Metrization of stochastic dominance rules

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Abstract

We consider a new approach towards stochastic dominance rules. It allows measuring the degree of domination or violation of a given stochastic order and represents a way of describing stochastic orders in general. Examples are provided for the n-th order stochastic dominance and stochastic orders based on a popular risk measure.

1 Introduction

From a historical perspective, stochastic dominance (SD) rules were first introduced in relation to the normative expected utility theory describing choice under uncertainty. The notions of first-order stochastic dominance (FSD) and second-order stochastic dominance (SSD) were used to prescribe the behavior of unsatiable investors and unsatiable, risk-averse investors, respectively. Since its introduction, the significance of SD analysis has increased enormously. In portfolio theory, for example, new families of risk measures have been introduced but consistency with FSD and SSD is always sought for. In areas other than finance, it finds application in diverse fields such as economics, insurance, agriculture, and medicine. For additional information, see Levy (2006).

In this paper, we propose a new concept describing SD relations which is based on the notion of a *quasi-semidistance*. It allows measuring by how much a given prospect X dominates another prospect Y or, in case they are incomparable, a quasi-semidistance allows measuring the degree of violation of the SD rule. There is a close connection between this concept and the concept of *almost stochastic dominance* discussed in Leshno and Levy (2002) and Bali et al. (2009).

The new concept distinguishes between a few types of stochastic orders nested in each other such that a stochastic order from a given category cannot imply a stochastic order from categories in which it is nested. Here is an example. The smallest category includes stochastic orders based on certain characteristics of the underlying prospects. For example, the mean-variance order belongs to this category as it is based on inequalities between the means and the variances of the corresponding prospects. The second smallest category includes stochastic orders based on inequalities between certain transformations of the cumulative distribution functions (cdfs). Both FSD and SSD belong to this category. As a consequence, the mean-variance order can imply neither FSD nor SSD. The same holds for any mean-risk order, where risk is measured by an arbitrary risk measure.

Comparing the mean-variance, or more generally the mean-risk, approach and the SD approach, we can conclude that the former leads to optimization problems that are practical. Even though the SD approach is more general, it does not provide a method for construction of a portfolio from several individual securities, see, for example, Levy (2006). We believe that the current framework is a step towards resolving this shortcoming.

In this paper, we consider a class of stochastic dominance rules which we call *metrizable*. It includes the stochastic dominance rules commonly used in theory and practice. Our goal is to describe the metrizable stochastic orders by means of the universal Hausdorff construction which is well-known in the field of probability metrics. We provide examples for n-th order stochastic dominance, stochastic orders based on average value-at-risk (AVaR), and stochastic orders arising from classes of investors.

2 Metrization of preference relations

In this section, we introduce notation and discuss some basic terms. Let S denote the space of all combinations of goods, services and assets, which we also call *baskets*. A preference relation on S, denoted by $\preceq$, is introduced by a binary relation such that $x \preceq y$, if y is at least as preferable as x. There are a few assumptions that are usually made:

1. The binary relation is assumed *reflexive*, i.e. $x \preceq x$, for all $x \in S$.
2. The binary relation is assumed *transitive*, i.e. if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for any $x, y, z \in S$.

If $x \preceq y$ and $y \preceq x$, then we say that x and y are indistinguishable or equivalent from the standpoint of the preference order.

In this paper, we do not discuss the adequacy of the reflexivity and the transitivity assumptions. We assume that they characterize every preference relation[1] and, as a consequence, the preference relation $\preceq$ represents a pre-order defined on S.

The most direct way to describe a preference relation defined in this way is through the corresponding binary relation. However, this is not practical because we have to make a list of all pairs (x, y) such that $x \preceq y$. A generic and more practical approach to describe a preference relation is by means of a quasi-semidistance. Quasi-semidistances are introduced axiomatically. A quasi-semidistance is a function $d(x, y) : S \times S \to [0, \infty]$ satisfying the properties:

[1]For a detailed discussion of these axioms, see Anand (1995).

i. The identity property: if $x = y$, then $d(x, y) = 0$.

ii. The triangle inequality: $d(x, y) \leq K(d(x, z) + d(z, y))$ for any $x, y, z \in S$ in which $K \geq 1$.

If $K = 1$, then the quasi-semidistance turns into a quasi-semimetric.

Every quasi-semidistance defines a pre-order and, therefore, a preference relation in the following way. The basket y is at least as preferable as another basket x, $x \preceq_d y$, if $d(x, y) = 0$. It is straightforward to verify that the transitivity property of $\preceq_d$ is a consequence of the triangle inequality for $d(x, y)$ and reflexivity follows from the identity property.

Since every quasi-semidistance defines a preference relation, we can ask the converse question. The answer, however, is in the negative. Therefore, the set of all preference relations can be divided into two parts – those that arise from quasi-semidistances and those that do not arise in this fashion. The preference relations that do arise from quasi-semidistances we call *quasi-metrizable* or simply *metrizable*.

3 Quasi-semidistances and preference relations

In this section, we discuss in more detail the connection between quasi-semidistances and preference relations. The discussion is generic with no assumptions about the nature of the space S.

We noted that the preference order $\preceq_d$ defined through a quasi-semidistance is a pre-order and, therefore, a preference relation. Every preference relation induces a dual one by considering the converse relation. The dual of $\preceq_d$, denoted by $\preceq_{d^{-1}}$, is introduced in the following way: $x \preceq_{d^{-1}} y$ if and only if $y \preceq_d x$. It turns out that if a given preference relation is generated by a quasi-semidistance, then its dual is also generated by a quasi-semidistance.

Theorem 1. *Suppose that $\preceq_d$ is generated by the quasi-semidistance $d(x, y)$. Then, the dual preference relation $\preceq_{d^{-1}}$ is generated by $d^{-1}(x, y) = d(y, x)$ which is also a quasi-semidistance.*

Proof. According to the definition of the dual relation, for any $x, y \in S$, if $d(y, x) = 0$, then $x \preceq_{d^{-1}} y$. As a next step, it is straightforward to verify that the function $d^{-1}(x, y) := d(y, x)$ is a quasi-semidistance. □

The quasi-semidistance generating the dual order is monotonic with respect to primary order $\preceq_d$ in the following sense.

Theorem 2. *Suppose that $x \preceq_d y \preceq_d z$, where $x, y, z \in S$. Then, $d^{-1}(x,y) \leq d^{-1}(x,z)$ and also $d^{-1}(y,z) \leq d^{-1}(x,z)$ in which $d^{-1}(x,y) = d(y,x)$.*

Proof. Consider the triangle inequality $d(z,y) \leq d(z,x)+d(x,y)$. According to the assumed relationship, $d(x,y) = 0$ and, therefore, $d(z,y) \leq d(z,x)$ which proves the second inequality. Starting from $d(y,x) \leq d(y,z) + d(z,x)$ and using that $d(y,z) = 0$, we obtain the first inequality. □

This result shows how the quasi-semidistances concept can be used to construct monotonic functionals relative to a given metrizable preference relation, which can be exploited in approximation problems.

Another advantage of the theoretical framework is that it provides a way of comparing preference relations if there exists an inequality between the corresponding quasi-semidistances.

Theorem 3. *Suppose that $d_1(x,y)$ and $d_2(x,y)$ are two quasi-semidistances and that $d_1(x,y) \leq d_2(x,y)$. Under these assumptions, if $x \preceq_{d_2} y$, then $x \preceq_{d_1} y$.*

Proof. The proof is a simple consequence of the definition. If $x \preceq_{d_2} y$, then $d_2(x,y) = 0$. Because of the assumed inequality, this implies that $d_1(x,y) = 0$ and, therefore, $x \preceq_{d_1} y$. □

Note that this result does not imply the converse, i.e. if $x \preceq_{d_2} y \Rightarrow x \preceq_{d_1} y$ for all $x, y \in S$, then there exists an inequality between the corresponding quasi-semidistances.

Finally, note that from a given quasi-semidistance, we can always construct a semidistance using the dual. One approach to do that is to calculate the maximum between the quasi-semidistance and its dual,

$$\rho(x,y) = \max(d(x,y), d(y,x)). \tag{1}$$

It is straightforward to verify that in addition to the identity property and the triangle inequality, μ satisfies the symmetry property $\mu(x,y) = \mu(y,x)$. The representation in (1) implies that the quasi-semidistance $d(x,y)$

is consistent with any convergence in $\rho(x, y)$. That is, if $x_1, x_2, \ldots$ is a sequence converging to x in $\rho(x, y)$, $\lim_{n\to\infty} \rho(x_n, x) \to 0$, then necessarily $\lim_{n\to\infty} d(x_n, x) \to 0$. We exploit this property in the classification of stochastic orders which we link to the corresponding classification of the metric $\rho(x, y)$ which is obtained through the symmetrization transform in (1).

4 The Hausdorff metric structure

In the discussion so far, we have not specified the nature of the points in the space S. Suppose that S is the space of one-dimensional random variables defined on a probability space $(\Omega, \mathfrak{A}, \Pr)$ taking values in $(\mathbb{R}, \mathfrak{B}_1)$, where $\mathfrak{B}_1$ is the σ-field of all Borel subsets of $\mathbb{R}$.[2] In this setting, a quasi-metrizable preference order on S turns into a stochastic order with a quasi-semidistance $d(X, Y)$ defined on the space of all joint distributions S^2 generated by the pairs of random variables (X, Y) which we denote with capital letters. We proceed with the definition of the universal Hausdorff representation of quasi-semidistances on S^2 which we also call probability quasi-semidistances as they metrize preference relations between random quantities. Our discussion is based on the universal Hausdorff structure of probability distances, see Rachev (1991).

Consider the Hausdorff metric $r(A, B)$ defined on the space of all subsets of $\mathbb{R}$. Let $\mathfrak{B} \subseteq \mathfrak{B}_1$ and define a function $\phi : S^2 \times \mathfrak{B}^2 \to [0, \infty]$ satisfying the following relations:

I. If $P(X = Y) = 1$, then $\phi(X, Y; A, B) = 0$ for all $A = B \in \mathfrak{B}$.

II. There exists a constant $K_\phi \geq 1$ such that for all $A, B, C \in \mathfrak{B}$ and random variables X, Y, Z

$$\phi(X, Y; A, B) \leq K_\phi(\phi(X, Z; A, C) + \phi(Z, Y; C, B))$$

Let $d(X, Y)$ be a probability quasi-semidistance. The representation of $d(X, Y)$ in the following form

[2] The discussion will not change in a fundamental way if we consider general random elements taking values in a general functional space. We consider one-dimensional random variables for the sake of simplicity.

$$d(X,Y) = h_{\lambda,\phi,\mathfrak{B}}(X,Y) := \sup_{A\in\mathfrak{B}} \inf_{B\in\mathfrak{B}} \max\left\{\frac{1}{\lambda}r(A,B), \phi(X,Y;A,B)\right\} \quad (2)$$

is called the Hausdorff structure of $d(X,Y)$. In this representation, $r(A,B)$ is the Hausdorff metric in the set $\mathfrak{B}$, λ is a positive number, and the function ϕ satisfies properties I. and II.

It can be demonstrated that the function $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$ defined above is indeed a quasi-semidistance.

Theorem 4. *The function $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$ defined in equation (2) is a quasi-semidistance.*

Proof. The identity property and the triangle inequlity are essentially metric properties. The proof follows from the arguments in Rachev (1991) proving that the Hausdorff representation is a probability metric. □

Applying the symmetrization in equation (1) to the representation in (2), we obtain the Hausdorff representation of probability metrics. For more information, see Chapter 4 in Rachev (1991).

The following example illustrates the significance of the Hausdorff representation. It turns out that every probability quasi-semidistance is representable in the form in (2). Consider an arbitrary probability quasi-semidistance $\mu(X,Y)$. It has the trivial form $h_{\lambda,\phi,\mathfrak{B}}(X,Y) = \mu(X,Y)$ where the set $\mathfrak{B}$ is a singleton, for example $\mathfrak{B} = \{A_0\}$, and $\phi(X,Y;A_0,A_0) = \mu(X,Y)$.

In the limit cases $\lambda \to 0$ and $\lambda \to \infty$, the Hausdorff structure turns into a structure of a uniform type. The following limit relations hold

Theorem 5. *Let $d(X,Y)$ have the representation in (2). Then, as $\lambda \to 0$, $d(X,Y)$ has a limit equal to*

$$h_{0,\phi,\mathfrak{B}}(X,Y) = \sup_{A\in\mathfrak{B}} \phi(X,Y;A,A) \quad (3)$$

As $\lambda \to \infty$, the limit $\lim_{\lambda\to\infty} \lambda h_{\lambda,\phi,\mathfrak{B}}(X,Y) = h_{\infty,\phi,\mathfrak{B}}(X,Y)$ exists and equals

$$h_{\infty,\phi,\mathfrak{B}}(X,Y) = \sup_{A\in\mathfrak{B}} \inf_{B\in\mathfrak{B},\phi(X,Y;A,B)=0} r(A,B) \quad (4)$$

Proof. The arguments for the proof of (4) are the same as in Lemma 4.1.3. in Rachev (1991). In order to prove (3), note that

$$\lim_{\lambda\to 0} \max\left\{\frac{1}{\lambda}r(A,B), \phi(X,Y;A,B)\right\} = \begin{cases} \infty, & B \neq A \\ \phi(X,Y;A,A), & B = A \end{cases}$$

Therefore, the infimum in (2) is attained at $B = A$ and, as a result, $h_{0,\phi,\mathfrak{B}}(X,Y)$ is calculated by computing the supremum of $\phi(X,Y;A,A)$ with respect to A. □

The main building block of the Hausdorff representation, the function $\phi(X,Y;A,B)$, can be interpreted in the following way. It calculates the performance of X relative to Y over two events A and B. If $\phi(X,Y;A,B) = 0$ for some A and B, then, according to the preference order definition, Y performs at least as X with respect to the two events. As we demonstrate in the next section, in some cases there is a straightforward interpretation in the sense that ϕ calculates the deviation of the probability of X belonging to A relative to the probability of Y belonging to B. In other cases, the relationship of X to A and Y to B is not so direct.

Besides the function ϕ, the definition in (2) includes also the Hausdorff metric $r(A,B)$ in order take into account the degree of dissimilarity between the events A and B. If we want to calculate the degree of deviation between X and Y on one an the same events, i.e. $A = B$, then we can use the limit case given in (3). In this case, if Y outperforms X with respect to all events $A = B$, i.e. $\phi(X,Y;A,A) = 0$ for all A, then Y is at least as preferable as X.

The Hausdorff representation of a quasi-semidistance in (2) can be translated into a different form which is more open to interpretation.

Theorem 6. *Suppose that a probability quasi-semidistance admits the Hausdorff representation $h_{\lambda,\phi,\mathfrak{B}}$ given in (2). Then, the probability quasi-semidistance enjoys also the following representation*

$$h_{\lambda,\phi,\mathfrak{B}}(X,Y) = \inf\{\epsilon > 0 : v(X,Y;\lambda\epsilon) < \epsilon\} \tag{5}$$

where

$$v(X,Y;t) = \sup_{A\in\mathfrak{B}} \inf_{B\in A(t)} \phi(X,Y;A,B) \tag{6}$$

in which $A(t)$ is the collection of all elements B of $\mathfrak{B}$ such that the Hausdorff metric $r(A,B)$ is not greater than t.

Proof. The proof is constructed in the same way as the proof of Theorem 4.2.1 in Rachev (1991). □

We can interpret equation (6) in the following way. Fix an event A and a tolerance level $t > 0$. Using the Hausdorff metric, take all events that do not deviate from A more than as implied by the tolerance level, i.e. build the set $A(t) = \{B \in \mathfrak{B} : r(A,B) < t\}$. With A fixed, compute the minimum performance deviation between X and Y running through all events for Y, which are within the tolerance level. As a next step, compute the maximum of those minimal deviations by varying A.

By varying the tolerance level t, we control the size of the admissible sets relative to A. The larger t is, the more the admissible events may deviate from the event A and, therefore, the larger potential there is for deviation in the performance of Y relative to X. At the other extreme, when $t = 0$, the deviation in performance is estimated over one and the same event.

Finally, in equation (5) we calculate the smallest tolerance level such that the largest of those minimal performance deviations is smaller than it. Note that, depending on the nature of the random variables X and Y and the choice of ϕ, this smallest tolerance level may actually be infinite, i.e. the quasi-semidistance may be unbounded.

The parameter λ in both (2) and (5) allows calculating limit quasi-semidistances arising naturally from the general case. This is demonstrated in Theorem 5. If we view $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$ defined in (2) as a function of the parameter λ, it appears that it is a monotonic, non-increasing function.

Theorem 7. *The quasi-semidistance $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$ defined in* (2) *is a non-increasing function of $\lambda > 0$.*

Proof. For any fixed $A \in \mathfrak{B}$ and $0 < \lambda_1 < \lambda_2$,

$$\max\left\{\frac{1}{\lambda_2} r(A,B), \phi(X,Y;A,B)\right\} \leq \max\left\{\frac{1}{\lambda_1} r(A,B), \phi(X,Y;A,B)\right\}$$

for all $B \in \mathfrak{B}$. Therefore, the same inequality is preserved after computing sequentially the infimum with respect to B and the then the supremum with respect to A. In effect, $h_{\lambda_2,\phi,\mathfrak{B}}(X,Y) < h_{\lambda_1,\phi,\mathfrak{B}}(X,Y)$. $\square$

This result implies that the limit computed in (3) is an upper bound of $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$, i.e.

$$h_{\lambda,\phi,\mathfrak{B}}(X,Y) \leq h_{0,\phi,\mathfrak{B}}(X,Y).$$

Thus, if $h_{0,\phi,\mathfrak{B}}(X,Y)$ is finite, it means that $h_{\lambda,\phi,\mathfrak{B}}(X,Y)$ is finite as well.

5 Examples

In this section, we provide examples of quasi-semidistances with a Hausdorff structure. We also provide examples of quasi-semidistances metrizing different stochastic dominance orders.

The structure of the quasi-semidistance determines whether the induced stochastic order is based essentially on inequalities between certain characteristics such as mean, volatility, etc., inequalities based on the cumulative distribution function (cdf), or inequalities directly between functions of the corresponding random variables. In line with the theory of probability metrics, the first order type we call *primary*, the second – *simple*, and the third – *compound*. The formal definition is as follows.

Definition 1. *A metrizable stochastic order $\preceq_d$ is called primary, simple, or compound if the probability semidistance arising from a symmetrization transform, such as the one given in* (1)*, is primary, simple, or compound, respectively.*

From the point of view of finance, the stochastic order behind the mean-variance framework is primary. In contrast, FSD and SSD are simple orders, as we demonstrate below. A theoretical advantage of this categorization is the inclusion

$$\text{primary orders} \ \subset \ \text{simple orders} \ \subset \ \text{compound orders}$$

which implies that a primary order cannot induce a simple order which, in turn, cannot induce a compound order. This is a consequence of the corresponding relations between primary, simple, and compound probability metrics.

5.1 The Lévy quasi-semidistance and first-order stochastic dominance

Consider the choice $\phi(X, Y; (-\infty, x], (-\infty, y]) = (F_X(x) - F_Y(y))_+$, where $(x)_+ = \max(x, 0)$. In this case, the sets A and B are of the form $(-\infty, a], a \in \mathbb{R}$. The representation in (2) becomes

$$L^*_\lambda(X, Y) = \sup_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \max \left\{ \frac{1}{\lambda}|x - y|, (F_X(x) - F_Y(y))_+ \right\} \tag{7}$$

The quasi-semidistance defined above also equals

$$L^*_\lambda(X, Y) = \inf\{\epsilon > 0 : (F_X(x) - F_Y(x + \lambda\epsilon))_+ < \epsilon, \forall x \in \mathbb{R}\}$$

which can be demonstrated by applying the result in Theorem 6. Applying the symmetrization transform in (1) leads to the parametric version of the celebrated Lévy metric

$$L_\lambda(X, Y) = \inf\{\epsilon > 0 : F_X(x - \lambda\epsilon) - \epsilon \leq F_Y(x) \leq F_X(x + \lambda\epsilon) + \epsilon, \forall x \in \mathbb{R}\}.$$

and for this reason we call $L^*_\lambda(X, Y)$ the Lévy quasi-semidistance. The two limit cases in Theorem 5 can be calculated explicitly and they equal

$$L^*_0(X, Y) = \sup_{x \in \mathbb{R}} (F_X(x) - F_Y(x))_+$$

$$L^*_\infty(X, Y) = \sup_{t \in [0,1]} (F_Y^{-1}(t) - F_X^{-1}(t))_+$$

where $F_X^{-1}(t) = \sup\{x : F_X(x) < t\}$ is the inverse cdf of X.

The Lévy quasi-semidistance is an important example because it can be used to metrize FSD. In fact, the definition in (7) induces the dual order and is, therefore, monotonic with respect to FSD in the sense of Theorem 2. Recall that FSD can be introduced by an inequality between the corresponding cdfs,

$$X \preceq_{FSD} Y \quad \Longleftrightarrow \quad F_Y(x) \leq F_X(x), \forall x \in \mathbb{R}. \tag{8}$$

The dual order, $\preceq_{FSD^{-1}}$, can be expressed in a similar way

$$X \preceq_{FSD^{-1}} Y \quad \Longleftrightarrow \quad F_X(x) \leq F_Y(x), \forall x \in \mathbb{R}. \tag{9}$$

Theorem 8. *The functional $L^*_\lambda(X,Y)$ defined in* (7) *metrizes $\preceq_{FSD^{-1}}$.*

Proof. We demonstrate that $X \preceq_{FSD^{-1}} Y$ if and only if $L^*_\lambda(X,Y) = 0$. First, notice that $L^*_\lambda(X,Y) = 0$ if and only if

$$\inf_{y \in \mathbb{R}} \max \left\{ \frac{1}{\lambda}|x-y|, (F_X(x) - F_Y(y))_+ \right\} = 0, \ \forall x \in \mathbb{R}.$$

This, in turn, holds if and only if $(F_X(x) - F_Y(y))_+ = 0$ for $x = y$ since otherwise $|x-y| \neq 0$. As a result, $L^*_\lambda(X,Y) = 0$ if and only if $F_X(x) \leq F_Y(x), \forall x \in \mathbb{R}$ and, therefore, we can conclude that

$$X \preceq_{L^*_\lambda} Y \quad \Longleftrightarrow \quad F_X(x) \leq F_Y(x), \forall x \in \mathbb{R}.$$

Comparing to (9), we can conclude that $L^*_\lambda(X,Y)$ metrizes $\preceq_{FSD^{-1}}$. $\square$

As a corollary from this result, it follows that the dual quasi-semidistance $L^*_\lambda(Y,X)$ metrizes FSD. Also, note that the reasoning does not depend on the choice of λ and, therefore, the result is valid for any $\lambda > 0$. In effect, there are many metrics inducing FSD.

Since symmetrization of L^*_λ leads to the Lévy metric which is a simple probability metric, it follows that FSD is a simple order.

5.2 Higher order stochastic dominance

The reasoning in Section 5.1 can be applied to the more general case of higher order stochastic dominance. Stochastic dominance of order n, $\preceq_n$, can be introduced by means of an inequality involving the corresponding cdfs,

$$X \preceq_n Y \quad \Longleftrightarrow \quad F_X^{(n)}(x) \leq F_Y^{(n)}(x), \forall x \in \mathbb{R}. \tag{10}$$

where $F_X^{(n)}(x)$ stands for the n-th integral of the cdf of X which can be defined recursively as

$$F_X^{(n)}(x) = \int_{-\infty}^{x} F_X^{(n-1)}(t)dt. \tag{11}$$

Repeating the arguments in Theorem 9, it can be demonstrated that the quasi-semidistance

$$\mathbb{L}_\lambda^*(X,Y) = \sup_{x\in\mathbb{R}} \inf_{y\in\mathbb{R}} \max\left\{\frac{1}{\lambda}|x-y|, \left(F_X^{(n)}(x) - F_Y^{(n)}(y)\right)_+\right\} \tag{12}$$

metrizes the dual of the n-th order stochastic dominance and, as result, $\mathbb{L}_\lambda^*(Y,X)$ metrizes $\preceq_n$. In this more general case, however, it is not clear a priori if $\mathbb{L}_\lambda^*(X,Y) < \infty$. For the Lévy quasi-semidistance this question is redundant because the limit $L_0^*(X,Y)$ is always finite and Theorem 7 guarantees boundedness of the Lévy quasi-semidistance. The limit of (12) as $\lambda \to 0$ equals

$$\mathbb{L}_0^*(X,Y) = \sup_{x\in\mathbb{R}} \left(F_X^{(n)}(x) - F_Y^{(n)}(x)\right)_+$$

and by Theorem 7 we can conclude that $\mathbb{L}_\lambda^*(X,Y) < \infty$ if $\mathbb{L}_0^*(X,Y) < \infty$.

We develop a set of sufficient conditions involving another probability quasi-semidistance.

Theorem 9. *The following inequality holds true, provided that $E|X|^k = E|Y|^k,\ k = 1,2,\ldots,n-2$, $E|X|^{n-1} < \infty$ and $E|Y|^{n-1} < \infty$,*

$$\mathbb{L}_\lambda(X,Y) \le \int_{\mathbb{R}} \left(\int_{-\infty}^{x} \frac{(x-t)^{n-2}}{(n-2)!} d(F_X(t) - F_Y(t))\right)_+ dx < \infty \tag{13}$$

Proof. Integrating by parts $\int_{-\infty}^{x}(F_X^{(n-1)}(t) - F_Y^{(n-1)}(t))dt$ and using the equality of all moments up to order $n-2$, we obtain

$$\int_{-\infty}^{x} \left(F_X^{(n-1)}(t) - F_Y^{(n-1)}(t)\right) dt = \int_{-\infty}^{x} \frac{(x-t)^{n-1}}{(n-1)!} d(F_X(t) - F_Y(t)).$$

Therefore,

$$
\begin{aligned}
\sup_{x\in\mathbb{R}}\left(\int_{-\infty}^{x}\left(F_X^{(n-1)}(t)-F_Y^{(n-1)}(t)\right)dt\right)_+ &\leq \sup_{x\in\mathbb{R}}\int_{-\infty}^{x}\left(F_X^{(n-1)}(t)-F_Y^{(n-1)}(t)\right)_+ dt\\
&= \int_{\mathbb{R}}\left(\int_{-\infty}^{x} F_X^{(n-2)}(t)-F_Y^{(n-2)}(t)dt\right)_+ dx\\
&= \int_{\mathbb{R}}\left(\int_{-\infty}^{x} \frac{(x-t)^{n-2}}{(n-2)!}d(F_X(t)-F_Y(t))\right)_+ dx
\end{aligned}
$$

in which the first equality follows since due to the positivity of the integrand, the corresponding integral is a non-decreasing function of the upper bound. The boundedness in (13) follows from the assumed finite absolute moments of order $n-1$.

□

The inequality in (13) is an inequality between two quasi-semidistances. The upper bound is the Zolotarev quasi-semidistance

$$
\zeta_{n-1}^{*}(X,Y)=\int_{\mathbb{R}}\left(\int_{-\infty}^{x}\frac{(x-t)^{n-2}}{(n-2)!}d(F_X(t)-F_Y(t))\right)_+ dx
$$

which can also be represented as

$$
\zeta_{n-1}^{*}(X,Y)=\int_{\mathbb{R}}\left(E(x-X)_+^{n-1}-E(x-Y)_+^{n-1}\right)_+ dx
$$

The Zolotarev quasi-semidistance itself can be used to metrize the n-order stochastic dominance under the assumed moment conditions. However, $\mathbb{L}_\lambda(X,Y)$ is strictly weaker as there is no lower bound of it that can be expressed in terms of $\zeta_{n-1}^{*}(X,Y)$. Therefore, this can be viewed as an illustration how one and the same stochastic order can arise from two different quasi-semidistances.

The conclusion that FSD is a simple order can be extended to the n-th order stochastic dominance by noticing that symmetrizing ζ_{n-1}^{*} leads to a simple probability semidistance.

The approach discussed in this section can be applied without modification to the fractional and the inverse orders discussed in Ortobelli et al. (2009). From a theoretical viewpoint, they belong to the class of simple stochastic orders as the probability semidistances arising from applying the symmetrization transform are simple.

5.3 AVaR generated stochastic orders

Rockafellar et al. (2006) provide an axiomatic description of convex dispersion measures called *deviation measures*. Any functional defined on the space of random variables which is non-negative, positively homogeneous, sub-additive and translation invariant is called a deviation measure. Stoyanov et al. (2008) demonstrate that there is a close relationship between deviation measures and probability metrics. In fact, it is possible to show that all deviation measures can be generated from probability metrics.

Deviation measures are closely related to the concept of coherent risk measures introduced in Artzner et al. (1998). Expectation bounded coherent risk measures can generate deviation measures and, therefore, probability quasi-metrics. The converse is also possible, see Rockafellar et al. (2006) and Stoyanov et al. (2008).

In this section, we provide an example of a probability quasi-metric generated from a coherent risk measure that admits the representation given in (3). The coherent risk measure is the average value-at-risk (AVaR), also known as conditional value-at-risk, which is defined as

$$AVaR_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F_X^{-1}(t)dt \tag{14}$$

where $0 < \epsilon < 1$ is called tail probability and X is a random variable describing the return distribution of an investment. AVaR is interpreted as the average loss provided that the loss is larger than the ϵ-quantile. For additional interpretations, see Rachev et al. (2008).

Another representation of (14), which is essentially a consequence of the general representation of coherent risk measures given in Artzner et al. (1998), equals

$$AVaR_\epsilon(X) = \sup_{A\in\mathfrak{A}_\epsilon} -\int_0^1 F_X^{-1}(t)d\nu_A = -\inf_{A\in\mathfrak{A}_\epsilon}\int_0^1 F_X^{-1}(t)d\nu_A \tag{15}$$

where $\mathfrak{A}_\epsilon = \{A \subset [0,1] : \lambda(A) = \epsilon\}$ in which $\lambda(A)$ is the Lebesgue measure of A and ν_A is a uniform probability measure on the set A. The family of sets A_ϵ can be interpreted as the collection of all sets A such that $F_X^{-1}(A)$ is an ϵ-probability event, $P(X \in F_X^{-1}(A)) = \epsilon$. The interval $[0,\epsilon] \in \mathfrak{A}_\epsilon$ yields the AVaR at tail probability ϵ.

Consider the following choice for the building block ϕ of the Hausdorff representation in (2)

$$\phi(X,Y;A,B) = \left(\int_0^1 F_X^{-1}(t)d\nu_B - \int_0^1 F_Y^{-1}(t)d\nu_A\right)_+ \tag{16}$$

in which $A, B \in \mathfrak{B}$ where $\mathfrak{B} = [0,\epsilon] \cup \mathfrak{B}_1 \subseteq \mathfrak{A}_\epsilon$ because the interval $[0,\epsilon]$ needs to be in $\mathfrak{B}$. It is easy to verify that the axiomatic properties hold and this is a valid choice for ϕ in the Hausdorff representation. The resulting quasi-semidistance

$$\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}(X,Y) = \sup_{A\in\mathfrak{B}} \inf_{B\in\mathfrak{B}} \max\left\{\frac{1}{\lambda}r(A,B), \left(\int_0^1 F_X^{-1}(t)d\nu_B - \int_0^1 F_Y^{-1}(t)d\nu_A\right)_+\right\} \tag{17}$$

is an AVaR generated quasi-semidistance. In the special case when $\mathfrak{B} = \{[0,\epsilon]\}$, then

$$\mathbb{AV}_{\lambda,\epsilon,\{[0,\epsilon]\}}(X,Y) = (AVaR_\epsilon(Y) - AVaR_\epsilon(X))_+.$$

The stochastic order $\preceq_{\mathbb{AV}_\mathfrak{B}}$ induced by the quasi-semidistance $\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}$ can be interpreted in the following way. Suppose that X and Y are two random variables describing the returns of two stocks. If $X \preceq_{\mathbb{AV}_\mathfrak{B}} Y$, then the average loss of X in events occurring with probability ϵ is always not smaller than the corresponding average loss of Y. The events that we consider in this comparison depend on the choice of $\mathfrak{B}$ but the most extreme ones, $F_X^{-1}([0,\epsilon])$ and $F_Y^{-1}([0,\epsilon])$, are always included.

A couple of properties are collected in the next theorem.

Theorem 10. *The following relations hold true.*

1. *If* $X \preceq_{\mathbb{AV}_\mathfrak{B}} Y$, *then* $AVaR_\epsilon(Y) \leq AVaR_\epsilon(X)$ *for any admissible choice of* $\mathfrak{B}$.

2. *The limit of* $\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}(X,Y)$ *as* $\lambda \to 0$ *equals*

$$\mathbb{AV}_{0,\epsilon,\mathfrak{B}}(X,Y) = \sup_{A\in\mathfrak{B}} \left(\int_0^1 F_X^{-1}(t)d\nu_A - \int_0^1 F_Y^{-1}(t)d\nu_A\right)_+ \tag{18}$$

3. *If* $X = EY$ *is a constant, then* $\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}(EY,Y) = AVaR_\epsilon(Y - EY)$

and, thus, equals the deviation measure behind the AVaR risk measure.

4. *Suppose that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$. Then, the stochastic order $\preceq_{\mathbb{AV}_{\mathfrak{B}_2}}$ implies the stochastic order $\preceq_{\mathbb{AV}_{\mathfrak{B}_1}}$.*

5. *If $X \preceq_{FSD} Y$, then $X \preceq_{\mathbb{AV}_{\mathfrak{B}}} Y$ for any admissible choice of $\mathfrak{B}$. The converse is not true.*

Proof. We prove one by one the claims in the theorem.

1. Repeating the arguments in Theorem 8, we find out that $\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}(X,Y) = 0$ if and only if the function in (16) equals zero for each $A \in \mathfrak{B}$ and $B = A$ which, in turn, implies that

$$\int_0^1 F_X^{-1}(t) d\nu_A \leq \int_0^1 F_Y^{-1}(t) d\nu_A, \quad \forall A \in \mathfrak{B}$$

   Since $[0,\epsilon] \in \mathfrak{B}$,

$$-\int_0^1 F_X^{-1}(t) d\nu_{[0,\epsilon]} \geq -\int_0^1 F_Y^{-1}(t) d\nu_{[0,\epsilon]} =$$

   which proves that $AVaR_\epsilon(Y) \leq AVaR_\epsilon(X)$.

2. The proof is a simple application of the reasoning in Theorem 5.

3. First, notice that $\int_0^1 F_{EY}^{-1}(t) d\nu_A = EY$ irrespective of A. As a result,

$$\begin{aligned}
\mathbb{AV}_{\lambda,\epsilon,\mathfrak{B}}(EY,Y) &= \sup_{A\in\mathfrak{B}} -\int_0^1 F_Y^{-1}(t) d\nu_A + EY \\
&= AVaR_\epsilon(Y) + EY \\
&= AVaR_\epsilon(Y - EY)
\end{aligned}$$

4. If we can demonstrate that there is an inequality between two quasi-semidistances metrizing $\preceq_{\mathbb{AV}_{\mathfrak{B}_1}}$ and $\preceq_{\mathbb{AV}_{\mathfrak{B}_2}}$, the rest is a consequence of Theorem 3. Consider $\mathbb{AV}_{0,\epsilon,\mathfrak{B}_1}(X,Y)$ and $\mathbb{AV}_{0,\epsilon,\mathfrak{B}_2}(X,Y)$. The inequality $\mathbb{AV}_{0,\epsilon,\mathfrak{B}_1}(X,Y) \leq \mathbb{AV}_{0,\epsilon,\mathfrak{B}_2}(X,Y)$ follows from (18) and the assumed inclusion $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$.

5. If $X \preceq_{FSD} Y$, then $F_X^{-1}(t) \leq F_Y^{-1}(t)$, $\forall t \in [0,1]$. As a result, $\int_0^1 F_X^{-1}(t) d\nu_A \leq \int_0^1 F_Y^{-1}(t) d\nu_A$, $\forall A \in \mathfrak{B}$, where $\mathfrak{B}$ is any admissible family of sets. Therefore, $X \preceq_{\mathbb{AV}_{\mathfrak{B}}} Y$.

   The fact that the converse does not hold follows essentially from the granularity of the sets in the family $\mathfrak{B}$. We can always construct an example in which $F_X^{-1}(t) \geq F_Y^{-1}(t), \forall t \in [t_0 - \delta, t_0 + \delta]$, where $\delta < \epsilon$, and yet $\int_0^\epsilon F_X^{-1}(t)dt \leq \int_0^\epsilon F_Y^{-1}(t)dt$.

□

An expected corollary from the results above is that AVaR is consistent with FSD. Assuming that the random variables X and Y describe stock returns, it is the structure of the admissible family $\mathfrak{B}$ which determines whether only events including losses are considered in $\preceq_{\mathbb{AV}_{\mathfrak{B}}}$, i.e. negative returns, or both profits and losses, i.e. positive and negative returns.

5.4 Compound quasi-semidistances

The examples in the previous sections share a common feature. If X and Y are two random variables such that $F_X(x) = F_Y(x), \forall x \in \mathbb{R}$, then the corresponding quasi-semidistances equal zero. In this section, we consider compound quasi-semidistances in the form in (5) which are essentially characterized by the following feature: if $X = Y$ in almost sure sense, then they turn into zero.

Consider a function $d(x, y)$ defined on $\mathbb{R}\times\mathbb{R}$, which is a quasi-semidistance. Define the function v in the representation in (5) to be

$$v(X, Y; t) = P(d(X, Y) > t).$$

Then, the functional

$$\mu_\lambda(X, Y) = \inf\{\epsilon > 0 : P(d(X, Y) > \lambda\epsilon) < \epsilon\}$$

is a compound quasi-semidistance.

The stochastic order generated from $\mu_\lambda(X, Y)$ is of a compound type. Suppose that $d(X, Y) = (X - Y)_+$. Under this assumption, $\mu_\lambda(X, Y) = 0$ if and only if $P((X - Y)_+ > \epsilon) = 0, \forall \epsilon > 0$ which means that $X \leq Y$ in almost sure sense.

There are also other ways to construct compound quasi-semidistances which do not enjoy a non-trivial Hausdorff representation. For additional information, see Stoyanov et al. (2008).

6 Utility-type representations

From the point of view of the economic theories describing choice under uncertainty, some stochastic orders arise from the preferences of a given class of economic agents. For example, according to classical expected utility theory, FSD arises from the class of *non-satiable* investors who have non-decreasing utility functions. Thus, if all non-satiable investors do not prefer Y to X, then $X \preceq_{FSD} Y$. Likewise, second-order stochastic dominance arises from the non-satiable, risk-averse investors who have non-decreasing, concave utility functions. In the same manner, n-th order stochastic dominance can be introduced through the preference relations of a class of investors the utility functions of whom are characterized by certain properties involving derivatives of higher order.

Consider the preference relation of an investor with a utility function $u(x), x \in \mathbb{R}$. The preference relation is characterized by the expected utility, i.e. $X \preceq_u Y$ if and only if $Eu(X) \leq Eu(Y)$. As a result, one natural quasi-semidistance metrizing the preference relation is

$$\zeta_u^*(X,Y) = (Eu(X) - Eu(Y))_+.$$

Indeed, it can be directly verified that $X \preceq_{\zeta_u^*} Y \Leftrightarrow X \preceq_u Y$.

This approach can be generalized to a given class of investors $\mathcal{U}$. The arising stochastic order $\preceq_{\mathcal{U}}$ is introduced in the following way: $X \preceq_{\mathcal{U}} Y$ if and only if $X \preceq_u Y, \forall u \in \mathcal{U}$. In this case, one natural quasi-semimetric metrizing $\preceq_{\mathcal{U}}$ has the form

$$\zeta_{\mathcal{U}}^*(X,Y) = \sup_{u \in \mathcal{U}} \left(\int_{\mathbb{R}} u(x) d(F_X(x) - F_Y(x)) \right)_+ \tag{19}$$

which equals $\zeta_{\mathcal{U}}^*(X,Y) = \sup_{u \in \mathcal{U}} (Eu(X) - Eu(Y))_+$ if the corresponding expected utilities are finite. Thus, the condition $\zeta_{\mathcal{U}}^*(X,Y) = 0$ guarantees that $X \preceq_u Y, \forall u \in \mathcal{U}$, and therefore the stochastic order generated by the quasi-semidistance in (19) coincides with the stochastic order of the class $\mathcal{U}$. Since the representation in (19) is directly liked to the class $\mathcal{U}$, we call it a *utility-type* representation.

Some properties of (19) are collected in the following theorem.

Theorem 11. *Suppose that the functional defined in* (19) *is finite. Under this assumption, it is a probability quasi-semidistance which metrizes the stochastic order* $\preceq_{\mathcal{U}}$.

Proof. The identity property is obvious, if $F_X(x) = F_Y(x), \forall x \in \mathbb{R}$, then $\zeta^*_{\mathcal{U}}(X,Y) = 0$. The triangle inequality follows from the properties of the $(y)_+$ function.

From the definition in (19), it follows that if $\zeta^*_{\mathcal{U}}(X,Y) = 0$, then $X \preceq_u Y, \forall u \in \mathcal{U}$. Therefore, $\preceq_{\zeta^*_{\mathcal{U}}} \Rightarrow \preceq_{\mathcal{U}}$. The converse relationship follows by construction, if $Eu(X) \leq Eu(Y), \forall u \in \mathcal{U}$, then $\sup_{u\in\mathcal{U}}(Eu(X) - Eu(Y))_+ = 0$. As a result, $\preceq_{\zeta^*_{\mathcal{U}}} \Leftrightarrow \preceq_U$. The assumption of boundedness of $\zeta^*_{\mathcal{U}}(X,Y)$ is technical and is required to make sure the order $\preceq_{\zeta^*_{\mathcal{U}}}$ well-defined over all pairs (X,Y). □

Additional properties for the functions in $\mathcal{U}$ have to be specified in order to guarantee that $\zeta^*_{\mathcal{U}}(X,Y)$ is finite. Usually this is done by imposing certain growth conditions. For additional details, see Rachev (1991).

Stoyanov et al. (2009) consider a functional similar to (19) which is constructed to be consistent with cumulative prospect theory. They demonstrate that the class of investors with balanced views, introduced in Stoyanov et al. (2009), is sufficient to metrize FSD. In order to be consistent with the definition in (19), we illustrate this with a sub-class. Consider all investors with bounded, non-decreasing Lipschitz utility functions, $u(x) : |u(x) - u(y)| \leq K|x - y|, \forall x, y \in \mathbb{R}$, where $0 < K \leq 1$. Denote this class of utility functions with $\mathcal{U}_L$. Under these assumptions, the quasi-semidistance $\zeta^*_{\mathcal{U}_L}(X,Y)$ is bounded,

$$\zeta^*_{\mathcal{U}_L}(X,Y) \leq \int_{\mathbb{R}} (F_Y(x) - F_X(x))_+ dx,$$

and metrizes FSD.

Note that both $\zeta^*_{\mathcal{U}_L}$ from this example and the Lévy quasi-semidistance in (7) metrize FSD. This does not necessarily mean that there is a certain relationship between $\zeta^*_{\mathcal{U}_L}$ and L^*_λ. The topologies generated by the two quasi-semimetrics may be completely different and yet their specialization orders can be the same.

The quasi-semidistance in (19) is not a universal representation as the Hausdorff construction in (2). Therefore, even though any metrizable stochastic order is generated by a quasi-semidistance, there may not exist a quasi-semidistance with a utility-type representation metrizing it. An example of a stochastic order which implies second-order stochastic dominance but for which no representation in terms of a class of investors is known can be found in Rachev et al. (2008).

Finally, whether a utility-type order is primary or simple depends on how rich the family $\mathcal{U}$ is. This is illustrated in Stoyanov et al. (2009) in a discussion concerning how rich $\mathcal{U}$ can be in order for $\zeta^*_{\mathcal{U}}$ to metrize FSD. As an extreme example, if $\mathcal{U}$ contains only one utility function, i.e. there is only one investor, $\zeta^*_{\mathcal{U}}$ generates a primary order.

7 Almost stochastic orders and degree of violation

A way to address some of the paradoxes arising from expected utility theory is discussed in Leshno and Levy (2002) and Bali et al. (2009). They suggest considering a sub-set of the corresponding investors set because, as they argue, paradoxes arise from non-realistic choices of utility functions. The stochastic order arising from this smaller set of investors is called *almost stochastic order*.

The general idea is to develop conditions that the utility functions in a given set need to satisfy which depend on the degree of violation of the stochastic order arising from the larger investors set. For instance, consider $s_1 = \{x : F_X(x) - F_Y(x) < 0\}$ and $s_2 = \{x : F_Y(x) - F_X(x) < 0\}$. The degree of violation of $X \preceq_{FSD} Y$ is defined as the ratio

$$\epsilon = \frac{\int_{s_1} (F_Y(x) - F_X(x))dx}{\int_{\mathbb{R}} |F_X(x) - F_Y(x)|dx}$$

and the corresponding condition on the non-decreasing utility functions is derived to be $u'(x) \leq \inf_x u'(x)(1/\epsilon - 1)$.

The degree of violation of FSD can be expressed in terms of a quasi-semidistance metrizing FSD. Consider the Kantorovich quasi-semidistance

$$k^*(X,Y) = \int_{\mathbb{R}} (F_Y(x) - F_X(x))_+ dx.$$

It can be demonstrated that it metrizes FSD by repeating the arguments in Theorem 8. For additional information, see also Stoyanov et al. (2009). The degree of violation ϵ can be related to $k^*(X,Y)$ in the following way

$$\epsilon/(1-\epsilon) = k^*(X,Y)/k^*(Y,X).$$

As a result, the corresponding condition becomes

$$u'(x) \leq \inf_x u'(x) \frac{k^*(Y,X)}{k^*(X,Y)}.$$

This example is interesting as it illustrates a generic property. If X and Y are two prospects such that their cdfs do not coincide completely, then the ratio

$$r = \frac{\mu(X,Y)}{\mu(Y,X)}$$

in which $\mu(X,Y)$ is some quasi-semidistance measures the degree of violation of the stochastic order $X \preceq_\mu Y$ metrized by μ.

8 Conclusion

In this paper, we considered a general systematic approach towards describing stochastic dominance rules by means of quasi-semidistances. We provided a universal representation of quasi-semidistances, which we call the Hausdorff representation in line with a similar universal representation in the theory of probability metrics. The theoretical framework allows for a categorization of stochastic orders to a primary, simple, and compound type. A number of examples supporting the theoretical construct were discussed pertaining to FSD and the n-th order stochastic dominance in general. We introduced a stochastic order based on average value-at-risk which illustrates how the quasi-semidistances approach can be used to generate new stochastic orders. We also considered stochastic orders arising from classes of investors and a utility-type quasi-semidistance metrizing them. An expected outcome from the theoretical framework is that not all metrizable stochastic orders have a utility type representation. Finally, we discussed a way to measure the degree of violation of a stochastic order and how it is related to the notion of almost stochastic dominance.

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