A NOTE ON THE IMPACT OF NON LINEAR REWARD AND RISK MEASURES

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Abstract: In this note, we examine the impact of non linear reward and risk measures on portfolio selection. In particular, we compare the ex-post final wealth sample paths of strategies based on the Sharpe ratio and strategies based on non-linear reward/risk measures. As suggested by the recent literature, we model dependencies with an asymmetric t copula estimated on the innovations of the marginals that follow an ARMA-GARCH type model. Therefore, we first simulate future scenarios, on the basis of which allocation decisions are made, and then we compare the ex-post final wealth obtained with non-linear risk and reward strategies and the wealth obtained with classic portfolio strategies .
Key words: Sharpe ratio, portfolio choice, asymmetric t copula.

1 Introduction

This paper critically reviews the non-linearity property of risk and reward measures in order to determine its effects on portfolio strategies. In particular, we show with an ex-post empirical analysis how important it is to model it appropriately.

In portfolio-choice theory, a consistency between risk measures and investor's preferences is viewed as desirable and has been explored in the literature. See Rachev et al (2008) and the reference therein for a classification of basic portfolio selection problems with respect to types of investor's behavior. In particular, Ortobelli et al. (2008b) discuss how to compute optimal choices consistent with any admissible preference ordering. Clearly, in portfolio selection theory the criteria of choice represent the factor with highest impact for the investors. Therefore, we should guarantee that these criteria are consistent with orderings of preferences that consider all intuitive characteristics of investors' attitude.

It is well known that a measure of uncertainty is not necessarily adequate in measuring risk which is asymmetric as it is related to downside outcomes only. In particular, any realistic way of optimizing risk should maximize upside potential outcomes and minimize the downside outcomes. Artzner et al. (1999) define some intuitive characteristics of investment risk in the concept of a coherent risk measure. However, as observed by Balzer (2001), Rachev et al. (2008) even coherent risk measures cannot consider exhaustively all investment characteristics. According to recent studies (see Balzer (2001), Okuyama and Francis (2007), Farinelli et al. (2008)), investor's attitude is non-linear with respect to different sources of risk . Thus, even the celebrated expected shortfall, which is a coherent risk measure, does not take into account that most investors perceive a low probability of a large loss to be far more risky than a high probability of a small loss. Therefore, investors perceive risk to be non-linear (see Olsen (1997)).

In this note, we value the impact of non-linearity of risk perception and we propose a new different reward/risk ratio that takes into account some standardized moments of upside and downside outcomes. Moreover, in order to value correctly the evolution of wealth, we generate the future multivariate returns as suggested by Biglova et al. (2008) (see also Sun et al. (2008) and Ortobelli et al. (2008a)). Thus, we approximate the behavior of the corresponding marginals with an ARMA(2,0)-GARCH(0,2) model with stable paretian innovations and we approximate the dependencies of the innovation of the marginals with an asymmetric t copula. Finally, we compute the ex post final wealth sample paths obtained with the new ratios and with the Sharpe ratio (see Sharpe (1994)). The empirical analysis demonstrates the superiority of the new reward/risk measures with respect to the classical mean variance analysis.

Section 2 describes the empirical comparison and we summarize our principal findings in the last Section.

2 On the portfolio selection problem: choices consistent with investor behavior and empirical evidence

Suppose we have a frictionless market in which no short selling is allowed and all investors act as price takers. Given a risk-free benchmark with log-return r_b and n risky securities with a vector of log-returns $r = [r_1, ..., r_n]'$, the classical portfolio selection problem in the reward-risk plane consists of minimizing a given risk measure ρ provided that the expected reward v is constrained by some minimal value m, i.e.:

$$\begin{gathered} \min_x \rho\left(x'r - r_b\right) \\ s.t. \\ v\left(x'r - r_b\right) \geq m \\ x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \end{gathered}$$

where the vector notation $x'r = \sum_{i=1}^{n} x_i r_i$ stands for the returns of a portfolio with composition $x = (x_1, ..., x_n)'$. Along the efficient choices obtained by varying the value of the constraint m, there is a portfolio (often called the *market portfolio*) that provides the maximum expected reward v per unit of risk ρ. So, assuming that the reward and risk are both positive the *market portfolio* is obtained as the solution of the optimization problem

$$\begin{array}{c}\max\limits_{x} \frac{v(x'r-r_b)}{\rho(x'r-r_b)} \\ s.t. \\ x_i \geq 0, \sum\limits_{i=1}^{n} x_i = 1\end{array} \quad (1)$$

Starting from the original Markowitz' analysis, Sharpe suggestedthat investors should maximize the so called Sharpe ratio (see Sharpe (1994) and the reference therein).

Sharpe ratio (SR). The Sharpe ratio computes the expected excess return for unity of risk, i.e.:

$$SR(x'r) = \frac{E(x'r - r_b)}{STD(x'r - r_b)}.$$

In the Sharpe ratio, risk is proxied by the standard deviation $STD(x'r - r_b)$ of excess returns. Thus, maximizing the Sharpe ratio, we get a market portfolio that is not dominated in the sense of second-order stochastic dominance and therefore it should be optimal for non-satiable risk averse investors.

Almost in contrast with the tendency of these first studies, behavioral finance (see Friedman and Savage (1948), Markowitz (1952), Tversky and Kahneman (1992), Levy and Levy (2002), Ortobelli et al. (2008b)) suggests that most investors are clearly non-satiable but they are neither risk averse nor risk loving. To identify optimal portfolios selected by these investors, Rachev et al. (2008) propose to maximize reward-risk ratios $G(X) = \frac{v(X)}{\rho(X)}$, (where v is a positive reward measure, ρ is a positive generic risk measure for all X) isotonic with non-satiable investors' preferences (i.e. if $X \geq Y$, then $G(X) \geq G(Y)$), and that are not isotonic neither with risk averse investors' choices (that is $G(X+Y)$ is not greater or equal to $G(X)+G(Y)$ for all admissible X and Y) nor with risk lover investors' preferences (that is $G(X+Y)$ is not smaller or equal to $G(X)+G(Y)$ for all admissible X and Y) (see also Ortobelli et al. (2008b) and Bauerle and Müller (2006) for a classification of risk measures consistent with orderings).

Next we propose a new type of reward/risk ratio that takes into account the non-linearity of reward and risk measures according to the evidence on investors' risk perception (see Balzer (2001), Okuyama and Francis (2007), Olsen (1997)).

Rachev High Moments Ratio ($RHMR$).With this performance ratio, we propose a reward-risk ratio isotonic with the preferences of non-satiable investors who are neither risk averse nor risk loving. Moreover, we suggest to approximate the non-linearity attitude to risk of decision makers considering the first four moments of the standardized tails of the return distribution. Rachev high moments ratio is given by:

$$RHMR(x'r) = \frac{v_1(x'r - r_b)}{\rho_1(x'r - r_b)}$$

where

$$v_1(x'r - r_b) = E(x'r - r_b / x'r - r_b > F^{-1}_{x'r-r_b}(p_1)) +$$

$$+\sum_{i=2}^{4} a_i E\left(\left(\frac{x'r - r_b}{\sigma_{x'r-r_b}}\right)^i \big| x'r - r_b > F^{-1}_{x'r-r_b}(p_i)\right);$$

$$\rho_1(x'r - r_b) = -E(x'r - r_b / x'r - r_b < F^{-1}_{x'r-r_b}(q_1)) -$$

$$-\sum_{i=2}^{4} b_i E\left(\left(\frac{x'r - r_b}{\sigma_{x'r-r_b}}\right)^i \big| x'r - r_b < F^{-1}_{x'r-r_b}(q_i)\right),$$

$F_X^{-1}(q) = \inf\{x \,|P(X \leq x) > q\}$; $\sigma_{x'r-r_b}$ is the standard deviation of $x'r - r_b$, a_i, $b_i \in \mathbb{R}$ and $p_i, q_i \in (0,1)$. As we can observe from the definition, the Rachev high moments ratio is very versatile and depends on many parameters. To simplify our analysis in the following empirical comparison, we assume $a_i = b_i = 2$ for $i = 2, 3$, $a_4 = b_4 = 1$; $p_1 = 0.99$; $p_2 = 0.97$; $p_3 = 0.95$; $p_4 = 0.5$; and $q_i = 0.5$, $i = 1, 2, 3, 4$ (here the values of q_i are big enough to guarantee that the risk measure $\rho_1(x'r - r_b)$ is positive for all portfolios).

2.1 An empirical comparison between two different portfolio strategies

In order to value the impact of non-linear reward-risk measures, we provide an empirical ex-post comparison among the above strategies. The data includes the daily return series of the three-months treasury bill used as benchmark and the daily returns of the following US stock indexes: DJIA, NYSE, Major Market Index, DJ composite, DJAIG Commodity. As observed by Ortobelli et al. (2008a), we can generate future scenarios by fitting to these marginal series an ARMA(2,0)-GARCH(0,2) model with stable innovations and then estimating the dependence structure for the innovations with an asymmetric t-copula. Using the fitted model, we generate future scenarios for the vector of returns $r_{T+1,s} = [r_{T+1}^{(1,s)}, ..., r_{T+1}^{(n,s)}]'$ (where $r_{T+1}^{(i,s)}$ is the s-th scenario of the i-th return at time $T + 1$). In this way, we consider realistic models for marginals and the dependence structure (see also, Sun et al. (2008), Biglova et al. (2008)).

As a next step, we suppose that decision makers invest their wealth purchasing the *market portfolio* determined by maximizing either the Sharpe ratio or the Rachev high moments ratio. For any optimal portfolio chosen on a daily basis, we consider a window of one year $T = 250$ of historical observations. We use these observations to generate $S = T$ future scenarios of returns according to the algorithm proposed by Ortobelli et al. (2008a). We assume that the investor has an initial wealth W_0 equal to 1 and an initial cumulative return CR_0 equal to 0 at the date 12/8/1993.

Therefore, for both ratios (Sharpe ratio and Rachev high moments ratio), we can compute the optimal portfolio as a solution of the optimization problem (1). Since we want to compare the ex post sample path of the final wealth and of the cumulative return obtained from the two approaches, we assume that investors recalibrate their portfolio every day investing their wealth in the market portfolio obtained from solving (1) on the simulated data. Therefore, after k days we compute the ex-post final wealth and cumulative return determining first the

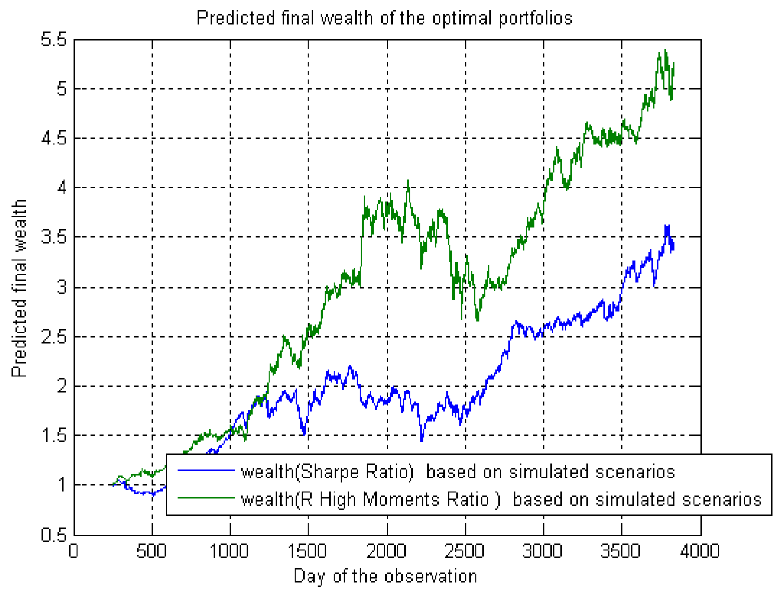

Figure 1: This figure compares the ex-post final wealth obtained by maximizing the RHM ratio and the Sharpe ratio.

market portfolio $x_M^{(k)}$ that maximizes one of the two performance ratios (the solution of problem (1)). The ex-post final wealth is given by:

$$W_{k+1} = W_k \left(\left(x_M^{(k)} \right)' (1 + r_{k+1}) \right),$$

and the ex-post cumulative return is given by:

$$CR_{k+1} = CR_k + \left(x_M^{(k)} \right)' r_{k+1}.$$

where r_{k+1} is the vector of observed returns at $(k+1)$-th day. We repeat this computation for both ratios (Sharpe ratio and Rachev high moments ratio) till the end of the period.

The output of this analysis is represented in Figures 1,2. The two figures show the superiority of the approach based on Rachev high moments ratio with respect to the classic approach. Therefore, we can conclude that the application of non-linear reward and risk measures which are consistent with realistic investors' preferences[1] has an important impact in portfolio selection theory.

[1] We implicity consider non-satiable investors who are neither risk averse nor risk loving.

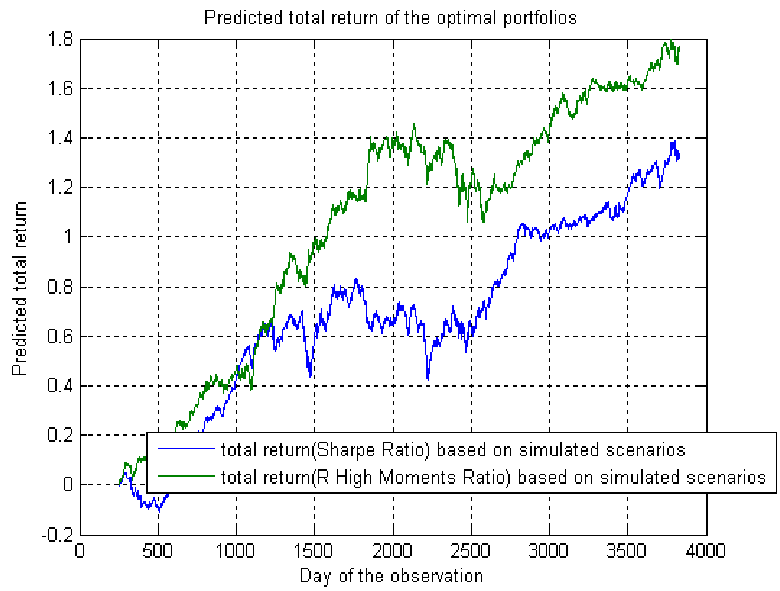

Figure 2: This figure compares the ex-post cumulative return obtained by maximizing the RHM ratio and the Sharpe ratio.

3 Conclusions

This paper examines and shows the impact of non-linear reward, risk measures in portfolio selection theory. In particular, we first discuss the use of opportune reward/risk criteria to select optimal portfolios. Then, we simulate realistic future scenarios using a copula approach and, finally, we compare the ex-post final wealth and cumulative return processes obtained using either a new reward/risk ratio or the classical Sharpe ratio. As anticipated, the ex-post empirical comparison shows the greater predictable capacity of the non-linear reward and risk measures.

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