Construction of probability metrics on classes of investors

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Abstract

We introduce functionals with metric properties defined on classes of investors allowing inference about relations between prospects. In this context, we introduce the class of investors with balanced views. Our approach is consistent with Cumulative prospect theory.

Key words: probability metrics, cumulative prospect theory

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Introduction

Expected utility theory (EUT) is an accepted model describing choice under uncertainty. However, a number of alternative behavioral models for human choice have been proposed. One of them is Cumulative Prospect Theory (CPT) as proposed by Tversky and Kahneman (1992). CPT is built upon the following main observations. First, investors usually think about possible outcomes relative to a certain reference point rather than the final outcome. This is referred to as the framing effect by behavioral finance theorists. Second, investors have different attitude towards gains (outcomes which are larger than the reference point) and losses (outcomes which are less than the reference point), referred to by behavioral finance theorists as loss aversion. Finally, investors tend to overweight extreme events and underweight events with higher probability.

CPT arises as an alternative theory to EUT on the basis of the observations outlined above in which the utility function is replaced by a value function and the cumulative probabilities are replaced by weighted cumulative probabilities. The value function, $v(x)$, assigns values to the possible outcomes. It is non-decreasing and $v(0) = 0$ since the outcome equal to the reference point brings no value to the individual. Different functional forms for $v(x)$ have been suggested. It is often assumed that the $v(x)$ has an S-shaped form, i.e. $v(x)$ is convex for $x < 0$ and it is concave for $x > 0$, see Kahneman and Tversky (1979). From a financial viewpoint, the framing effect means that the value function can be constructed for returns rather than wealth as in EUT. See Starmer (2000) for a description of other non-EUT frameworks.

The weighted cumulative probabilities are usually modeled as transformations of the cumulative distribution function (c.d.f.) of the prospect $F_X(x) = P(X \leq x)$ and the tail $1 - F_X(x) = P(X > x)$ depending on whether $x < 0$ or $x > 0$, respectively. The transformation for losses is denoted by $w^-(p)$ and the one for profits by $w^+(p)$. Both weighting functions are non-decreasing and satisfy the following conditions

$$w^-(0) = w^+(0) = 0 \text{ and } w^-(1) = w^+(1) = 1.$$

Empirical studies suggest that the general shape of the weighting functions is inverse S-shaped, see for example Tversky and Kahneman (1992).

According to CPT, individuals make a choice between two risky prospects X and Y by computing the subjective expected values according to

$$V(X) = \int_{-\infty}^{0} v(x) d[w^{-}(F_X(x))] + \int_{0}^{\infty} v(x) d[-w^{+}(1 - F_X(x))] \qquad (1)$$

and then compare $V(X)$ and $V(Y)$, see for example Baucells and Heukamp (2006). If $V(X) \geq V(Y)$, then Y is not preferred to X. If $V(X) = V(Y)$, then the individual is indifferent. Note that if the individuals do not weight the cumulative probabilities, then $V(X)$ reduces to $V(X) = Ev(X)$.

The expression (1) implies that we can map all individuals to pairs (v, w) where v is the corresponding value function and w is a shorthand for both w^{-} and w^{+}. Consider a set of individuals represented by $(v_j, w_j),\ j \in \mathcal{J}$ and two prospects X and Y.

The range of questions that we discuss in this paper is whether it is possible to draw a conclusion about the dissimilarity between the two random variables X and Y by looking only at how a set of individuals value the two prospects. If $F_X(x) = F_Y(x),\ \forall x \in \mathbb{R}$, then the individuals are indifferent between the two prospects. The converse does not necessarily follow because it depends on how many and diverse they are. As an extreme example, if there is only one investor then $V_1(X) = V_1(Y)$ implies equality of certain characteristics of X and Y but not the entire c.d.f.s. In this paper, we provide sufficient conditions for v_j that guarantee coincidence of the corresponding c.d.f.s on condition that all individuals in $\mathcal{J}$ are indifferent between X and Y. We study this problem through a functional on the set of investors which we also extend to a functional consistent with first-order stochastic dominance (FSD) order. Finally, we introduce the class of investors with balanced views which ensures the functionals are bounded and is also large enough to allow one to infer the relation between X and Y. We regard the considerations in this paper as a step towards defining measures of dispersion and, eventually, risk measures ideal for a particular class of investors, see the related discussions in Stoyanov et al. (2008) and Rachev et al. (2008).

1 Metrics construction

We begin by introducing some notation. The class of bounded S-shaped value functions we denote by $\mathcal{S}$. The elements of $\mathcal{S}$ are bounded real-valued functions $v(x) : \mathbb{R} \to \mathbb{R}$ with the following property,

$$v(x) = \begin{cases} v^{-}(x), & x < 0 \\ 0, & x = 0 \\ v^{+}(x), & x > 0 \end{cases}$$

where $v^-(x) < 0$ is a monotonically increasing convex function and $v^+(x) > 0$ is a monotonically increasing concave function.

Suppose that all investors which we consider are indifferent between X and Y, $V_j(X) = V_j(Y)$, for all $j \in \mathcal{J}$. Note that $\mathcal{J}$ is a general set, not necessarily countable. In order to study the implications of this assumption on the distribution functions of X and Y, we consider the functional,

$$\zeta_{\mathcal{J}}(X,Y) = \sup_{j\in\mathcal{J}} |V_j(X) - V_j(Y)|\,, \text{ where} \tag{2}$$

$$\begin{aligned} V_j(X) - V_j(Y) = &\int_{-\infty}^{0} v_j(x) d[w_j^-(F_X(x)) - w_j^-(F_Y(x))] \\ &+ \int_{0}^{\infty} v_j(x) d[w_j^+(1-F_Y(x)) - w_j^+(1-F_X(x))]. \end{aligned}$$

Note that in the case of no subjective weighting, this expression reduces to

$$V_j(X) - V_j(Y) = \int_{-\infty}^{\infty} v_j(x) d(F_X(x) - F_Y(x)).$$

The functional $\zeta_{\mathcal{J}}(X,Y)$ is the largest difference between the values assigned by the investors to X and Y running through all investors. If the functional in (2) equals zero, then this means that all investors that we consider are indifferent between X and Y. In fact, $\zeta_{\mathcal{J}}(X,Y)$ has metric properties, see Rachev (1991). More precisely, it is a probability semimetric. Table 1 provides definitions of the key terms. In particular, if $X \overset{d}{=} Y$, then $\zeta_{\mathcal{J}}(X,Y) = 0$ although the converse may not hold. Therefore, the fact that $\zeta_{\mathcal{J}}(X,Y) = 0$ does not necessarily imply equality in distribution between X and Y as it depends on how rich the set $\mathcal{J}$ is. The next theorem establishes a sufficient condition for the converse relationship.

Theorem 1. *Suppose that the set $\mathcal{V}_{\mathcal{J}} = \{v_j, j \in \mathcal{J}\} \subseteq \mathcal{S}$ contains the functions*

$$v_{x_0,n}^-(x) = \begin{cases} -1/n, & x < x_0 \\ x_0 - x - 1/n, & x \in [x_0, x_0 + 1/n) \\ 0, & x \geq x_0 + 1/n \end{cases} \tag{3}$$

where $n = 1, 2, ...$ and $x_0 + 1/n \leq 0$ and

$$v_{x_0,n}^+(x) = \begin{cases} 0, & x < x_0 - 1/n \\ x - x_0 + 1/n, & x \in [x_0 - 1/n, x_0) \\ 1/n, & x \geq x_0 \end{cases} \tag{4}$$

where $n = 1, 2, ...$ and $x_0 - 1/n \geq 0$. Suppose also that the weighting functions w^- and w^+ are continuous. Then, $\zeta_{\mathcal{J}}(X, Y)$ is a simple probability metric which means that $\zeta_{\mathcal{J}}(X, Y) = 0 \Longleftrightarrow X \stackrel{d}{=} Y$.

Proof. First, since $\mathcal{V}_{\mathcal{J}} \subseteq \mathcal{S}$, then $\zeta_{\mathcal{J}}(X, Y) < \infty$ because the class $\mathcal{S}$ contains by construction bounded functions. In order to prove the claim, it suffices to demonstrate that $\zeta_{\mathcal{F}}(X, Y) = 0$ implies $F_X(y) = F_Y(y), \forall y \in \mathbb{R}$. We consider two cases and take advantage of the inequalities

$$V(X) \leq \zeta_{\mathcal{J}}(X, Y) + V(Y) \tag{5}$$

$$\begin{aligned} -w^-(F_X(x_0 + 1/n)) &\leq n \int_{-\infty}^0 v^-_{x_0,n}(x) dw^-(F_X(x)) < -w^-(F_X(x_0)) \\ w^+(1 - F_X(x_0)) &< n \int_{-\infty}^0 v^+_{x_0,n}(x) d[-w^+(1 - F_X(x))] \\ &\leq w^+(1 - F_X(x_0 - 1/n)) \end{aligned} \tag{6}$$

Case I, $y < 0$. Assume $\zeta_{\mathcal{J}}(X, Y) = 0$, apply (5) for $v(x) = n.v^-_{x_0,n}(x)$, and use the first chain of inequalities in (6),

$$\begin{aligned} -w^-(F_X(y + 1/n)) &\leq n \int_{-\infty}^0 v^-_{y,n}(x) dw^-(F_X(x)) \\ &\leq n \int_{-\infty}^0 v^-_{y,n}(x) dw^-(F_Y(x)) < -w^-(F_Y(y)), \end{aligned}$$

and because of the symmetry of $\zeta_{\mathcal{J}}(X, Y)$, $w^-(F_Y(x_0 + 1/n)) > w^-(F_X(y))$. At the limit, as $n \to \infty$, we obtain $w^-(F_X(y)) = w^-((F_Y(y))$ and because of the monotonicity and continuity of w^-, it follows that $F_X(y) = F_Y(y)$.

Case II, $y > 0$. By the same reasoning as in Case I, but using the second chain of inequalities in (6), we obtain $F_X(y) = F_Y(y)$.

Combining Case I and Case II, we conclude that $P(X \in A) = P(Y \in A)$ on all events A such that $0 \notin A$. Since the distribution functions are continuous from the right, computing the limit $\lim_{y \to 0^+} F_X(y) = \lim_{y \to 0^+} F_Y(y)$ we obtain $F_X(y) = F_Y(y),\ y \in \mathbb{R}$. □

This result implies that if the set of investors is so large that it contains the value functions defined in (3) and (4), then $\zeta_{\mathcal{J}}(X, Y) = 0$ indicates that the c.d.f.s of X and Y coincide. Note that the particular form of the weighting functions is immaterial. The only properties needed are that they are non-decreasing and continuous.

The reasoning outlined above can be used to construct a functional consistent with the FSD order. Consider

$$\zeta^+_{\mathcal{J}}(X,Y) = \sup_{j\in\mathcal{J}} (V_j(X) - V_j(Y))_+ \,, \tag{7}$$

where $(x)_+ = \max(x, 0)$. The interpretation of (7) is as follows. The distance between X and Y equals the largest difference $V_j(X) - V_j(Y)$ running through all investors who do not prefer Y to X. In this case, the condition $\zeta^+_{\mathcal{J}}(X,Y) = 0$ implies that all investors prefer Y to X because in this case $V_j(X) \leq V_j(Y),\ \forall j \in \mathcal{J}$.

Theorem 2. *$\zeta^+_{\mathcal{J}}(X,Y)$ is a probability quasi-semimetric and if $F_Y(x) \leq F_X(x), \forall x \in \mathbb{R}$, then $\zeta^+_{\mathcal{J}}(X,Y) = 0$. Furthermore, if the set of value functions contains the set defined in* (3) *and* (4), *and the weighting functions are continuous, then $\zeta^+_{\mathcal{J}}(X,Y) = 0$ implies $F_Y(x) \leq F_X(x), \forall x \in \mathbb{R}$.*

Proof. First, we demonstrate that $\zeta^+_{\mathcal{J}}(X,Y)$ is a probability semimetric by checking the defining axioms, see Rachev (1991): (i) if $X \overset{d}{=} Y$, then $\zeta^+_{\mathcal{J}}(X,Y) = 0$ due to the monotonicity and continuity of the weighting functions, (ii) the triangle inequality holds due to the properties of the $(x)_+$ function, and (iii) $\zeta^+_{\mathcal{J}}(X,Y) \neq \zeta^+_{\mathcal{J}}(Y,X)$.

Next, consider

$$\begin{aligned} V_j(X) - V_j(Y) = &\int_{-\infty}^{0} v_j(x) d[w_j^-(F_X(x)) - w_j^-(F_Y(x))] \\ &+ \int_0^{\infty} v_j(x) d[w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))]. \end{aligned}$$

Due to the assumed boundedness of the integrand, the properties of the weighting functions, and the fact that $v(0) = 0$, integration by parts leads to

$$\begin{aligned} V_j(X) - V_j(Y) = &- \int_{-\infty}^{0} [w_j^-(F_X(x)) - w_j^-(F_Y(x))] dv_j(x) \\ &- \int_0^{\infty} [w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x))] dv_j(x). \end{aligned}$$

By assumption $F_X(x) \geq F_Y(x)$ and $v_j(x)$, $w_j^-(x)$, and $w_j^+(x)$ are non-decreasing. Therefore, the integrand is non-positive on the entire real line and the increments of $v(x)$ are non-negative. As a result, $V_j(X) - V_j(Y) \leq 0, \forall j \in \mathcal{J}$ and $\zeta^+_{\mathcal{J}}(X,Y) = 0$.

The proof of the last claim repeats the arguments in Cases I and II of Theorem 1. The inequality between the distribution functions appears as a result of the lack of symmetry of $\zeta^+_{\mathcal{J}}(X,Y)$.

$\square$

Similarly to $\zeta_{\mathcal{J}}(X,Y)$, if the class $\mathcal{J}$ is not rich enough, then the condition $\zeta^+_{\mathcal{J}}(X,Y) = 0$ does not imply inequality between the distribution functions but only between certain characteristics of X and Y.

2 Investors with balanced views

One last condition we need to check is whether we can choose a class of investors which is sufficiently large and at the same time (2) and (7) are bounded. Otherwise, if (2) and (7) take only two values — zero and infinity, the construct is meaningless. In this section, we derive upper bounds on $\zeta_{\mathcal{J}}(X,Y)$ and $\zeta^+_{\mathcal{J}}(X,Y)$ introducing additional assumptions which concern the rate of change of $v_j(x)$ and the weighting functions. From a mathematical viewpoint, they can be regarded as smoothness assumptions but because of the particular relationship between $v_j(x)$ and $w(x)$, we call the set $\mathcal{J}$ investors with balanced views. The main result is provided below.

Theorem 3. *Consider the set $\mathcal{V}_{\mathcal{J}}$ of value functions $v_j \in \mathcal{S}$ satisfying the Lipschitz condition $|v_j(x) - v_j(y)| \leq K_{v_j}|x-y|$ and the weighting functions satisfy the Lipschitz conditions $|w^-_j(x) - w^-_j(y)| \leq K_{w_j}|x-y|$ and $|w^+_j(x) - w^+_j(y)| \leq K_{w_j}|x-y|$ where $0 < K_{v_j}K_{w_j} \leq 1$. The following inequalities hold*

$$\zeta_{\mathcal{J}}(X,Y) \leq \int_{\mathbb{R}} |F_X(x) - F_Y(x)|dx \tag{8}$$

$$\zeta^+_{\mathcal{J}}(X,Y) \leq \int_{\mathbb{R}} (F_Y(x) - F_X(x))_+ dx \tag{9}$$

Proof. We demonstrate directly (8). Consider the expression we used in the proof of Theorem 2,

$$\begin{aligned}|V_j(X) - V_j(Y)| = \bigg| &\int_{-\infty}^{0} [w^-_j(F_X(x)) - w^-_j(F_Y(x))]dv_j(x) \\ &+ \int_0^{\infty} [w^+_j(1-F_Y(x)) - w^+_j(1-F_X(x))]dv_j(x)\bigg|\end{aligned}$$

Then $|V_j(X) - V_j(Y)| \leq \int_{-\infty}^{0} \left| w_j^-(F_X(x)) - w_j^-(F_Y(x)) \right| dv_j(x)$

$$
\begin{aligned}
&+ \int_{0}^{\infty} \left| w_j^+(1 - F_Y(x)) - w_j^+(1 - F_X(x)) \right| dv_j(x) \\
&\leq K_{v_j} K_{w_j} \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)|\, dx \\
&\leq \int_{-\infty}^{\infty} |F_X(x) - F_Y(x)|\, dx
\end{aligned}
$$

The inequality in (9) follows from the same arguments and taking advantage of the inequality $(w_j^{-/+}(x) - w_j^{-/+}(y))_+ \leq K_{w_j}(x - y)_+$ which holds because of the monotonic properties of the weighting functions. □

The Lipschitz conditions imply that the value function and the weighting functions do not change too quickly. For example, if we compare two outcomes x and $x + h$, $h > 0$, then the Lipschitz condition suggests that $v_j(x + h) - v(x) \leq K_{v_j} h$ which means that the difference between the assigned values by v_j of the j-th investor is bounded by $K_{v_j} h$. Likewise, we can interpret the Lipschitz condition for the weighting function.

The condition in the theorem, $0 < K_{w_j} K_{v_j} \leq 1$, means that if the value function of a given investor is changing too quickly (K_{v_j} is high), then the weighting functions of the corresponding investor should have a constant K_{w_j} bounded from above by $1/K_{v_j}$. In effect, the combined condition in the theorem means that the individuals that we consider are balanced in their views. A steeper value function should be compensated by a more flat weighting function and vice versa. If the value function and the weighting functions are differentiable, then the Lipschitz conditions translate into bounds on their first derivatives, $|dv_j(x)/dx| \leq K_{v_j}$ and $|dw^{-/+}(x)/dx| \leq K_{w_j}$.

The class of Lipschitz value functions includes (4) and (3) with a constant $K_v = 1/n \leq 1$. Thus, investors with balanced views are a sufficiently large class with suitable properties. On the basis of this class using (2) and (7), we can draw conclusions about the relation between X and Y.

3 Conclusion

We discussed the possibility of defining functionals directly on classes of investors and provided sufficient conditions ensuring we can draw decisive conclusions about relations between prospects. In particular, we considered the class of investors with balanced views.

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A mapping $\mu : \mathcal{L}\mathfrak{X}_2 \to [0, \infty]$ from the space of all joint distributions generated by the pairs X, Y from the set of all real-valued r.v.s on a given probability space is said to be

a) a probability metric if ID, SYM and TI hold,

b) a probability semimetric if $\widetilde{\text{ID}}$, SYM, TI hold,

c) a probability quasi-semimetric if $\widetilde{\text{ID}}$ and TI hold,

where

ID.	$\mu(X, Y) \geq 0$ and $\mu(X, Y) = 0,$ if and only if $X \sim Y$
$\widetilde{\text{ID}}$.	$\mu(X, Y) \geq 0$ and $\mu(X, Y) = 0,$ if $X \sim Y$
SYM.	$\mu(X, Y) = \mu(Y, X)$
TI.	$\mu(X, Y) \leq \mu(X, Z) + \mu(Z, Y)$ for any X, Y, Z

Table 1: The axiomatic definitions of probability metric functionals.