Hockmann (1979)[d]	Jan 1970 - Jun 1976	daily	40 stocks	chi-square	rejection of normal distribution[b]
Kaiser (1997)	Jul 1990 - May 1994	daily	30 stocks	1) Kiefer-Salomon test (p=.01), 2) Kurtosis, 3) Ljung-Box of returns (p=.01), 4) Ljung-Box of squared returns (p=.01), 5) LM test, 6) loglikelihood ratio for student-t $/N(\mu, \sigma)$ of GARCH(1,1)-M noise	1) reject normal, 2) Kurtosis very high for all, 3) reject autocorrelation, 4) squared returns autocorrelated (at least lag 1), 5) significant ARCH, 6) student-t significantly better than normal (p=.01)
Krämer and Runde (1991)	Mar 1980 - Mar 1990	daily	14 stocks	corrected Box-Pierce test (for *iid* X_i with infinite variance)	autocorrelation of α-stable returns not significant in most cases (p=.01)
Lux (1996a)	Jan 1988 - Sep 1994	daily	30 stocks	1) Kurtosis, 2) Hill estimator (various tail sizes), 3) parametric test for stable α, 4) chi-square against stable 5) chi-square homogeneity test for $\alpha_1 = \ldots = \alpha_{30}$	1) Kurtosis>3 in all cases, 2) all $\alpha_{Hill} > 2$ and all 95% confidence lower bounds for $\alpha_{Hill} > 2$ 3) stable $\alpha \in [1.42, 1.748]$ $\rightarrow$ non-Gaussian, 4) 14/30: reject stable Paretian 5) cannot be rejected (p=.05)
Lux (2001)	Nov 1988 - Dec 1995	high frequency (minute)	DAX index	1) Kurtosis, 2) Hill estimator (various tail sizes), 3) parametric test for stable α, 4) homogeneity chi-square test for $\alpha_1 = \ldots = \alpha_7$ for each stock	1) Kurtosis $>> 3$, 2) $3 \leq \alpha_{Hill,i} \leq 4$, 3) stable $\alpha_i < 2$, 4) homogeneity of $\alpha_{j,i}$ not rejected
Möller (1984)[a]					

Appendix: Summary of research for the German stock market (concluded)

Study (German)	Period covered	Frequency	Equity instrument	Methodology used	Conclusion
Mhlbrandt (1978)[d]	Jan 1967 - Dec 1975	daily	46 stocks	chi-square	rejection of normal distribution[b]
Reiß (1974)[d]	Jan 1961- Dec 1972	daily	50 stocks	chi-square	rejection of normal distribution[b]
Röder and Bamberg (1995)	Jan 1991 - Dec 1993	high frequency (minute)	DAX	1) chi-square of homogeneous σ for (Mo - Fr) 2) chi-square of homogeneous σ for 12 periods (15 min.) during each trading day	1) not rejected at p = .05 2) rejected at p = .05
Ronning (1974)[d]	Jan 1961- May 1973	weekly	18 stocks	chi-square	rejection of normal distribution[b]
Scheicher (1996b)	Feb 1963 - Dec 1993	monthly	12 stocks	1) Wald test for heteroscdasticity 2) student-t modeling stand. GARCH noise	1) 11/12: significant p=.01 2) degree of freedom $\nu <$ Gaussian -¿ heavy tails
Schlag (1991)	Jan 1986 - Dec 1990	daily	5 stocks	1) skewness 2) kurtosis 3) Shapiro-Wilk test 4) Ljung-Box 5) log-likelihood ratio of GARCH vs no GARCH 6) t-test of GARCH coefficients	1) significant 2) significant 3) reject normal at p=.01 4) no significant autocorrelation for returns autocorrelation of squared returns significant p=.01 5) ratio significantly (p=.01) in favor of GARCH 6) significant (p=.01) for several lags
Schmitt (1994)	Jan 1987 - Dec 1992	daily	DAX and 11 (constituent) stocks	1) Kurtosis (p=.01) 2) AD test (p=.01) 3) Ljung-Box for squared errors 4) Schwarz IC of ARCH(1) 5) normal GARCH(1,1) vs GARCH(1,1)-t	1) reject normal in all cases 2) reject normal in all cases 3) significant in all cases 4) SIC higher for ARCH (1) than normal 5) SIC of GARCH(1,1)-t $>$ SIC of normal GARCH(1,1)

Appendix: Summary of research for the Austrian stock market

Study (Austrian)	Period covered	Frequency	Equity instrument	Methodology used	Conclusion
Geyer and Hauer (1991)	Jan 1986 - Dec 1988	daily	28 stocks (Austrian)	1) K-S test (p=.01), 2) Kurtosis ($2 \times \sigma_{Kurtosis}$-bounds), 3) skewness ($2 \times \sigma_{skew}$), 4) Ljung-Box (p=.01), 5) LM test (p=.01)	1) reject normal, 2) 27/28: Kurtosis > 3 and 25/28: $\|\text{Kurt} - 3\| > 2 \times \sigma_{Kurt}$, 3) 14/28: $\|\text{skew} - 0\| > 2 \times \sigma_{skew}$, 4) all squared returns exhibit significant autocorrelation, 5) 26/28: reject no ARCH
Scheicher (1996)	Sep 1986 - May 1992	daily	Vienna Stock Index	1) LM test (p=.01), 2) Ljung-Box of returns for various lags (p=.01), 3) Ljung-Box of squared returns (p=.01), 4) Jarque-Bera of AR(1) regression innovations[b], 5) Kurtosis of innovations, 6) MLE for student-t fitting of residuals	1) significant, 2) significant for high order lags, 3) significant (at least lag 1), 4) reject normal, 5) very high, 6) degree of freedom less than 4.3

a = no research done by author himself, just compilation of tests and results. b = confidence unknown. c = uncertain about specific months. d = cited in Möller (1984).

(1) WKN	(2) Kurtosis	(3) H	(4) P	(5) KSSTAT	(6) CV
500340	5.9	1	$7.25 \cdot 10^{-6}$	7.57	4.11
515100	6.3	1	$3.37 \cdot 10^{-14}$	6.53	2.23
519000	9.0	1	$1.43 \cdot 10^{-20}$	7.90	2.23
543900	6.2	1	$2.74 \cdot 10^{-5}$	5.05	2.90
550000	11.0	1	$6.37 \cdot 10^{-9}$	5.98	2.60
550700	40.5	1	$2.25 \cdot 10^{-12}$	8.42	3.08
551200	15.2	1	$4.17 \cdot 10^{-15}$	7.78	2.57
575200	11.8	1	$1.35 \cdot 10^{-16}$	7.07	2.23
575800	8.1	1	$2.99 \cdot 10^{-11}$	6.50	2.50
593700	14.2	1	$1.38 \cdot 10^{-11}$	5.88	2.23
604843	11.7	1	$9.42 \cdot 10^{-19}$	7.53	2.23
627500	12.7	1	$8.44 \cdot 10^{-13}$	6.54	2.35
648300	9.8	1	$3.28 \cdot 10^{-20}$	7.83	2.23
656000	11.1	1	$1.68 \cdot 10^{-12}$	6.87	2.50
660200	21.5	1	$3.27 \cdot 10^{-18}$	1.14	3.43
695200	9.1	1	$3.96 \cdot 10^{-11}$	6.36	2.46
703700	11.6	1	$3.46 \cdot 10^{-20}$	8.37	2.38
716463	8.4	1	$2.63 \cdot 10^{-5}$	6.22	3.56
717200	6.1	1	$1.99 \cdot 10^{-14}$	6.58	2.23
723600	11.0	1	$3.49 \cdot 10^{-13}$	7.09	2.51
725750	5.4	1	$4.35 \cdot 10^{-3}$	4.49	3.48
748500	7.2	1	$1.11 \cdot 10^{-6}$	5.05	2.56
761440	7.7	1	$1.46 \cdot 10^{-20}$	7.34	2.07
762620	15.1	1	$2.55 \cdot 10^{-21}$	8.76	2.42
766400	8.1	1	$4.06 \cdot 10^{-8}$	4.88	2.23
781900	13.7	1	$4.74 \cdot 10^{-6}$	5.48	2.93
802000	18.7	1	$1.59 \cdot 10^{-10}$	6.65	2.65
802200	12.5	1	$3.27 \cdot 10^{-23}$	8.40	2.23
803200	10.4	1	$2.03 \cdot 10^{-17}$	7.25	2.23
804010	12.2	1	$7.30 \cdot 10^{-14}$	7.26	2.51
804610	14.7	1	$3.41 \cdot 10^{-23}$	9.43	2.50
823210	9.1	1	$7.10 \cdot 10^{-8}$	5.93	2.75
823212	5.6	0	$9.37 \cdot 10^{-2}$	3.45	3.79
840400	9.9	1	$1.01 \cdot 10^{-18}$	7.53	2.23
843002	6.8	1	$7.20 \cdot 10^{-6}$	6.41	3.48

Table 1: Nonparametric Estimates: Kurtosis and Kolmogorov-Smirnov Test.
Column (2): Kurtosis measurements of the returns with over 1,000 trading days (Jan 1988 - Sep 2002).
Columns (3) - (6): Kolmogorov-Smirnov test results. H=0: normal hypothesis not rejected. H=1: normal hypothesis rejected. P is the significance level, $KSSTAT$ is the value of the KS statistic, and CV is the critical value.

WKN	$\hat{\alpha}_{Hill}$ 15%	$\hat{\alpha}_{Hill}$ 10%	$\hat{\alpha}_{Hill}$ 5%	$\hat{\alpha}_{Hill}$ 2.5%	WKN	$\hat{\alpha}_{Hill}$ 15%	$\hat{\alpha}_{Hill}$ 10%	$\hat{\alpha}_{Hill}$ 5%	$\hat{\alpha}_{Hill}$ 2.5%	WKN	$\hat{\alpha}_{Hill}$ 15%	$\hat{\alpha}_{Hill}$ 10%	$\hat{\alpha}_{Hill}$ 5%	$\hat{\alpha}_{Hill}$ 2.5%
500340	1.8616	2.4965	3.3593	4.7388	648300	1.8024	2.3913	2.7990	3.6715	766400	2.1288	2.3991	2.8568	3.6100
	-0.1481	-0.2447	-0.4746	-0.9842		-0.0768	-0.1250	-0.2080	-0.3913		-0.0907	-0.1254	-0.2123	-0.3847
	(1.5749;2.1483)	(2.257;2.9673)	(2.4633;4.2552)	(2.9513;6.5262)		(1.6525;1.9524)	(2.1477;2.6350)	(2.3957;3.2024)	(2.9212;4.4217)		(1.9517;2.3059)	(2.1546;2.6435)	(2.4451;3.2684)	(2.8724;4.3477)
515100	2.2716	2.5888	3.2018	3.6164	656000	2.0611	2.3120	2.6687	3.1270	781900	2.1301	2.4259	3.6467	4.9462
	-0.0968	-0.1353	-0.2380	-0.3854		-0.0988	-0.1362	-0.2239	-0.3763		-0.1196	-0.1674	-0.3592	-0.7059
	(2.826;2.4606)	(2.3250;2.8526)	(2.7404;3.6632)	(2.8774;4.3554)		(1.8683;2.2539)	(2.0468;2.5772)	(2.2358;3.1016)	(2.4097;3.8444)		(1.8971;2.3631)	(2.1009;2.7509)	(2.9557;4.3377)	(3.6145.6.2778)
519000	1.8129	2.5960	2.7820	2.9701	660200	1.6200	1.8110	2.1767	2.2613	802000	2.0424	2.2791	2.7775	3.0806
	-0.0772	-0.1077	-0.2068	-0.3165		-0.1070	-0.1474	-0.2547	-0.3871		-0.1037	-0.1422	-0.2474	-0.3942
	(1.6620;1.9637)	(1.8498;2.2695)	(2.3811;3.1829)	(2.3632;3.5770)		(1.4120;1.8280)	(1.5259;2.0961)	(1.6905;2.6629)	(1.5423;2.9802)		(1.8402;2.2446)	(2.0026;2.5556)	(2.3000;3.2549)	(2.3317;3.8295)
543900	2.1379	2.5170	3.7595	3.8560	695200	2.0248	2.3918	3.0154	3.2034	802200	1.8774	2.1146	2.7013	3.2240
	-0.1190	-0.1720	-0.3668	-0.5448		-0.0954	-0.1383	-0.2487	-0.3800		-0.0800	-0.1105	-0.2008	-0.3436
	(1.9062;2.3697)	(2.1829;2.8511)	(3.538;4.4653)	(2.8276;4.8845)		(1.8385;2.2110)	(2.1225;2.6611)	(2.5344;3.4963)	(2.4785;3.9284)		(1.7212;2.0336)	(1.8992;2.3301)	(2.3120;3.0905)	(2.5652;3.8828)
550000	2.8910	2.4401	2.9580	3.4210	703700	1.8443	2.2362	2.5567	3.0567	803200	2.0268	2.3021	2.9800	3.5650
	-0.1039	-0.1490	-0.2574	-0.4274		-0.0841	-0.1252	-0.2040	-0.3505		-0.0863	-0.1203	-0.2215	-0.3799
	(1.8864;2.2918)	(2.1501;2.7300)	(2.4609;3.4552)	(2.6079;4.2341)		(1.6801;2.0084)	(1.9924;2.4801)	(2.1618;2.9517)	(2.3869;3.7265)		(1.8582;2.1954)	(2.0675;2.5367)	(2.5506;3.4094)	(2.8365;4.2934)
550700	1.8320	2.1401	2.9854	3.2290	716463	2.2053	2.7453	2.9038	3.2880	804010	1.9459	2.2807	2.8516	2.9843
	-0.1085	-0.1557	-0.3112	-0.4862		-0.1515	-0.2320	-0.3520	-0.5800		-0.0935	-0.1346	-0.2401	-0.3617
	(1.6208;2.0433)	(1.8382;2.4420)	(2.3882;3.5826)	(2.3155;4.1425)		(1.9112;2.4994)	(2.2969;3.1936)	(2.2331;3.5745)	(2.2139;4.3620)		(1.7635;2.1284)	(2.0186;2.5427)	(2.3874;3.3157)	(2.2949;3.6736)
551200	1.8241	2.2767	2.8059	3.6056	717200	1.9554	2.3835	3.1413	3.2798	804610	1.8021	2.2054	2.7932	3.4286
	-0.0899	-0.1378	-0.2424	-0.4470		-0.0833	-0.1246	-0.2335	-0.3495		-0.0863	-0.1297	-0.2344	-0.4125
	(1.6488;1.9994)	(2.85;2.5448)	(2.3377;3.2740)	(2.7549;4.4564)		(1.7927;2.1181)	(2.1407;2.6264)	(2.6886;3.5939)	(2.6096;3.9500)		(1.6337;1.9705)	(1.9529;2.4580)	(2.3401;3.2462)	(2.6421;4.2151)
575200	2.1572	2.5713	3.1801	3.3505	723600	1.9545	2.2557	2.5070	3.0554	823210	1.9243	2.5392	3.2903	4.5295
	-0.0919	-0.1344	-0.2364	-0.3571		-0.0940	-0.1334	-0.2111	-0.3703		-0.1016	-0.1646	-0.3041	-0.6048
	(1.9777;2.3367)	(2.3093;2.8333)	(2.7219;3.6384)	(2.6659;4.0351)		(1.7710;2.1379)	(1.9961;2.5154)	(2.0990;2.9151)	(2.3497;3.7612)		(1.7264;2.1223)	(2.2193;2.8591)	(2.7040;3.8766)	(3.3834.5.6756)
575800	2.1258	2.2923	2.8145	3.5725	725750	2.4456	2.8363	3.2696	3.8904	823212	2.3372	2.6067	3.6152	4.6111
	-0.1018	-0.1348	-0.2362	-0.4299		-0.1641	-0.2339	-0.3878	-0.6759		-0.1709	-0.2350	-0.4703	-0.8848
	(1.9272;2.3244)	(2.298;2.5548)	(2.3579;3.2710)	(2.7530;4.3920)		(2.1268;2.7645)	(2.3840;3.2887)	(2.5296;4.0095)	(2.6368.5.1439)		(2.0057;2.6686)	(2.1534;3.0601)	(2.7225;4.5079)	(2.9879.6.2343)
593700	2.1192	2.4859	2.9345	3.2762	748500	2.3886	2.6807	2.9423	3.3119	840400	2.0676	2.4309	2.7247	2.8750
	-0.0903	-0.1299	-0.2181	-0.3491		-0.1170	-0.1611	-0.2523	-0.4074		-0.0881	-0.1271	-0.2025	-0.3064
	(1.9429;2.2955)	(2.2326;2.7392)	(2.5116;3.3573)	(2.6067;3.9456)		(2.1604;2.6167)	(2.3672;2.9941)	(2.4549;3.4297)	(2.5360;4.0877)		(1.8955;2.2396)	(2.1832;2.6786)	(2.3321;3.1173)	(2.2875;3.4625)
604843	1.8024	2.1875	2.9855	3.3556	761440	1.8189	2.2911	2.5757	3.1915	843002	2.2079	2.6067	3.1357	3.9076
	-0.0768	-0.1143	-0.2219	-0.3576		-0.0720	-0.1113	-0.1777	-0.3144		-0.1478	-0.2150	-0.3720	-0.6789
	(1.6524;1.9524)	(1.9647;2.4104)	(2.5553;3.4158)	(2.6699;4.0413)		(1.6782;1.9596)	(2.0741;2.5082)	(2.2306;2.9208)	(2.5868;3.7962)		(1.9207;2.4951)	(2.1910;3.0225)	(2.4260;3.8454)	(2.6485;5.1667)
627500	1.9455	2.4141	3.8200	3.4590	762620	1.7494	2.0657	2.3389	2.7752					
	-0.0876	-0.1335	-0.2429	-0.3915		-0.0812	-0.1177	-0.1897	-0.3225					
	(1.7745;2.1166)	(2.1540;2.6742)	(2.6117;3.5522)	(2.7103;4.2077)		(1.5909;1.9079)	(1.8365;2.2949)	(1.9719;2.7060)	(2.1593;3.3910)					

Table 2: Hill estimates of log-returns (with over 1,000 observations). Standard errors and 95% confidence bounds in parentheses, respectively.

(1) WKN	(2) α	(3) γ	(4) β	(5) μ	(6) α	(7) γ	(8) β	(9) μ	(10) α	(11) γ	(12) β	(13) μ
	MLE				MCE				CFE			
500340	1.6182	0.0145	0.0817	-0.0010	1.4871	0.0137	0.0250	0.0003	1.7577	0.0150	0.4577	0.0011
515100	1.6811	0.0091	-0.0590	0.0007	1.5301	0.0086	-0.0556	0.0001	1.7849	0.0093	0.0185	0.0007
519000	1.5444	0.0098	0.0470	0.0003	1.4471	0.0094	0.0687	0.0006	1.6623	0.0101	0.1269	0.0010
543900	1.7370	0.0100	0.1411	-0.0002	1.5511	0.0094	0.0810	0.0005	1.8236	0.0102	0.4648	0.0009
550000	1.7014	0.0093	0.0951	0.0002	1.5907	0.0088	0.1129	0.0005	1.7796	0.0094	0.1920	0.0007
550700	1.6590	0.0105	0.1502	-0.0005	1.5079	0.0098	0.1109	0.0008	1.7347	0.0107	0.2990	0.0008
551200	1.6113	0.0089	0.0826	0.0001	1.5016	0.0086	0.0760	0.0005	1.7085	0.0092	0.2673	0.0011
575200	1.6770	0.0093	-0.1332	0.0008	1.5225	0.0087	-0.0740	0	1.7673	0.0095	-0.1633	0.0002
575800	1.6429	0.0089	0.0983	0.0002	1.5758	0.0087	0.0732	0.0007	1.7482	0.0092	0.2151	0.0010
578580	1.7085	0.0139	-0.1553	-0.0003	1.5886	0.0130	-0.1739	-0.0012	1.6986	0.0051	-0.0032	0.0024
593700	1.7214	0.0109	0.0195	0.0002	1.5727	0.0103	-0.0012	0	1.7986	0.0111	-0.0143	0.0002
604843	1.9117	0.0288	0.3086	-0.0054	1.4607	0.0080	0.0830	0.0006	1.7145	0.0088	0.1391	0.0006
627500	1.6642	0.0098	0.1014	-0.0001	1.5298	0.0094	0.0829	0.0005	1.7533	0.0101	0.2704	0.0008
648300	1.5633	0.0079	0.0259	0.0002	1.44	0.0074	0.0010	0	1.6877	0.0082	0.1071	0.0006
656000	1.6542	0.0103	0.0302	0.0011	1.5759	0.0100	0.0488	0.0011	1.7342	0.0106	0.0002	0.0011
660200	1.4605	0.0104	0.0024	-0.0006	1.3975	0.0100	-0.0444	-0.0005	1.5377	0.0107	0.0859	0.0001
695200	1.6738	0.0102	-0.0125	0.0001	1.535	0.0097	-0.0149	-0.0001	1.7740	0.0105	-0.0715	-0.0001
703700	1.5466	0.0081	0.1107	0.0000	1.4195	0.0076	0.0434	0.0003	1.6731	0.0085	0.1602	0.0007
716463	1.6716	0.0181	-0.0954	0.0020	1.5149	0.0168	-0.1392	0	1.7723	0.0185	-0.1897	0.0008
717200	1.6366	0.0089	0.0408	0.0005	1.4674	0.0083	0.0211	0.0004	1.7561	0.0092	0.0657	0.0007
723600	1.6416	0.0080	0.0370	0.0005	1.5781	0.0077	0.0421	0.0006	1.7118	0.0081	0.0326	0.0006
725750	1.8217	0.0143	-0.2349	0.0006	1.6935	0.0136	-0.1593	-0.0008	1.8623	0.0143	-0.2041	-0.0001
748500	1.7807	0.0101	0.0884	0.0004	1.6827	0.0096	0.1061	0.0004	1.8327	0.0102	0.1927	0.0008
761440	1.5851	0.0088	-0.0260	0.0004	1.4715	0.0084	-0.0566	-0.0001	1.7084	0.0091	-0.0786	0.0001
762620	1.5657	0.0083	0.0817	0.0002	1.4622	0.0078	0.0764	0.0005	1.6652	0.0086	0.0322	0.0004
766400	1.7218	0.0117	-0.0793	0.0008	1.6684	0.0114	-0.0574	0.0003	1.8055	0.0119	-0.1804	0.0002
781900	1.7502	0.0093	0.1419	0.0000	1.5893	0.0089	0.1238	0.0006	1.8280	0.0095	0.4307	0.0008
802000	1.6891	0.0079	0.1689	0.0001	1.5673	0.0074	0.1477	0.0006	1.7618	0.0080	0.2513	0.0008
802200	1.5478	0.0099	-0.0058	0.0003	1.4277	0.0093	0.0002	0	1.6495	0.0102	0.0742	0.0006
803200	1.6271	0.0092	-0.0135	0.0002	1.4822	0.0085	0.0228	0.0002	1.7225	0.0094	0.0272	0.0003
804010	1.6105	0.0085	0.0372	0.0005	1.5283	0.0083	0.0488	0.0006	1.7137	0.0088	0.1020	0.0009
804610	1.5021	0.0080	0.1207	0.0002	1.4138	0.0076	0.1188	0.0009	1.6198	0.0083	0.2581	0.0015
823210	1.7218	0.0109	0.1965	-0.0002	1.569	0.0102	0.1777	0.0010	1.8090	0.0111	0.3514	0.0009
823212	1.8506	0.0159	-0.0166	-0.0002	1.8019	0.0157	0.1195	0.0004	1.8828	0.0159	0.3694	0.0003
840400	1.6281	0.0098	-0.0338	0.0004	1.5274	0.0094	-0.0376	-0.0002	1.7120	0.0100	-0.0236	0.0003
843002	1.6863	0.0146	-0.0223	0.0004	1.5244	0.0136	-0.0348	-0.0003	1.7871	0.0148	0.0721	0.0006

Table 3: Parametric estimation results: Maximum likelihood estimate MLE (columns 2-5), McChulloch quantile estimation results (MCE) (columns 6-9), and characteristic function based estimation results (CFE) (columns 10-13).

(1) WKN	(2) AD normal	(3) AD skew-t	(4) AD stable	(5) WKN	(6) AD normal	(7) AD skew-t	(8) AD stable
500340	$2.91 \cdot 10^{01}$ (right end)	$6.21 \cdot 10^{-02}$ (right end)	$1.48 \cdot 10^{-01}$ (left end)	717200	$8.02 \cdot 10^{02}$ (left end)	$6.00 \cdot 10^{-02}$ (median)	$6.19 \cdot 10^{-02}$ (right end)
515100	$7.24 \cdot 10^{47}$ (left end)	$6.49 \cdot 10^{-01}$ (left end)	$5.59 \cdot 10^{-02}$ (right end)	723600	$3.32 \cdot 10^{10}$ (left end)	$3.37 \cdot 10^{-01}$ (left end)	$7.43 \cdot 10^{-02}$ (right end)
519000	$8.50 \cdot 10^{10}$ (right end)	$1.13 \cdot 10^{-01}$ (left end)	$8.59 \cdot 10^{-02}$ (right end)	725750	$3.28 \cdot 10^{00}$ (left end)	$4.62 \cdot 10^{-02}$ (left end)	$5.01 \cdot 10^{-02}$ (right end)
543900	$1.71 \cdot 10^{02}$ (left end)	$5.06 \cdot 10^{-02}$ (right end)	$7.81 \cdot 10^{-02}$ (left end)	748500	$6.28 \cdot 10^{04}$ (right end)	$1.21 \cdot 10^{-01}$ (left end)	$4.62 \cdot 10^{-02}$ (left quartile)
550000	$1.67 \cdot 10^{02}$ (left end)	$5.04 \cdot 10^{-02}$ (right end)	$7.71 \cdot 10^{-02}$ (left end)	761440	$9.24 \cdot 10^{13}$ (right end)	$8.65 \cdot 10^{-02}$ (left end)	$6.62 \cdot 10^{-02}$ (left end)
550700	$7.00 \cdot 10^{22}$ (right end)	$3.87 \cdot 10^{-01}$ (left end)	$9.12 \cdot 10^{-02}$ (median)	762620	∞ (right end)	$1.36 \cdot 10^{-01}$ (left end)	$6.91 \cdot 10^{-02}$ (left end)
551200	$5.45 \cdot 10^{19}$ (right end)	$1.26 \cdot 10^{-01}$ (left end)	$5.95 \cdot 10^{-02}$ (right end)	766400	$8.57 \cdot 10^{11}$ (left end)	$3.63 \cdot 10^{-01}$ (left end)	$5.98 \cdot 10^{-02}$ (median)
575200	$1.17 \cdot 10^{19}$ (left end)	$2.39 \cdot 10^{-01}$ (left end)	$6.36 \cdot 10^{-02}$ (median/left)	781900	$1.26 \cdot 10^{08}$ (left end)	$1.23 \cdot 10^{-01}$ (left end)	$8.19 \cdot 10^{-02}$ (right quartile)
575800	$2.07 \cdot 10^{09}$ (left end)	$2.46 \cdot 10^{-01}$ (left end)	$7.21 \cdot 10^{-02}$ (right end)	802000	∞ (right end)	$2.04 \cdot 10^{-01}$ (left end)	$5.01 \cdot 10^{-02}$ (right end)
593700	$1.39 \cdot 10^{18}$ (right end)	$2.98 \cdot 10^{-01}$ (left end)	$4.87 \cdot 10^{-02}$ (right end)	802200	$1.27 \cdot 10^{09}$ (right end)	$1.12 \cdot 10^{-01}$ (left end)	$8.69 \cdot 10^{-02}$ (left end)
604843	$1.67 \cdot 10^{24}$ (left end)	$2.68 \cdot 10^{-01}$ (left end)	$6.99 \cdot 10^{-02}$ (right end)	803200	$1.99 \cdot 10^{16}$ (left end)	$1.57 \cdot 10^{-01}$ (left end)	$6.80 \cdot 10^{-02}$ (right end)
627500	$4.30 \cdot 10^{13}$ (left end)	$1.35 \cdot 10^{-01}$ (left end)	$7.45 \cdot 10^{-02}$ (right end)	804010	$1.73 \cdot 10^{12}$ (left end)	$3.48 \cdot 10^{-01}$ (left end)	$7.58 \cdot 10^{-02}$ (right end)
648300	$3.63 \cdot 10^{14}$ (left end)	$1.46 \cdot 10^{-01}$ (left end)	$6.73 \cdot 10^{-02}$ (right end)	804610	$7.36 \cdot 10^{13}$ (left end)	$1.92 \cdot 10^{-01}$ (left end)	$9.88 \cdot 10^{-02}$ (right end)
656000	$1.33 \cdot 10^{14}$ (left end)	$2.78 \cdot 10^{-01}$ (left end)	$6.97 \cdot 10^{-02}$ (median)	823210	$1.26 \cdot 10^{07}$ (right end)	$9.74 \cdot 10^{-02}$ (median)	$6.44 \cdot 10^{-02}$ (right quartile)
660200	$1.33 \cdot 10^{14}$ (left end)	$2.78 \cdot 10^{-01}$ (left end)	$6.97 \cdot 10^{-02}$ (median)	823212	$2.35 \cdot 10^{01}$ (left end)	$1.28 \cdot 10^{-01}$ (left end)	$7.43 \cdot 10^{-02}$ (left end)
695200	$6.08 \cdot 10^{07}$ (right end)	$1.35 \cdot 10^{-01}$ (left end)	$5.35 \cdot 10^{-02}$ (right end)	840400	∞ (left end)	$7.97 \cdot 10^{-02}$ (left end)	$6.05 \cdot 10^{-02}$ (left end)
703700	∞ (left/right end)	$1.19 \cdot 10^{-01}$ (left end)	$6.37 \cdot 10^{-02}$ (left/right end)	843002	$3.61 \cdot 10^{04}$ (left end)	$1.34 \cdot 10^{-01}$ (right end)	$7.53 \cdot 10^{-02}$ (left end)
716463	$1.33 \cdot 10^{11}$ (left end)	$3.15 \cdot 10^{-01}$ (left end)	$1.04 \cdot 10^{-01}$ (median)				

Table 4: Values of AD statistic of $S^{\delta}_{\alpha,\beta}$GARCH and $t^{\delta}_{\nu,\lambda}$GARCH innovations, respectively, and positions of the AD statistic in parentheses.