Conditional Valuation of Barrier Options with Incomplete Information

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In this paper, we investigate the role of reduced available information on the valuation of single and double barrier options. We present closed-form analytical solutions for barrier options with three different types of information about the underlying: full information arriving continuously, delayed continuous observations and multiple discrete observations. Such information structures are the typical information sets available in liquid markets for mark-to-model valuations, market risk and credit risk mangement. We also consider the case where additional observation noise is layered over the underlying asset price – information set that is common for some investors in unlisted stocks and other illiquid assets.

Key words: single and double barrier options; incomplete information; conditional valuation; risk management; value-at-risk (VaR); potential future exposure (PFE)

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1 Introduction

A barrier option is a path-dependent derivative, similar to a standard option with the additional feature that the final payoff depends on whether the underlying has crossed a pre-specified barrier level. For the knock-out barrier option, rebate is the payoff to the option holder if the barrier level is reached, otherwise the payoff is that of a European option. In the knock-in barrier option, the option holder receives a European option if the barrier is hit. In contrast to standard barrier options, the payoff of digital barrier options is one if the pre-specified barrier condition is met and zero otherwise.

Barrier options exist on underlying assets from almost every asset class with the underlyings being either spot or forward prices (rates), futures prices or, less frequently, average prices. Barrier options are popular derivative instruments among hedge funds and directional traders since their premiums are lower than those of European options and, thus, they allow for greater leverage. They are also used to express a more complex directional view enhanced with a view on the probability of the barrier-crossing the underlying asset. Another ussage is for hedging of corporate exposure, with popular structures in which barrier options are attached to swaps and forward contracts. Barrier options are frequently used in foreign exchange (FX) margin and carry trades in order to hedge exposures to clients and counterparties.

Barrier options started to be actively traded on the over-the-counter markets in the late 1960s, which inspired a significant academic interest in their valuations. Inital research focused on the valuation of single barrier options on assets with lognormal dynamics and continuous monitoring of the barrier. Merton (1973) obtained closed-form analytical solutions for the most popular, down-and-out barrier call option. Reiner and Rubinstein (1991) provided analytical solutions for all eight types of single barrier European options. Broadie et al. (1997) developed a methods for continutity correction of the barrier level for barrier options with discrete monitoring of the barrier, moving the valuation problem back to the classical setting. Although the valuation of more complex barrier options has been solved in by several researchers (see, e.g., Carr (1995) and Lou (2001)) and there are studies that investigated valuation under alternative stochastic processes (see, e.g., Mitov et al. (2009)), there is no systematic study on barrier options valuation under incomplete information.

Valuation with incomplete information is very important, because the barrier option value depends on the whole path of the underlying asset from the issue date to the valuation date, but in many situations, investors and risk managers have imperfect information about the sample path of the underlying. To our knowledge, the only study dealing with barrier option valuation with incomplete information is Lomibao and Zhu (2005) who focus on credit exposure computations, but employ a highly restrictive type of information sets that consist of only two observations — at a start date and at a single future simulation date – which is not the information set available in the most credit risk systems. Their typical information structure consists of multiple underlying asset observations at all the preceding discrete simulation dates.

In this paper, we provide analytical valuation formulas of barrier options with three different types of information: continuous observations of the underlying asset value, delayed continuous observations, and multiple discrete observations. The first type of information is used for derivative valuations on underlyings with high-frequency data, such as in the foregn exchange (FX) market. The second type of information is the information available for most Monte Carlo-based market risk systems which compute the value-at-risk (VaR) metric. The third type of information is the information available in most of the credit risk systems. The time of barrier crossing is a predictable time in the full information filtration generated by continuous observations of the underlying. Jeanblanc and Valchev (2005) demonstrate that the time of barrier hitting changes to a totally inaccessible random time in the filtrations generated by delayed continuous observations and observations at discrete simulation (observation) times and derive closed-form analytical expressions for the conditional probability of barrier-crossing in a multi-period setting. In this paper, we provide multi-period conditional valuations of single and double barrier options. We show that the multiplicative separability of the barrier option value in the case of no barrier hit and the probability of no hit is valid only if the option value is replaced in the decomposition by a newly issued barrier option which retains the other terms and conditions of the old barrier option.

In addition, we consider the barrier option valution in cases where the discrete asset value information is further obscured by some observation noise – an information structure typical for unlisted stocks and other illiquid underlying assets. In the context of lognormal structural credit risk models, the influence of such information on default probabilities has been studied in Duffie and Lando (2001). For normal asset returns, we present explicit solutions based on the multiperiod barrier crossing probability of a Brownian bridge observed in noise.

Market and credit risk management applications of the valuations with incomplete information are also discussed. Panayotov and Bakshi (2009) studied the intra-horizon risk in VaR models for general Levy processes by using numerical solutions for the first passage probability of a Levy bridge process to a boundary. However, these authors treat only the cases with discrete information consisting of observations at just two dates – the start date and the single risk horizon date. We obtain explicit analytical solutions in geometric Brownian motion setting for more general information structures with delayed continuous and discrete information generated of multiple observations. The first two incomplete information filtrations converge to the full information as the distance between the observation dates converges. The non-conditional valuation of barrier options is based on the wrong assumption that the barrier crossing can occur only on the discrete observation dates, but not between them. We find that the difference in risk numbers increases with the observation lags. Also, the convergence of the risk numbers is faster for barrier options, whose barriers are set initially far away from the underlying asset values. For barrier options with barrier levels very close to the underlying asset value, using non-conditional valuation could prove costly and could lead to an incorrect assessment of the risk of portfolios containing barrier options.

The rest of this paper is organized as follows. Section 2 describes the classical case with full, continuous information about the underlying asset. Section 3 deals with the case of delayed continuous information, a reduction of the full information, where continuous historical information is obtained with some observation lag. Using this information, we derive the conditional probability of a first passage to a barrier level and obtain closed-form conditional valuation formulas for digital barrier options and discuss the valuation errors caused by not using conditional valuation consistent with the available information structure. Section 4 investigates the case of discrete observations of the underlying asset at multiple discrete times and computes the conditional probability of barrier crossing in this filtration. Section 5 offers explicit closed-form solutions for the main types of single barrier options with discrete information and parity results. In Section 6, we investigate the case of discrete and obscure information where the underlying asset value is observed with noise. This is the usual information set for unlisted stocks and illiquid assets. Section 7 presents credit risk management applications of the methodology as well as specific examples based on Monte Carlo simulations for FX barrier options. Section 8 provides valuation results for double barrier options and Section 9 concludes our paper.

2 Full information: continuous observations

Consider an option underlying, V, and a barrier level α, which is lower than the initial underlying value $V_0 = v$. Assume that under a risk-neutral probability, P, in a lognormal model, the underlying satisfies

$$dV_t = V_t((r-\delta)dt + \sigma dW_t), \tag{1}$$

where r is the short-term interest rate,[1] $(W_t, t \geq 0)$ is a (P, F_t^V)-Brownian motion. The solution of (1) is $V_t = ve^{\sigma(W_t + \nu t)}$, where $\nu = \frac{1}{\sigma}(r - \delta - \frac{\sigma^2}{2})$. We rewrite the solution in the more convenient form

[1]If the underlying is a dividend-paying stock, δ is the continuous dividend rate. In FX models, δ is the foreign short-term interest rate, while in the spot commodity models, it is the continuous storage cost.

$$V_t = ve^{\sigma X_t},$$

where $X_t = \nu t + W_t$ is a (P, F_t^V)-Brownian motion with drift. It is important to notice that the filtration F^V is equal to the filtrations F^X and F^W generated by X and W, respectively.

The time of hitting the barrier τ is given by

$$\tau = \inf\{t: V_t \le \alpha\} = \inf\{t: X_t \le a\} \tag{2}$$

with $X_t = \frac{1}{\sigma}\ln(V_t/v)$ and $a = \frac{1}{\sigma}\ln(\alpha/v)$.

We denote the probability that the process $(X_s = W_s + \nu s, s \ge 0)$ remains above the barrier z till time t by

$$\Phi(\nu, t, z) = P\left(\inf_{s\le t} X_s > z\right). \tag{3}$$

It is important to notice that this probability depends on the drift rate ν. The reflection principle and elementary considerations lead to

$$\Phi(\nu, t, z) = N\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} N\left(\frac{z + \nu t}{\sqrt{t}}\right), \qquad \textit{for } z < 0, t > 0,$$

$$\Phi(\nu, t, z) = 0, \qquad \textit{for } z \ge 0, t \ge 0,$$

$$\Phi(\nu, 0, z) = 1, \qquad \textit{for } z < 0.$$

Note that Φ, being a probability, satisfies $0 \le \Phi \le 1$.

The event that the barrier has not been reached up to some future date T can be expressed as

$$\{\tau > T\} = \left\{\inf_{s\le T} V_s > \alpha\right\} = \left\{\inf_{s\le t} V_s > \alpha\right\} \cap \left\{\inf_{t<s\le T} V_s > \alpha\right\}$$

$$= \left\{\inf_{s\le t} X_s > a\right\} \cap \left\{\inf_{t<s\le T} X_s - X_t > a - X_t\right\}$$

$$= \left\{ \inf_{s \le t} X_s > a \right\} \cap \left\{ \inf_{s \le T-t} \widetilde{X}_{s-t} > a - X_t \right\},$$

where $\widetilde{X} = (\widetilde{X}_u = X_{t+u} - X_t, u \ge 0)$ is independent of F_t^V. Because of the stationarity and the independence of the increments of the Brownian motion, the process $(\widetilde{X}_u, u \ge 0)$ is a $(P, F_{t+u}^V)-$Brownian motion with drift.

The conditional probability of not reaching the barrier until T, $P(\tau > T \mid F_t^V)$, can be derived as follows

$$P(\tau > T \mid F_t^V) = P\left(\left\{ \inf_{s \le t} X_s > a \right\} \cap \left\{ \inf_{t < s \le T} V_s > \alpha \right\} \mid F_t^V \right)$$

$$= 1\!\!1_{t<\tau} P\left(\inf_{t \le s \le T} V_s > \alpha \mid F_t^V \right)$$

$$= 1\!\!1_{t<\tau} P\left(\inf_{s \le T-t} \widetilde{X}_s > a - X_t \mid F_t^V \right)$$

$$= 1\!\!1_{t<\tau} P\left(\inf_{s \le T-t} \widetilde{X}_s > a - x \right)|_{x=X_t}$$

$$= 1\!\!1_{t<\tau} \Phi(\nu, T-t, a-X_t), \tag{4}$$

where $\widetilde{X}_s = (X_{t+s} - X_t, t \ge 0)$. In what follows, we will suppress the first coordinate of $\Phi(\nu, T-t, a-X_t)$ and will write $\Phi(T-t, a-X_t)$.

The value of a down-and-out digital call option with maturity T in the filtration F^V at time t is

$$DIG_C(t) = E(e^{-r(T-t)} 1\!\!1_{\tau>T} \mid F_t^V) = e^{-r(T-t)} 1\!\!1_{\tau>t} \Phi(T-t, a-X_t). \tag{5}$$

3 Delayed continuous information

In many situations, it is costly or impossible to monitor the value of the underlying continuously and information about all the past values of V is collected on discrete dates. Such cases arise for less liquid underlying assets, such as unlisted stocks or illiquid equity, whose prices can be obtained with some observation lag. Other cases are in the market risk VaR Monte Carlo computations. On the risk horizon date, barrier options should be evaluated conditional on delayed continuous historical information and the simulated underlying value.

Denote by $F^1 = (F_t^1, t \geq 0)$ the filtration generated by the observations of current and all the past V at times $t_1, \ldots, t_n$ with $t_n \leq t < t_{n+1}$

$$F_t^1 = \{\emptyset, \Omega\} \qquad \text{for } t < t_1,$$

$$F_t^1 = F_{t_1}^V = \sigma(V_s, s \leq t_1) \quad \text{for } t_1 \leq t < t_2,$$

$$F_t^1 = F_{t_n}^V = \sigma(V_s, s \leq t_n) \quad \text{for } t_n \leq t < t_{n+1}.$$

We write $D_t^1 := P(\tau \leq t \mid F_t^1)$ for the F^1-conditional barrier-hitting probability. We easily extend the previous result for full information to the case of discrete observation times. For $t_j \leq t < t_{j+1}$, we can write

$$D_t^1 = P(\tau \leq t \mid F_t^1) = 1 - P(\tau > t \mid F_{t_j}^V)$$

$$= 1 - P\left(\inf_{s<t} X_s > a \mid F_{t_j}^V\right) = 1 - 1\!\!1_{\tau > t_j} P\left(\inf_{t_j \leq s < t} X_s > a \mid F_{t_j}^V\right)$$

$$= 1 - 1\!\!1_{\tau > t_j} \Phi(t - t_j, a - X_{t_j}).$$

The value of a down-and-out digital call option with maturity T in the filtration F^1 at time t is

$$DIG_C^1(t) = E(e^{-r(T-t)} 1\!\!1_{\tau > T} \mid F_t^1) = e^{-r(T-t)} 1\!\!1_{\tau > t_j} \Phi(T - t_j, a - X_{t_j}). \tag{6}$$

Figure 1 compares the values of a digital FX barrier call option with full and delayed continuous information arriving every 1.5 months. While $DIG_C(t)$ varies with the underlying asset price, $DIG_C^1(t)$ value is almost constant between two observation dates and is affected only by the changes of the discount factor with time. At each observation date, $DIG_C^1(t_j)$ jumps to the full information price $DIG_C(t_j)$. Thus, discrete information arrivals induce jump-discontinuities in $DIG_C^1(t)$ at the observation dates.

The amount at risk at the barrier level is known as "parity". For a regular knock-out (down-and-out) option, parity is small. Under incomplete information, the risk exposure of the option increases dramatically. The pricing errors when using complete information formulas under conditions of imperfect information are larger for deep out-of-the money regular knock-out options.

3.1 Market risk applications

Let us consider a down-and-out digital call option with maturity T. Write $DIG_C(t)$ for its value at time t and $DIG_C^{CV-mr}(t)$ for its conditional value in the information set available to the market risk manager. In the market risk VaR computations, based on Monte Carlo simulations, there are two sets of present value computations: on the initial date, t_j and on the risk horizon date, t_{j+1} While the first set of present value computations is in the full information filtration F, the second set of present value computations should be in the delayed continuous information filtration F^1, enlarged by the simulated underlying value $V_{t_{j+1}}$.

The conditional value of a down-and-out digital call option with the available market risk information $F^1_{t_j} \vee V_{t_{j+1}}$ at time t_{j+1} is

$$DIG_C^{CV\_mr}(t_{j+1}) = E(e^{-r(T-t_{j+1})} 1\!\!1_{\tau>T} \mid F^1_{t_j} \vee X_{t_{j+1}})$$

$$= DIG_C(t_{j+1}) 1\!\!1_{\tau>t_j} P\left(\tau > t_{j+1} \mid F^1_{t_j} \vee X_{t_{j+1}}\right)$$

$$= e^{-r(T-t_{j+1})} 1\!\!1_{\tau>t_j} \left(1 - e^{-\frac{2}{t_{j+1}-t_j} a(a-X_{t_{j+1}})}\right) \Phi(T-t_{j+1}, a - X_{t_{j+1}}), \tag{7}$$

where the second equality follows, because of the event decomposition

$$\{\tau > T\} = \{\tau > t_j\} \cap \{\tau \notin (t_j, T]\},$$

(5), because $F^1_{t_j} \vee V_{t_{j+1}} \subset F^V_{t_{j+1}}$ and because on the interval $(t_j, t_j + 1]$, the process X becomes a Brownian bridge. The barrier crossing probability of a Brownian bridge, conditional on its start and end point, is well known and the exact computations can be found, for example, in Jeanblanc and Valchev (2005).

The currently available market risk systems do not use conditional valuation, that is, they omit the term $\left(1 - \exp\left(-\frac{2}{t_{j+1}-t_j} a(a - X_{t_{j+1}})\right)\right)$, ignoring the probability of barrier crossing between the initial date and the risk horizon date. This leads to overpricing of knock-out barrier options and underpricing of knock-in options. The incorrect valuations are larger for longer horizon market risk measures, such as 10-day VaR and 30-day VaR since the conditional probability of barrier crossing is increasing in the first coordinate.

4 Discrete information

Suppose that information is generated by observations of the underlying value on a discrete sequence of dates.

4.1 Information structure

We denote by $F^2 = (F_t^2, t \geq 0)$ the filtration generated by V at dates $t_1, \ldots, t_n$. It follows that F_t^2 is trivial for $t < t_1$,

$$F_t^2 = \{\emptyset, \Omega\} \qquad \text{for } t < t_1,$$

$$F_t^2 = F_{t_1}^2 = \sigma(V_{t_1}) = \sigma(X_{t_1}) \quad \text{for } t_1 \leq t < t_2,$$

$$F_t^2 = F_{t_2}^2 = \sigma(V_{t_1}, V_{t_2}) = \sigma(X_{t_1}, X_{t_2}) \quad \text{for } t_2 \leq t < t_3,$$

and so on. At each observation date t_j, the filtration F^2 is enlarged with the observation of the underlying at that date — that is, $F_{t_j}^2 = F_{t_{j-1}}^2 \vee \sigma(V_{t_j})$.

Therefore the modeller has access to a subfiltration of the filtration F^V (i.e., $F^2 \subset F^V$). We denote the conditional barrier-hitting probability with respect to F^2 by $D_t^2 = P(\tau \leq t \mid F_t^2)$, where the superscript stands for the second type of information.

4.2 Conditional barrier-hitting probability with multiple discrete observation times

The general formula for the multi-step barrier-hitting conditional probability in the filtration F^2 is derived in Jeanblanc and Valchev (2005) by taking sequential conditional expectations and is

$$P(\tau \le t_j \mid F_{t_j}^2) = 1 \qquad if \;\; X_{t_k} < a \text{ for at least one } t_k, t_k \le t_j$$

$$= 1 - P_j^a, \quad \text{otherwise}$$

where

$$P_j^a = \left(1 - \exp\left(-\frac{2}{t_1} a(a - X_{t_1})\right)\right)$$

$$\times \left(1 - \exp\left(-\frac{2}{t_2 - t_1}(a - X_{t_1})(a - X_{t_2})\right)\right)$$

$$\cdots$$

$$\times \left(1 - \exp\left(-\frac{2}{t_j - t_{j-1}}(a - X_{t_{j-1}})(a - X_{t_j})\right)\right).$$

Jeanblanc and Valchev prove that the Doob-Meyer decomposition of the conditional barrier-hitting probability D^2 is

$$D_t^2 = \zeta_t^2 + (D_t^2 - \zeta_t^2),$$

where ζ^2 is an F^2-martingale and $D^{2,c} := D^2 - \zeta^2$ is a continuous, hence predictable, increasing process. From the properties of F^2, ζ^2 is the pure jump process defined by

$$\zeta_t^2 = \sum_{i, t_i \le t} \Delta D_{t_i}^2.$$

5 Valuation of single barrier options with discrete information

Let us consider a discrete grid of simulation times $t_k, k = 1, \ldots, K$. The problem is to evaluate a barrier option at simulation time t_j, belonging to the grid of simulation times conditional on the filtration F^2. Let V_{t_0} be the initial underlying value and V_t its time t value,

$X_t = \dfrac{1}{\sigma}\ln(V_t/V_{t_0})$ — the transformed underlying.[2] Likewise, α is the original barrier, while a is the transformed barrier defined as $a = \dfrac{1}{\sigma}\ln(\alpha/V_{t_0})$.

Consider an "out" call barrier option. The time $t_j, t_j < T$ value of the option in the full information filtration F^V is

$$C_{bar}^{O}(t_j) = E\left\{e^{-r(T-t_j)}\left((V_T - K)^{+}1\!\!1_{\tau > T} + R1\!\!1_{\tau \le T}\right) | F_{t_j}^{V})\right\} \tag{8}$$

where R is the rebate, which is paid at the maturity date. The conditional value of the option at time t_j in F^2 is

$$CV_{bar}^{O}(t_j) = E\left\{e^{-r(T-t_j)}\left((V_T - K)^{+}1\!\!1_{\tau > T} + R1\!\!1_{\tau \le T}\right) | F_{t_j}^{2}\right\}$$

It can be developed further by taking into account that

$$1\!\!1_{\tau \le T} = 1\!\!1_{\tau \le t_j} + 1\!\!1_{t_j < \tau \le T} = 1\!\!1_{\tau \le t_j} + 1\!\!1_{\tau > t_j}1\!\!1_{\tau \le T}$$

and

$$1\!\!1_{\tau > T} = 1\!\!1_{\tau > t_j}1\!\!1_{\tau > T},$$

which leads to

$$\begin{aligned}
CV_{bar}^{O}(t_j) &= E\left\{e^{-r(T-t_j)}\left((V_T - K)^{+}1\!\!1_{\tau > T} + R1\!\!1_{\tau \le T}\right) | F_{t_j}^{2})\right\} \\
&= E\left\{e^{-r(T-t_j)}\left((V_T - K)^{+}1\!\!1_{\tau > T} + R(1\!\!1_{\tau \le t_j} + 1\!\!1_{t_j < \tau \le T})\right) | F_{t_j}^{2})\right\} \\
&= E\left\{e^{-r(T-t_j)}\left((V_T - K)^{+}1\!\!1_{\tau > t_j}1\!\!1_{\tau > T} + R1\!\!1_{t_j < \tau \le T}\right) | F_{t_j}^{2}\right\} \\
&\quad + e^{-r(T-t_j)}R \times P(\tau \le t_j \,|\, F_{t_j}^{2})
\end{aligned}$$

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[2]Note that $X_{t_0} = 0$.

$$= E\left\{e^{-r(T-t_j)}\left((V_T - K)^+ 1\!\!1_{\tau > t_j} 1\!\!1_{\tau > T} + R 1\!\!1_{\tau > t_j} 1\!\!1_{\tau \le T}\right) | F^2_{t_j}\right\}$$

$$+ e^{-r(T-t_j)} R \times P(\tau \le t_j \mid F^2_{t_j})$$

$$= E\left\{e^{-r(T-t_j)}\left((V_T - K)^+ 1\!\!1_{\tau > T} + R 1\!\!1_{\tau \le T}\right) 1\!\!1_{\tau > t_j} \mid F^2_{t_j}\right\}$$

$$+ e^{-r(T-t_j)} R \times P(\tau \le t_j \mid F^2_{t_j})$$

$$= C^{O_{new}}_{bar}(t_j) P(\tau > t_j \mid F^2_{t_j}) + e^{-r(T-t_j)} R \times P(\tau \le t_j \mid F^2_{t_j}), \tag{9}$$

where $C^{O_{new}}_{bar}(t_j)$ is a new barrier option with the same terms and the conditions as the old one, except that it is a newly starting option with issue date t_j. This new concept is introduced because $C^{O}_{bar}(t_j) 1\!\!1_{\tau > t_j} = C^{O_{new}}_{bar}(t_j) 1\!\!1_{\tau > t_j}$ and since the old barrier option value $C^{O}_{bar}(t_j)$ is not $F^2_{t_j}$-measurable, while $C^{O_{new}}_{bar}(t_j)$ is, and can be taken out of the expectation.[3] The equation before the last follows since $F^2_{t_j} \subseteq F^V_{t_j}$.

5.1 Knock-out options

An "out" barrier option is similar to a plain European option except that it becomes worthless if the underlying asset price rises above (for up options) or falls below (for down options) the barrier. In that event, the rebate is paid out either immediately or at maturity, depending on the setting of the rebate.

5.1.1 Down-and-out option

[3] The old barrier option could have been knocked out on its path, an event which is not necessarily observable in the discrete information filtration F^2.

The hitting time for the barrier is defined as

$$\tau = \inf\{t: V_t \le \alpha\} = \inf\{t: X_t \le a\}.$$

Before being knocked out on its path, the option value is that of a standard down-and-out barrier option C_{bar}^{DO}, while after the knock out time, its value is the rebate R. Thus the value of the barrier option at simulation time t_j can be decomposed as

$$CV_{bar}^{DO}(t_j) = \begin{cases} C_{bar}^{DO}(t_j, V_{t_j}) & \text{on the set } \{\tau > t_j\} \\ Re^{-r(T-t_j)} & \text{on the set } \{\tau \le t_j\} \end{cases}. \tag{10}$$

Using (9), the conditional value of the down-and-out option in the filtration F^2 on the simulation date t_j is the weighted average of the two values above with the respective probabilities of the two sets

$$CV_{bar}^{DO}(t_j) = \begin{cases} C_{bar}^{DO_{new}}(t_j, V_{t_j}) \times P_j^a + Re^{-r(T-t_j)} \times (1 - P_j^a) & if\ X_{t_k} \ge a \text{ for all } t_k,\, t_k \le t_j \\ Re^{-r(T-t_j)} & if\ X_{t_k} < a \text{ for at least one } t_k,\, t_k \le t_j \end{cases}, \tag{11}$$

where

$$P_j^a = \prod_{k=1}^{j} \left(1 - \exp\left(-\frac{2}{t_k - t_{k-1}} (a - X_{t_{k-1}})(a - X_{t_k}) \right) \right)$$

is the conditional with $t_0 = 0$.

5.1.2 Up-and-out option

The hitting time for the up-and-out option is defined as

$$\tau = \inf\{t: V_t \ge \alpha\} = \inf\{t: X_t \ge a\}.$$

Before being knocked out on its path, the option value is that of a standard up-and-out barrier option C_{bar}^{UO}. After the knock out time, its value is the rebate R. Hence, the conditional value of the barrier option at simulation time t_j is

$$CV_{bar}^{UO}(t_j) = \begin{cases} C_{bar}^{UO}(t_j, V_{t_j}) & \text{on the set } \{\tau > t_j\} \\ Re^{-r(T-t_j)} & \text{on the set } \{\tau \le t_j\} \end{cases}. \tag{12}$$

The conditional value of the DO option in the filtration F^2 on the simulation date t_j is the weighted average of the two values above with the respective probabilities of the two sets

$$CV_{bar}^{UO}(t_j) = \begin{cases} C_{bar}^{UO_{new}}(t_j, V_{t_j}) \times P_j^a + Re^{-r(T-t_j)} \times (1 - P_j^a) & \text{if } X_{t_k} < a \text{ for all } t_k, t_k \le t_j \\ Re^{-r(T-t_j)} & \text{if } X_{t_k} \ge a \text{ for at least one } t_k, t_k \le t_j \end{cases},$$

(13)

where

$$P_j^a = \prod_{k=1}^{j} \left(1 - \exp\left(-\frac{2}{t_k - t_{k-1}} (a - X_{t_{k-1}})(a - X_{t_k}) \right) \right)$$

with $t_0 = 0$.

5.2 Knock-in options

A knock-in barrier option is similar to an out-barrier option except that the payout only occurs if the underlying asset price rises above (for up options) or falls below (for down options) the barrier. Otherwise, the rebate is paid at maturity.

5.2.1 Down-and-in option

Before being knocked in on its path, the option value is that of a standard down-and-in barrier option C_{bar}^{DI}. After the knock in time, it is transformed into a standard European option C_{Euro}. Hence, the conditional value of a barrier option at simulation time t_j is

$$CV_{bar}^{DI}(t_j) = \begin{cases} C_{bar}^{DI}(t_j, V_{t_j}) & \text{on the set } \{\tau > t_j\} \\ C_{Euro}(t_j) & \text{on the set } \{\tau \le t_j\} \end{cases}. \tag{14}$$

The conditional value of the down-and-in option in the filtration F^2 on the simulation date t_j is the weighted average of the two values in (14) with the respective probabilities of the two sets

$$CV_{bar}^{DI}(t_j) = \begin{cases} C_{bar}^{DI_{new}}(t_j, V_{t_j}) \times P_j^a + C_{Euro}(t_j) \times (1 - P_j^a) & if \;\; X_{t_k} > a \text{ for all } t_k, t_k \le t_j \\ C_{Euro}(t_j) & if \;\; X_{t_k} \le a \text{ for at least one } t_k, t_k \le t_j \end{cases}, \tag{15}$$

where P_j^a is defined as for the down-and-out option.

5.2.2 Up-and-in option

Before being knocked in on its path, the option value of an up-and-in options is that of a standard up-and-in barrier option C_{bar}^{UI}. After the knock in time, it is transformed into a standard European option C_{Euro}. Hence, the conditional value of the barrier option at simulation time t_j is

$$CV_{bar}^{UI}(t_j) = \begin{cases} C_{bar}^{UI}(t_j, V_{t_j}) \times (1 - P_j^a) & \text{on the set } \{\tau > t_j\} \\ C_{Euro}(t_j) & \text{on the set } \{\tau \le t_j\} \end{cases} \tag{16}$$

The conditional value of the up-and-in option in the filtration F^2 on the simulation date t_j is the weighted average of the two values in (16) with the respective probabilities of the two sets

$$CV_{bar}^{UI}(t_j) = \begin{cases} C_{bar}^{UI_{new}}(t_j, V_{t_j}) \times P_j^a + C_{Euro}(t_j) \times (1 - P_j^a) & if \;\; X_{t_k} < a \text{ for all } t_k, t_k \le t_j \\ C_{Euro}(t_j) & if \;\; X_{t_k} \ge a \text{ for at least one } t_k, t_k \le t_j \end{cases}, \tag{17}$$

where P_j^a is defined as for the up-and-out option.

5.3 Parity conditions for barrier options

Here we present derivation of the standard parity conditions for single barrier options. Consider a portfolio of one up-and-out option and one up-and-in barrier option with the same remaining terms and conditions. The value of this portfolio at simulation time t_j is

$$
\begin{aligned}
C_{bar}^{UO}(t_j) + C_{bar}^{UI}(t_j) &= \left(C_{bar}^{UO}(t_j) + C_{bar}^{UI}(t_j)\right)1\!\!1_{\tau \le T} + \left(C_{bar}^{UO}(t_j) + C_{bar}^{UI}(t_j)\right)1\!\!1_{\tau > T} \\
&= \left(C_{bar}^{UO}(t_j)1\!\!1_{\tau \le T} + C_{bar}^{UI}(t_j)1\!\!1_{\tau \le T}\right) + \left(C_{bar}^{UO}(t_j)1\!\!1_{\tau > T} + C_{bar}^{UI}(t_j)1\!\!1_{\tau > T}\right) \\
&= \left(0 + C_{Euro}(t_j)1\!\!1_{\tau \le T}\right) + \left(C_{Euro}(t_j)1\!\!1_{\tau > T} + 0\right) \\
&= C_{Euro}(t_j).
\end{aligned}
$$

Repeating similar computations for a portfolio of a down-and-out option and down-and-in option, one has

$$C_{bar}^{DO}(t_j) + C_{bar}^{DI}(t_j) = C_{Euro}(t_j).$$

These parity conditions are valid for any information structure. Thus, information reductions change the relative valuations of $C_{bar}^{UO}(t_j)$ and $C_{bar}^{UI}(t_j)$. It is important to note that the partial information sets change the value of the portfolio of the two options, which is the value of the European option with the same remaining terms and conditions, but without the barrier feathure $C_{Euro}(t_j)$ since, in general, $C_{Euro}(t_j) \neq E\left[C_{Euro}(t_j) \middle| F_t^k\right]$, where $k = 1,\ldots,3$ denotes the available partial information set.

6 Noisy observations of illiquid underlying assets

There are assets such as unlisted stocks and illiquid assets, for which the modeler does not observe the market prices exactly. Instead, he gets some estimate of the underlying asset value based on expert estimates or accounting information. Incomplete accounting information in the context of credit risk models has been introduced in Duffie and Lando (2001). The available asset price information is obscured by some noise process. We present a methodology for evaluation of barrier options with such information.

We assume that, at each observation date, the process X, which determines the underlying asset value, is observed in noise. Write

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i}, \qquad \varepsilon_t \perp X_t, \tag{18}$$

where ε_{t_i} is the observation noise noise. Write F^3 for the filtration generated by the observation process, i.e., $F^3 = (Y_t, t \geq 0)$, where $F_t^3 = (Y_s, s \leq t)$.

Our aim is to compute the probability of not reaching a lower barrier a conditional on this information, i.e.,

$$P(\inf_{s \leq T} X_s > a \mid Y_t).$$

Consider first information of one noisy observation of the underlying asset. For $t_1 \leq t < t_2$, $F_t^3 = \sigma(X_{t_1} + \varepsilon_{t_1})$. We have

$$P\left(\min_{s \leq T} X_s > a \mid X_{t_1} + \varepsilon_{t_1} = x_1\right) = \frac{P\left(\min_{s \leq T} X_s > a, X_{t_1} + \varepsilon_{t_1} \in dx_1\right)}{P\left(X_{t_1} + \varepsilon_{t_1} \in dx_1\right)}.$$

The numerator can be expressed as

$$P\left(\min_{s \leq T} X_s > a, X_{t_1} + \varepsilon_{t_1} \in dx_1\right) = \int_{-\infty}^{+\infty} P\left(\min_{s \leq T} X_s > a, X_{t_1} + u \in dx_1\right) f(u) du,$$

where $f(u)$ is the density of ε_{t_1}.

We can write

$$P\left(\min_{s \leq T} X_s > a, X_{t_1} + u \in dx_1\right) = P\left(\min_{s \leq T} X_s > a, X_{t_1} \in dx_1 - u\right)$$

$$= P\left(\min_{s \leq t_1} X_s > a, \min_{t_1 < s \leq T} X_s > a, X_{t_1} \in dx_1 - u\right)$$

$$= P\left(\min_{s \le t_1} X_s > a, \min_{t_1 < s \le T} X_s - X_{t_1} > a - X_{t_1}, X_{t_1} \in dx_1 - u \right)$$

$$= E\left(1\!\!1_{\min_{s \le t_1} X_s > a} 1\!\!1_{X_{t_1} \in dx_1 - u} 1\!\!1_{\min_{t_1 < s \le T} X_s - X_{t_1} > a - X_{t_1}} \right)$$

$$= E\left(1\!\!1_{\min_{s \le t_1} X_s > a} 1\!\!1_{X_{t_1} \in dx_1 - u} E\left(1\!\!1_{\min_{t_1 < s \le T} X_s - X_{t_1} > a - X_{t_1}} \mid F_{t_1}^X \right) \right)$$

$$= E\left(1\!\!1_{\min_{s \le t_1} X_s > a} 1\!\!1_{X_{t_1} \in dx_1 - u} E\left(1\!\!1_{\min_{t_1 < s \le T} X_s - X_{t_1} > a - X_{t_1}} \mid X_{t_1} \right) \right)$$

$$= E\left(1\!\!1_{\min_{t_1 < s \le T} X_s - X_{t_1} > a - X_{t_1}} \mid X_{t_1} = dx_1 - u \right) P\left(\min_{s \le t_1} X_s > a, X_{t_1} \in dx_1 - u \right)$$

$$= \Phi(T - t_1, a - x_1 + u) P\left(\min_{s \le t_1} X_s > a, X_{t_1} \in dx_1 - u \right)$$

$$= \left(1 - \exp\left(-\frac{2}{t_1} a(a - x_1 + u)\right) \right) \Phi(T - t_1, a - x_1 + u) dx_1.$$

Since the process X is an arithmetic Brownian motion with drift rate ν and volatility coefficient 1, $X_t \sim N(\nu t, t)$.

Assume that the noise is Gaussian and its variance is proportional to the observation lag $\varepsilon_{t_1} \sim N(0, \sigma_\varepsilon^2 t_1)$. Using the independence between X_{t_1} and the noise, we have

$$Y_{t_1} = X_{t_1} + \varepsilon_{t_1} \sim N(\nu t_1, (\sigma_\varepsilon^2 + 1) t_1).$$

Hence,

$$P\left(X_{t_1} + \varepsilon_{t_1} \in dx_1 \right) = \frac{1}{\sqrt{2\pi(\sigma_\varepsilon^2 + 1) t_1}} \exp\left(\frac{(X_{t_1} + \varepsilon_{t_1} - \nu t_1)^2}{2(\sigma_\varepsilon^2 + 1) t_1} \right) dx_1$$

$$= \frac{1}{\sqrt{2\pi(\sigma_\varepsilon^2+1)t_1}} \exp\left(\frac{(x_1+u-\nu t_1)^2}{2(\sigma_\varepsilon^2+1)t_1}\right) dx_1. \tag{19}$$

6.1 Multiple noisy observations

Consider the case where the available information set consists of noisy observations of the underlying asset value at multiple discrete dates times $t_1,\ldots,t_n$ with $t_n \le t < t_{n+1}$. In this case, the information set at time t is $F_t^3 = \sigma(X_{t_1}+\varepsilon_{t_1},\ldots,X_{t_n}+\varepsilon_{t_n})$ with $\varepsilon_{t_j} \sim N(0,\sigma_{\varepsilon_j}^2 t_j)$ for $j=1,\ldots,n$.

Using the computations from the beginning of this section, and taking consecutive conditional expectations, as in Jeanblanc and Valchev (2005) with discrete non-noisy observations, leads to

$$P\left(\min_{s\le T} X_s > a, X_{t_1}+u_1 \in dx_1,\ldots,X_{t_n}+u_n \in dx_n\right) = P\left(\min_{s\le T} X_s > a, X_{t_1} \in dx_1-u_1,\ldots,X_{t_n} \in dx_n-u_n\right)$$

$$= \Phi(T-t_n, a-x_n+u_n)$$
$$\times\left(1-\exp(-\frac{2}{t_1}a(a-x_1+u_1))\right)\times\ldots\times\left(1-\exp(-\frac{2}{t_n-t_{n-1}}(a-x_{n-1}+u_{n-1})(a-x_n+u_n))\right)dx_1\ldots dx_n. \tag{20}$$

With the assumption of independence of the estimation errors for different periods, we have

$$P\left(X_{t_1}+\varepsilon_{t_1} \in dx_1,\ldots,X_{t_n}+\varepsilon_{t_n} \in dx_n\right)=$$

$$= \frac{1}{\sqrt{\prod_{j=1}^{n} 2\pi\left[(\sigma_{\varepsilon_j}^2+1)t_j\right]}} \exp\left(\sum_{j=1}^{n}\frac{(X_{t_j}+\varepsilon_{t_j}-\nu t_j)^2}{2(\sigma_{\varepsilon_j}^2+1)(t_j-t_{j-1})}\right)dx_1\ldots dx_n$$

$$= \frac{1}{\sqrt{\prod_{j=1}^{n} 2\pi\left[(\sigma_{\varepsilon_j}^2+1)t_j\right]}} \exp\left(\sum_{j=1}^{n}\frac{(X_{t_j}+u_j-\nu t_j)^2}{2(\sigma_{\varepsilon_j}^2+1)t_j}\right)dx_1\ldots dx_n .$$

The conditional value of the digital barrier option in the filtration $F^3 = (Y_t, t \geq 0)$ can be computed as in (9) by taking conditional expectations with respect to F^3 instead of F^2:

$$CV_{bar}^{O,3}(t_j) = C_{bar}^{O_{new}}(t_j) P(\tau > t_j \mid F_{t_j}^3) + e^{-r(T-t_j)} R \times P(\tau \leq t_j \mid F_{t_j}^3). \tag{21}$$

It is easy to notice that the prices of knock-in barrier options are decreasing in the degree of the observation noise as measured by its volatility σ_ε.

7 Credit risk management applications

In the modern credit risk system available to banks, counterparty risk is measured not at a single time, but over some prespecified future period. The potential future exposure (PFE) measures the close-out value of a portfolio consisting of financial instruments at some confidence level (usually 95%). In contrast to market risk, the high-risk scenarios are those with high portfolio value because the higher the portfolio value is, the higher the loss associated with counterparty default. Thus PFE measures the credit exposure under plausible, but extreme risk-factor scenarios. The scenarios generating PFE are used also for Exposure at Default (EAD) and effective maturity computations, quantities which are used for minimum capital requirement for corporate exposure under the Basel II rules. For PFE computations, it is important to evaluate the derivative instruments in a bank's portfolio consistently with the information structure represented by risk factor scenarios. These scenarios are generated on a prespecified grid of discrete simulation dates and the information consists of financial variables observed at a sequence of discrete simulation times (i.e., the second type of incomplete information that we described in Section 4).

Lomibao and Zhu (2005) present credit risk applications of conditional valuation, but they consider the very restrictive case when the information consists of the underlying values at just two dates — the start date and the risk horizon date, referred to as the direct-jump to the simulation date (DJS) approach. However, the DJS approach is not used in commercially available credit risk systems since there is no consistency between scenarios at different horizons on the same path. Simply the value of the financial variable at time t_j depends on its value at t_{j-1}, a feature which is not captured in the DJS approach. Most credit exposure systems utlize path-dependent-simulation (PDS) with multiple simulation times. We provide solutions in a multi-period setting with PDS with scenario generations typical for a bank's credit risk system, requiring derivation of the credit exposure over the life of the trade.[4]

The underlying asset for a barrier option is some financial variable and, in turn, the financial variable is a function of one or more risk factors. A scenario for the financial variable is obtained by assigning values to all the financial variables used in the calculation at a particular future time, while a path is a sequence of scenarios for all future times. There are, logically, several steps to modeling the dynamics of a financial variable and they are:

1. Generate paths for the risk factors, such as exchange rates and certain interest rates.
2. Generate paths for the financial variable which depends directly on the risk factors, such as spot FX rates, stock prices and the yield curve.
3. Generate paths for the financial variables which depend directly on other financial variables (modeled in step 2), such as swap prices, caps, floors, swaptions and forwards.

[4] Some advanced market risk management system such as the new Risk Metrics (2006) model also use multi-step path-dependent simulations.

For FX single barrier options, the conditional probabilities of hitting the barriers between the simulation dates depend on the FX barrier, the spot FX rate level at the start of the simulations, and the historical FX rate drift and volatility coefficients, which are computed from historical data. For double barrier options, the conditional value depends on the two rebates and on the conditional probabilities of hitting the upper and lower barrier. In this case, there are four possible combinations of barrier-crossing events.

Figure 2 illustrates the PFE profile of an up-and-out call barrier option on the USD/EUR rate with barrier level approximately 9% above the initial spot rate with conditional and non-conditional valuation. The PFE 95 of the barrier option is convex in contrast to the concave 95-th quantile of a standard European option. In the scenarios with extremely high FX rates, the value of the underlying European call option is increasing, but the option gets knocked out more frequently. The price impact of the second events is stronger and the PFEs are decreasing for longer horizons. For short simulation times, the PFEs with conditional and non-conditional valuation are smoothly varying and are very close. However, for longer horizons, since the distance between the simulation gridpoints is increasing, the non-conditional valuation overestimates the normal and extreme price appreciations of the option (the mean and 95th quantile) and underestimates the extreme depreciations (5-th quantile) since it is based on the assumption that the barrier crossing can occur only on the simulation gridpoints, but not between them. Thus, it leads to scenarios with less barrier crossing than in the conditional valuation. The mispricing of barrier options using non-conditional valuation is larger the coarser the simulation grid is, a fact which is evident from the simulation grid points after year one. On the simulation points after a year and half, the decrease of the 95th quantile with conditional valuation is so sharp that we are back to the concave shape of the 95th quantile of a standard FX European option.

Figure 3 compares the risk numbers for conditional and non-conditional valuation for an up-and-out call option on the USD/EUR rate with the same simulation parameters, except that the barrier level is much closer to the initial spot FX rate. With a closer barrier, the difference between the conditional and non-conditional option, the 95th quantiles are much larger. With conditional valuation, the option profile is similar to the concave PFE profile of a standard European option, but later, there are so many paths with barrier crossing that the 95th quantile becomes concave and goes close to zero at the maturity date of the option. The PFE profile with non-conditional valuation is more irregular and has a sharp kink, going from extremely large values to almost zero. It is important to note that as the maturity date approaches, the option's PFEs become more irregular because the deltas of the option around the barrier level are discontinuous.

We investigate the convergence of risk numbers going to simulation times at high frequency by looking at different parts on non-equidistant simulation grids. With very fine partition, as in the first simulation gridpoints of a bank's credit exposure system, the discrete filtration converges to the continuous information filtration. Figure 4 presents the example of the option in Figure 2, but with equidistant daily simulation grid and option time to maturity of one year. Similarly, Figure 5 presents the example of the option from Figure 3 with daily grid and one-year maturity. It is clear that because of the convergence of the information, the means with conditional and non-conditional valuation are much more closer. What is important for risk managers are not so much the properties of the filtrations, but the risk numbers they generate. The results in Figures 4 and 5 demonstrate that the convergence is faster (1) the further out the barrier is from the spot FX rate and (2) the more distant the maturity of the option is. The extension of the conditional valuation to risk factors with time-dependent volatilities (piecewise constant between the simulation grid-points) is straightforward. In this case the modified barrier becomes time-dependent:

$$a_{t_j} = \frac{1}{\sigma_{t_j}} \ln(\alpha/\nu)$$

and one has to use different a_{t_j}-s in the expression for multiperiod barrier-hitting conditional probability P_j^a.

8 Double-barrier options

In this section, we propose analytical and efficient Monte Carlo-based methodologies for conditional valuation of double barrier options, the conditional Monte Carlo approach being applicable to all types of barrier options.[5] For conditional valuation of double barrier options it is necessary to compute the conditional barrier hitting probabilities of the upper and lower barrier and the conditional probability that the underlying stays in the range bounded by the two processes. The first two probabilities can be computed using the methodology in this paper. Abundo (2002) obtains analytical solutions for the probability that the underlying stays in the range between the two barriers in the Brownian motion case. The most complicated case is that

[5] The advantage of the analytical method is that the 95% PFE is smoother between the two values (in the hit and no-hit state). In the Monte Carlo method, in the path with 950-th (out of 1,000) highest exposure, the barrier options is either in the state of hit or no-hit and at certain point jumps sharply from one state to the other and stays there. Also, in the MC methods, there is some instability caused by uniform random numbers used to simulate doubly stochastic barrier hitting times.

of out-out barrier options with two different rebate levels, payable depending on which barrier is hit first. While the first three types of double barrier options can be valued analytically, no analytical formulas for out-out options appears in the option pricing literature.

Denote the lower barrier for asset return process X with a and the upper barrier with b. The times for reaching the lower and upper barriers are defined as

$$\tau^a = \inf\{t: V_t \le \alpha\} = \inf\{t: X_t \le a\} \tag{22}$$

and

$$\tau^b = \inf\{t: V_t \ge \beta\} = \inf\{t: X_t \ge b\} \tag{23}$$

with $X_t = \dfrac{1}{\sigma}\ln(V_t/v)$, $a = \dfrac{1}{\sigma}\ln(\alpha/v)$ and $b = \dfrac{1}{\sigma}\ln(\beta/v)$.

8.1 Main structures and conditional valuation

A double barrier is similar to a single barrier option but has two barriers which influence the option's payoff. The option can be "knocked out" by hitting an "out" barrier or deliver a standard European call or put if an "in" barrier is hit. Either barrier can be "in" or "out". If the upper barrier type is "out" and the price of the underlying asset rises above the barrier level, the option ceases to exist and the upper rebate R_b become payable according to its style. Similarly, if the lower barrier type, is "out" and the price of the underlying asset falls below the lower barrier level, the option ceases to exist and the lower rebate R_a become payable according to its style. If either the lower or upper barrier is "in", and the price of the underlying asset remains between the barriers (that is, neither barrier has been touched or breached over the option's lifetime), no hit rebate $R_{\bar{a}\bar{b}}$ is paid out at maturity.

Alternatively, if either of the barriers is "in" and if one of these was hit then the option is considered to be “in” and the payout is that of a standard European option; that is, for a call, $\max(S-K,0)$, and for a put, $\max(K-S,0)$. However, an "out" barrier is not disabled by the option knocking in; that is, the option can still be knocked out by the "out" barrier being hit. Finally, the option is considered to be "in" if both barriers are of type "out", and neither was hit; that is, the option failed to knock out. In this case, the payout is that of a standard European option.

In summary, the payoffs for double barrier options are as follows (upper/lower):

• in/in - if knocked in by hitting either barrier, the payout is that of a standard European option, otherwise it is the no-hit rebate $R_{\bar{a}\bar{b}}$.

• out/in - if knocked in by hitting the "in" barrier the effect is that of turning the double barrier option into a single barrier option. The out barrier is still active, and if not knocked out, the payout is that of a standard European option. If the option is knocked out by hitting the "out" barrier, the payout is the upper rebate R_b. Otherwise, if neither barrier is hit, the payout is the no hit rebate $R_{\bar{a}\bar{b}}$.

• in/out - if knocked in by hitting the "in" barrier the effect is that of turning the double barrier option into a single barrier option. The out barrier is still active, and if not knocked out, the payout is that of a standard European option. If the option is knocked out by hitting the “out” barrier, the payoff is the lower rebate R_a. Otherwise, if neither barrier is hit, the payoff is $R_{\bar{a}\bar{b}}$.

• out/out - if the option is not knocked out by hitting either barrier, the payoff is that of a standard European option. In contrast, if one of the barriers is hit then the option ceases to exist and the rebate of the barrier which was first hit is paid out.

By the classical result from Doob (1949) on the first-passage time of a standard Brownian motion to two-sided linear barriers, we have

$P(-(d_1+d_2 t)\le W_t \le d_3+d_4 t)=G(d_1,d_2,d_3,d_4)$ with

$$G(d_1,d_2,d_3,d_4)=1-\sum_{k=1}^{\infty}\left(e^{-2A_k}+e^{-2B_k}-e^{-2C_k}-e^{-2D_k}\right), \tag{24}$$

where

$$\begin{cases} A_k = & -k^2 d_3 d_4-(k-1)^2 d_1 d_2+k(k-1)(d_2 d_3+d_1 d_4) \\ B_k = & -k^2 d_3 d_4-(k-1)^2 d_1 d_2+k(k-1)(d_2 d_3+d_1 d_4) \\ C_k = & k^2(d_3 d_4+d_1 d_2)+k(k-1)d_3 d_2+k(k+1)d_1 d_4 \\ D_k = & k^2(d_3 d_4+d_1 d_2)+k(k+1)d_3 d_2+k(k-1)d_1 d_4 \end{cases}.$$

With the second type of incomplete information, F^2, the conditional probability of crossing the upper barrier is

$$P(\tau^b \le t_j \mid F^2_{t_j})=1 \qquad \textit{if } X_{t_k}>b \text{ for at least one } t_k,\, t_k\le t_j$$

$$=1-P^b_j, \text{ otherwise,}$$

where

$$P^b_j=\prod_{k=1}^{j}\left(1-\exp\left(-\frac{2}{t_k-t_{k-1}}(b-X_{t_{k-1}})(b-X_{t_k})\right)\right). \tag{25a}$$

The conditional probability of crossing the lower barrier is

$$P(\tau \le t_j \mid F^2_{t_j})=1 \qquad \textit{if } X_{t_k}<a \text{ for at least one } t_k,\, t_k\le t_j$$

$$=1-P^a_j, \quad \text{otherwise}$$

where

$$P_j^a = \prod_{k=1}^{j}\left(1-\exp\left(-\frac{2}{t_k - t_{k-1}}(a - X_{t_{k-1}})(a - X_{t_k})\right)\right). \tag{25b}$$

In contrast to the case of single barrier options, there are three possible payoffs to double barrier options depending on whether the upper or lower barrier has been crossed or there has been no hit up to the valuation time.

Consider an up-and-out/down-and-in double barrier option $C_{2bar}^{UO/DI}$. Its value at simulation time t_j is

$$C_{2bar}^{UO/DI}(t_j) = \begin{cases} C_{bar}^{UO}(t_j, V_{t_j}) & \text{on the set } \{\tau^a \le t_j, \tau^b > t_j\} \\ R_b e^{-r(T-t_j)} & \text{on the set } \{\tau^a \le t_j, \tau^b \le t_j\} \\ R_{\bar{a}\bar{b}} e^{-r(T-t_j)} & \text{on the set } \{\tau^a > T, \tau^b > T\} \end{cases}, \tag{26}$$

where $C_{bar}^{UO}(t_j, V_{t_j})$ is the value of a single barrier up-and-out option. The conditional value of the $CV_{2bar}^{UO/DI}(t_j)$ option in the filtration F^2 on the simulation date t_j is the weighted average of the three values above with the respective probabilities of the three sets.

In evaluating the probabilities above, it is important to note that

$$P(\tau^b \le t_j \mid F_{t_j}^2) = P(\tau^b \le t_j, \tau^a \le t_j \mid F_{t_j}^2) + P(\tau^b \le t_j, \tau^a > t_j \mid F_{t_j}^2) \tag{27}$$

$$P(\tau^a \le t_j \mid F_{t_j}^2) = P(\tau^a \le t_j, \tau^b \le t_j \mid F_{t_j}^2) + P(\tau^a \le t_j, \tau^b > t_j \mid F_{t_j}^2) \tag{28}$$

$$\begin{aligned} P(\tau^a > t_j, \tau^b > t_j \mid F_{t_j}^2) &= 1 - P(\tau^a \le t_j, \tau^b \le t_j \mid F_{t_j}^2) \\ &\quad - P(\tau^a > t_j, \tau^b \le t_j \mid F_{t_j}^2) - P(\tau^a \le t_j, \tau^b > t_j \mid F_{t_j}^2)' \end{aligned} \tag{29}$$

where the last equality follows by partitioning the probability space into four disjoint sets. The double barrier probabilities in (26) can be derived by single barrier hitting probabilities by solving this system of equations. The explicit solutions for double-barrier crossing probabilities are

$$\begin{aligned} P(\tau^a \le t_j, \tau^b \le t_j \mid F_{t_j}^2) &= P(\tau^a > t_j, \tau^b > t_j \mid F_{t_j}^2) + P(\tau^a \le t \mid F_{t_j}^2) \\ &\quad + P(\tau^b \le t_j \mid F_{t_j}^2) - 1 \end{aligned} \tag{30}$$

$$P(\tau^a > t_j, \tau^b < t_j \mid F_{t_j}^2) = 1 - P(\tau^a > t_j, \tau^b > t_j \mid F_{t_j}^2) - P(\tau^b \le t_j \mid F_{t_j}^2) \tag{31}$$

$$P(\tau^a \le t_j, \tau^b > t_j \mid F_{t_j}^2) = 1 - P(\tau^a > t_j, \tau^b > t_j \mid F_{t_j}^2) - P(\tau^a \le t_j \mid F_{t_j}^2) \tag{32}$$

where the single barrier crossing probabilities are given in (25a) and (25b).

Analytical probability that a standard Brownian bridge stays in a corridor with piecewise linear boundaries from the start to the end date $P(\tau^a > t_j, \tau^b > t_j \mid F_{t_j}^2)$ is obtained using the results of Abundo (2002) and some additional derivations. Indeed, from Abundo (2002),

$$P\left[\bigcap_{[0,T]} -(d_1 + d_2 t) < W_t < (d_3 + d_4 t) \mid W_T = \eta\right]$$

$$= P\left[\bigcap_{s \ge 0} -\left(d_1 s + d_2 + \frac{d_1 + \eta}{T}\right) < W_s < \left(d_3 s + \left(d_4 + \frac{d_3 - \eta}{T}\right)\right)\right]$$

$$= G\left(d_1, d_2 + \frac{d_1 + \eta}{T}, d_3, d_4 + \frac{d_3 - \eta}{T}\right).$$

The probability that the process X stays in the corridor (a,b) for a single simulation interval is

$$P\left[\bigcap_{[t_{j-1}, t_j]} a < X_t < b \middle| X_{t_j} = x_j\right]$$

$$= P\left[\bigcap_{[t_{j-1},t_j]} a < X_t - X_{t_{j-1}} < b \middle| X_{t_j} - X_{t_{j-1}} = x_j - X_{t_{j-1}} \right]$$

$$= P\left[\bigcap_{s\in[0,t_j-t_{j-1}]} a < \widetilde{X}_s < b \middle| \widetilde{X}_{t_j-t_{j-1}} = X_{t_j} - X_{t_{j-1}} \right]$$

$$= P\left[\bigcap_{s\in[0,t_j-t_{j-1}]} a - \nu s < \widetilde{W}_s < b - \nu s \middle| \widetilde{W}_{t_j-t_{j-1}} = X_{t_j} - X_{t_{j-1}} - \nu(t_j - t_{j-1}) \right]$$

$$= G\left(\nu, a + \frac{\nu + X_{t_j} - X_{t_{j-1}} - \nu(t_j - t_{j-1})}{t_j - t_{j-1}}, -\nu, b + \frac{-\nu - X_{t_j} + X_{t_{j-1}} + \nu(t_j - t_{j-1})}{t_j - t_{j-1}} \right). \tag{33}$$

Using (33) and consecutive conditional expectations, with the second type of incomplete information, F^2, the conditional probability of the process X staying in the corridor of the two barriers is

$$P(\tau^a > t_j, \tau^b > t_j \mid F^2_{t_j}) = 0 \qquad \text{if } X_{t_k} < a \text{ or } X_{t_k} > b \text{ for at least one } t_k,\, t_k \le t_j \tag{34}$$

$$= P_j^{a,b},$$

where

$$P_j^{a,b} = \prod_{k=1}^{j} G\left(\nu, a + \frac{\nu + X_{t_k} - X_{t_{k-1}} - \nu(t_k - t_{k-1})}{t_k - t_{k-1}}, -\nu, b + \frac{-\nu - X_{t_k} + X_{t_{k-1}} + \nu(t_k - t_{k-1})}{t_k - t_{k-1}} \right). \tag{35}$$

The conditional value of up-and-out/down-and-in double barrier option $C_{2bar}^{UO/DI}$ in the filtration F^2 on the simulation date t_j is the weighted average of the three values in (26) with the respective probabilities of the three sets

$$\begin{aligned} CV_{2bar}^{UO/DI}(t_j) &= C_{bar}^{UO\_new}(t_j, V_{t_j}) P(\tau^a \le t_j, \tau^b > t_j \mid F^2_{t_j}) \\ &+ R_b e^{-r(T-t_j)} P(\tau^a \le t_j, \tau^b \le t_j \mid F^2_{t_j}) + R_{\bar{a}\bar{b}} e^{-r(T-t_j)} P(\tau^a > T, \tau^b > T \mid F^2_{t_j}), \end{aligned} \tag{36}$$

where

$$P(\tau^a \le t_j, \tau^b > t_j \mid F^2_{t_j}) \text{ is as in (32);}$$

$$P(\tau^a \le t_j, \tau^b \le t_j \mid F^2_{t_j}) \text{ is as in (30);}$$

$$\begin{aligned}
P(\tau^a > T, \tau^b > T \mid F^2_{t_j}) &= P(\tau^a > t_j, \tau^b > t_j \mid F^2_{t_j}) + P(\tau^a, \tau^b \notin [t_j, T] \mid F^2_{t_j}) \\
&= P(\tau^a > t_j, \tau^b > t_j \mid F^2_{t_j}) + P\left[\bigcap_{[t_j,T]} a < X_t < b \middle| X_{t_j} = x_j\right] \\
&= P(\tau^a > t_j, \tau^b > t_j \mid F^2_{t_j}) P\left[\bigcap_{s\in[0,T-t_j]} a - \nu s < \widetilde{W}_s < b - \nu s\right]
\end{aligned}$$

$$= P(\tau^a > t_j, \tau^b > t_j \mid F^2_{t_j}) \int_{a-\nu(T-t_j)}^{b-\nu(T-t_j)} \frac{e^{-\eta^2} / 2(T-t_j)}{\sqrt{2\pi(T-t_j)}} G\left(-a, \nu + \frac{b+\eta}{T-t_j}, b, -\nu + \frac{b-\eta}{T-t_j}\right) d\eta \tag{37}$$

with the last equality following from Abundo (2002).

The double-no-touch knock out barrier option $C^{UO/DO}_{2bar\_nt}$ is a specific case of out/out double barrier option, where the two rebates are set at zero. Using (37), the digital type of this option $DIGC^{UO/DO}_{2bar\_nt}$ can be easily valued in the case of full information

$$\begin{aligned}
DIGC^{UO/DO}_{2bar_nt} &= E(e^{-r(T-t)} 1\!\!1_{\{\tau^a > T, \tau^b > T\}} \mid F^V_t) = \\
&= e^{-r(T-t)} 1\!\!1_{\tau > t} \int_{a-\nu(T-t)}^{b-\nu(T-t)} \frac{e^{-\eta^2} / 2(T-t)}{\sqrt{2\pi(T-t)}} G\left(-a, \nu + \frac{b+\eta}{T-t}, b, -\nu + \frac{b-\eta}{T-t}\right) d\eta \,.
\end{aligned} \tag{38}$$

8.2 Intensity-based valuation of double-barrier options

There are more complex types of double-barrier options, which depend on the order of the hitting times of the two barriers. Here, we present a more general Monte Carlo-based valuation algorithm which applies to all the types of barrier options, including the out-out options, for which analytical solutions are unavilable. The algorithm can be used for MC-based market and credit risk computations as well as for mark-to-market valuations.

Suppose that the conditional barrier hitting probability admits intensity representation. Assume also that the barrier hitting intensity is constant between two simulation grid-points, in $(t_{j-1}, t_j]$. Then we can write:

$$P(\tau^a \le t_j \mid F_{t_j}^2) - P(\tau^a \le t_{j-1} \mid F_{t_j}^2) = 1 - e^{-\lambda_j^a (t_j - t_{j-1})},$$

where λ_j is the intensity of hitting the lower barrier level a in the filtration F^2. Solving this equation leads to

$$\lambda_j^a = -\ln\left(1 - P(\tau^a \le t_j \mid F_{t_j}^2) + P(\tau^a \le t_{j-1} \mid F_{t_j}^2)\right)$$

The conditional barrier hitting probabilities have closed-form solutions as in Section 4, depending on the simulated risk factors at the simulation gridpoints. Consequently, the time of crossing the barrier a can be simulated on each simulation interval $(t_{k-1}, t_k]$ as follows

$$\tau^a = \inf\left\{ t : \exp\left(-\lambda_k^a (t - t_{k-1}) - \sum_{j=1}^{k-1} \lambda_j^a (t_j - t_{j-1}) \right) \ge U^a \right\}, \tag{39}$$

where U^a is a uniform $[0,1]$ random variable. For each path, the simulation should be run until the hitting time (i.e, as long as $t_k < \tau^a$). Simulation of the time of hitting the upper barrier b can be done in the same way

$$\tau^b = \inf\left\{ t : \exp\left(-\lambda_k^b (t - t_{k-1}) - \sum_{j=1}^{k-1} \lambda_j^b (t_j - t_{j-1}) \right) \ge U^b \right\}, \tag{40}$$

where U^b is a uniform $[0,1]$ random variable. The uniform random variables U_a and U_b are uncorrelated with each other and with anything else. This is one way to generate random barrier hitting times and to order them.

9 Conclusion

The value of a barrier option depends on the whole path of the underlying asset from the issue date to the valuation date, but in many practical situations, investors or modellers have reduced information about this path. In this paper, we study the implications of using delayed historical information, discrete and noisy observations for the underlying asset on the valuation of barrier options. We present new analytical solutions for barrier-crossing probability in a multi-period setting for future underlying asset scenarios based on path-dependent simulation. We apply the multi-step conditional valuation to single and double barrier options and obtain closed-form analytical valuation results.

Information reductions change the properties of the first passage time to a barrier from predictable to totally inaccessible random time, which affects the value of a barrier option. Market and credit risk implications of barrier option valuations under incomplete information are discussed. Most of the commercially available market and credit risk systems do not value barrier options at the risk horizon dates consistent with the information sets. We demonstrate in this paper that using valuation formulas inconsistent with the available information leads to incorrect valuations and to wrong assessment of the risk.

We obtain closed-form analytical solutions also in the case where additional observation noise is layered over the underlying asset price – information set that is common for some investors in unlisted stocks and other illiquid assets. Using perfect information valuation formulas under conditions of reduced information for barrier options underestimates both the risk exposures and the frequency of losses associated with barrier crossing. The pricing errors when using complete information formulas under conditions of imperfect information are greater for deep out-of-the money regular knock-out options and barrier options with short times to maturities.

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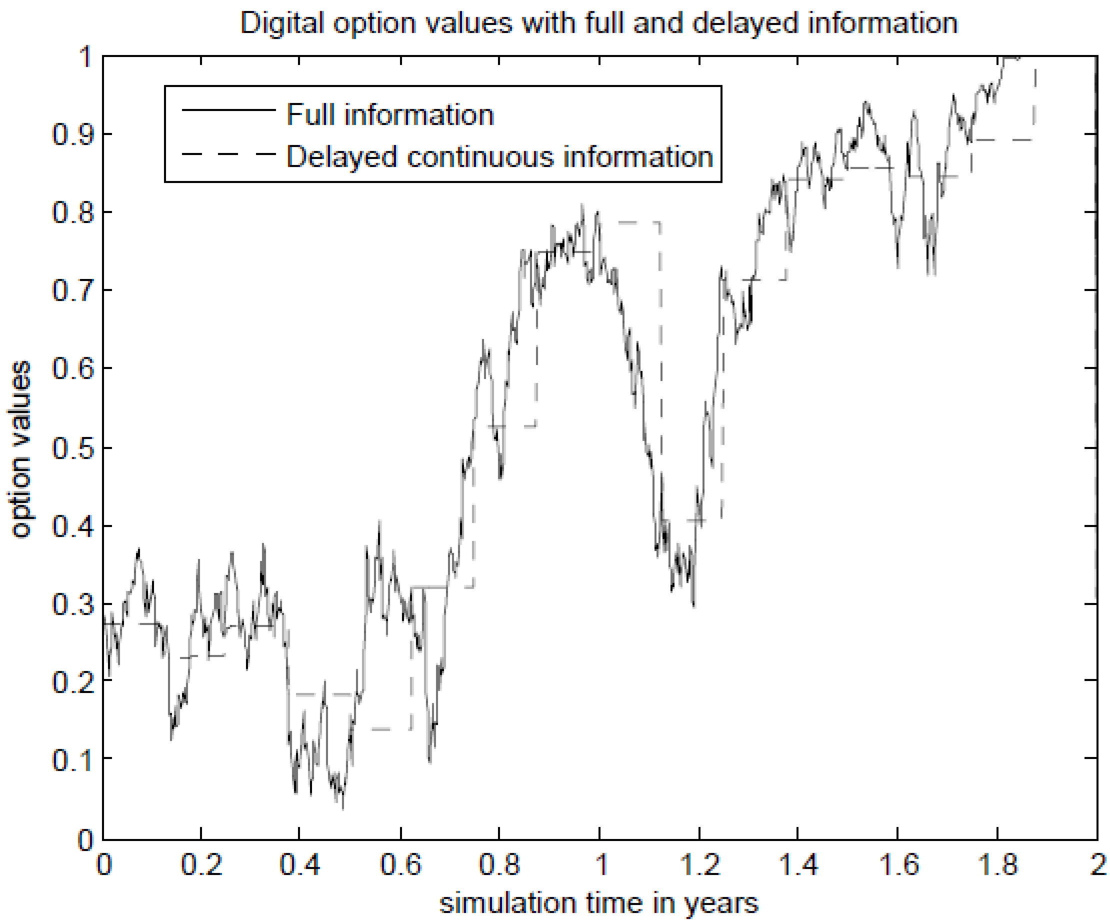
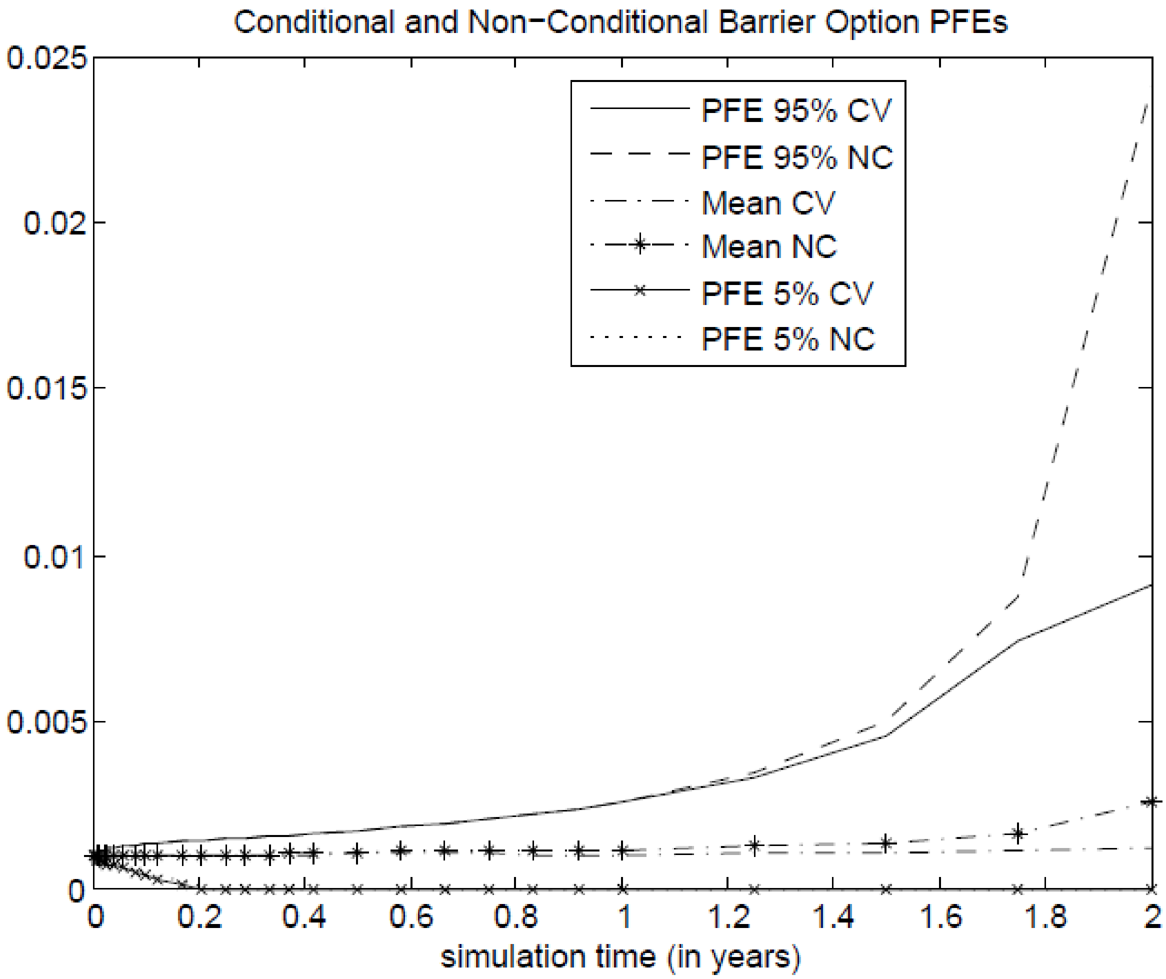
Figure 1: FX Digital Barrier Option Values with Full and Delayed Continuous Information



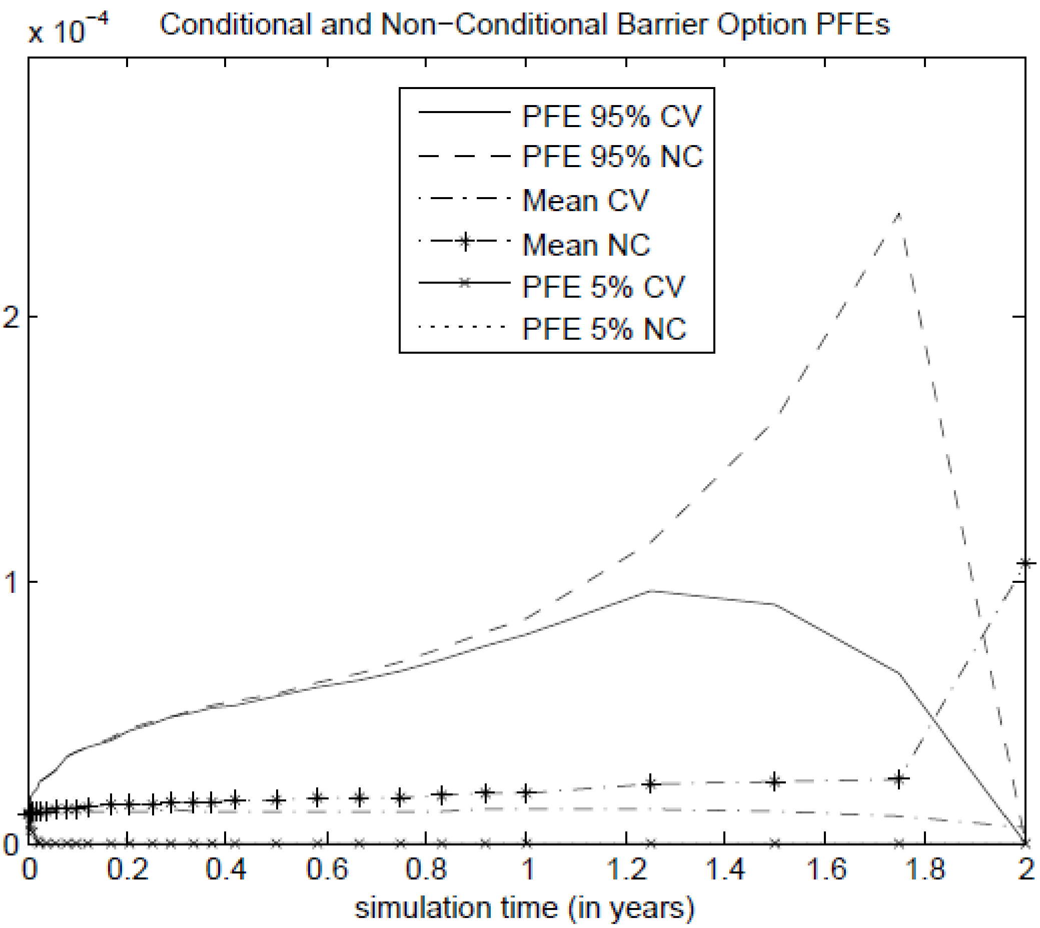
The starting USD/EUR rate is 0.75, the initial time to maturity is two years, σ and the option implied volatility are both 0.123, and the barrier level is 0.70. The domestic short rate is $r = 0.75\%$ and the foreign risk-free interest rate is $\delta = 1.00\%$. The observation lag is 0.125 years and the number of steps is 1,000.

Figure 2: Up-and-Out FX Barrier Option Mean and Risk Quantiles: Distant Barrier Level



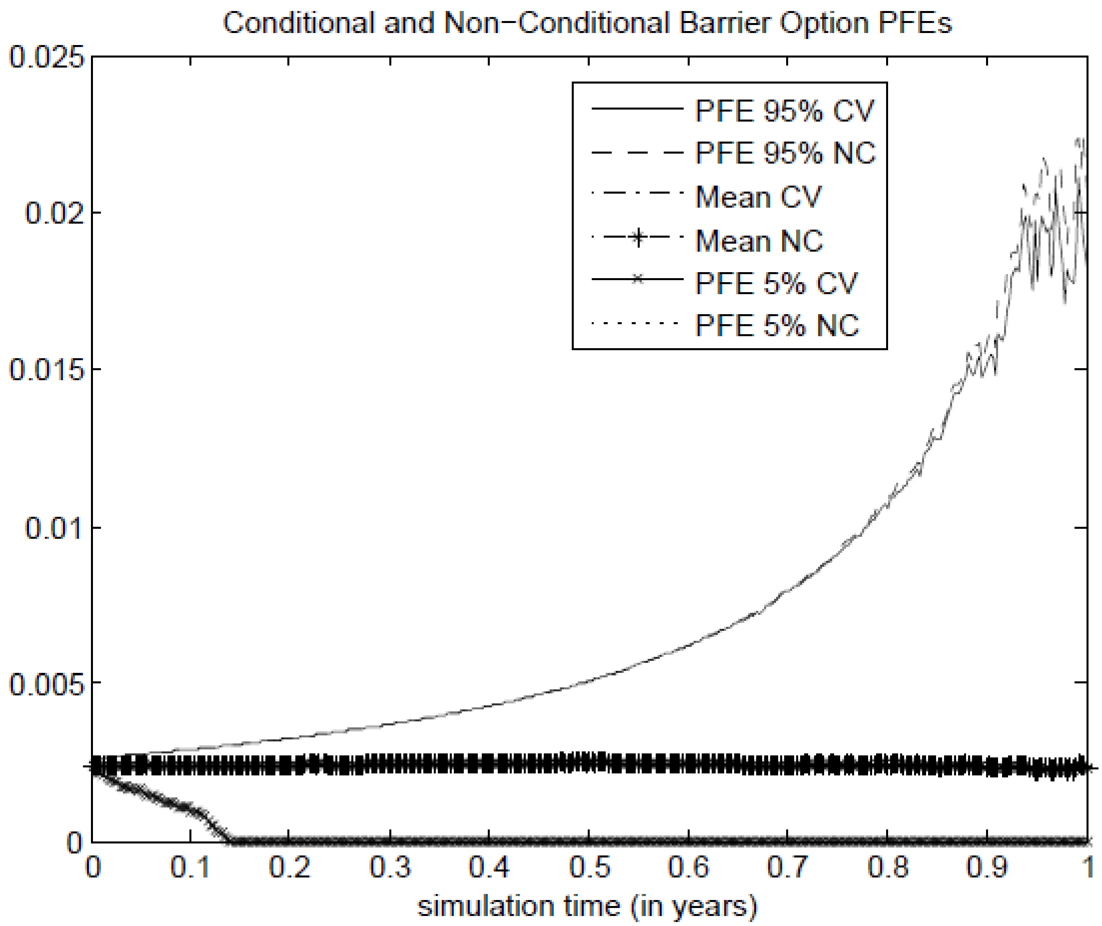
The number of paths is 1,000, the starting USD/EUR rate is 0.7518, the strike is 0.75, the initial time to maturity is two years, σ and the option implied volatility are both 0.123, and the barrier level is 0.82. The domestic short rate is $r = 0.75\%$ and the foreign risk-free interest rate is $\delta = 1.00\%$.The abbreviations CV and NC in the legend stand for conditional and non-conditional valuation.

Figure 3: Up-and-Out FX Barrier Option Mean and Risk Quantiles: Close Barrier Level



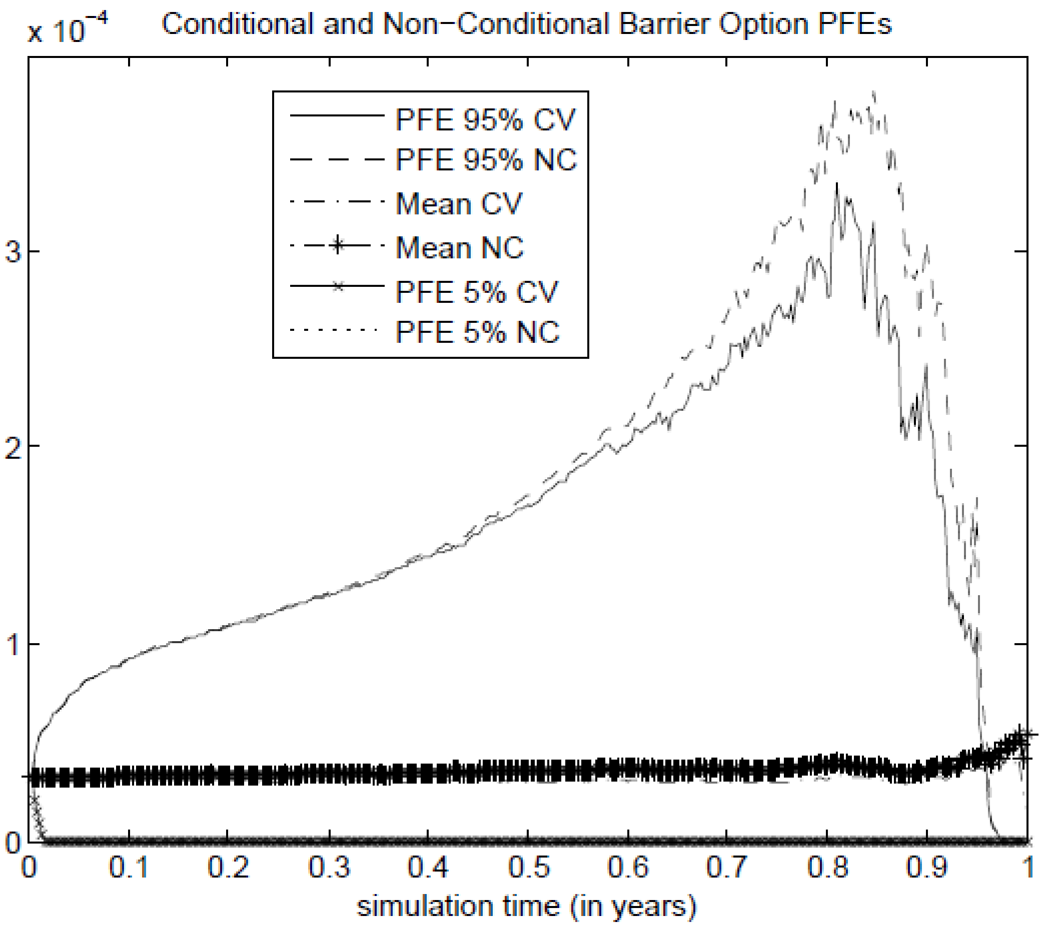
The number of paths is 1,000, the starting USD/EUR rate is 0.7518, the option strike is 0.75, the initial time to maturity is two years, the volatility coefficient of the FX process, σ and the option impled volatility are both 0.123, and the barrier level is 0.7718. The domestic short rate is $r = 0.75\%$ and the foreign risk-free interest rate is $\delta = 1.00\%$. The abbreviations CV and NC in the legend stand for conditional and non-conditional valuation.

Figure 4: Up-and-Out FX Barrier Option Mean and Risk Quantiles: Distant Barrier Level and Daily Simulation Grid



The number of paths is 1,000, the starting USD/EUR rate is $V_{t_0} = 0.7518$, the option strike is 0.75, the initial time to maturity is one year, σ and the option impled volatility are both 0.123, and the barrier level is 0.82. The domestic short rate is $r = 0.75\%$ and the foreign risk-free interest rate is $\delta = 1.00\%$. The abbreviations CV and NC in the legend stand for conditional and non-conditional valuation.

Figure 5: Up-and-Out FX Barrier Option Mean and Risk Quantiles: Close Barrier Level and Daily Simulation Grid



The number of paths is 1,000, the starting USD/EUR rate is 0.7518, the option strike is 0.75, the initial time to maturity is one year, σ and the option impled volatility are both 0.123, and the barrier level is 0.7718. The domestic short rate is $r = 0.75\%$ and the foreign risk-free interest rate is $\delta = 1.00\%$. The abbreviations CV and NC in the legend stand for conditional and non-conditional valuation.