Risk Management and Portfolio Optimization for Volatile Markets

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Abstract

We describe a framework of a system for risk estimation and portfolio optimization based on stable distributions and the average value-at-risk risk measure. In contrast to normal distributions, stable distributions capture the fat tails and the asymmetric nature of real-world risk factor distributions. In addition, we make use of copulas, a generalization of overly restrictive linear correlation models, to account for the dependencies between risk factors during extreme events. Using superior models, VaR becomes a much more accurate measure of downside risk. More importantly Stable Expected Tail Loss (SETL) can be accurately calculated and used as a more informative risk measure. Along with being a superior risk measure, SETL enables an elegant approach to risk budgeting and portfolio optimization Finally, we mention alternative investment performance measurement tools.

1. Introduction

The two main conventional approaches to modeling asset returns are based either on a historical or a normal (Gaussian) distribution for returns. Neither approach adequately captures unusual behavior of asset prices and returns. The historical model is bounded by the extent of the available observations and the normal distribution model inherently cannot produce extreme returns.

The inadequacy of the normal distribution is well recognized by the risk management community. To quote one major vendor:

> "It has often been argued that the true distributions returns (even after standardizing by the volatility) imply a larger probability of extreme returns than that implied from the normal distribution. Although we could try to specify a distribution that fits returns better, it would be a daunting task, especially if we consider that the new distribution

> would have to provide a good fit across all asset classes.“ (Technical Manual, RMG, 2001)

There are many studies exploring the non-normality of assets returns and suggesting alternative approaches. Among the well known candidates are Student's *t* distribution, generalized hyperbolic distributions (see Bibby and Sorensen (2003)) and stable Paretian distributions (see Rachev and Mittnik (2000)). At least some of their forms are subordinated normal models and thus provide a very practical and tractable framework. Rachev *et al* (2005) provide an introduction to heavy-tailed models in finance.

In response to these challenges, we use generalized multivariate stable distributions and generalized risk-factor dependencies, thereby creating a paradigm shift to consistent and uniform use of the most viable class of non-normal probability models in finance. Our paper discusses re-working of the classical approaches into a framework that allows for increased flexibility, accurate assets modeling, and sound risk measurement employing Generalized Stable Distributions together with average value-at-risk (AVaR) risk measure, see Rachev, *et al* (2007).

The paper is organized as follows. Section 1 discusses several heavy-tailed models with a special attention to the Generalized Stable Distributions. In Section 2, we discuss multivariate modeling. Section 3 provides a summary of risk and performance measures properties and describes the AVaR measure. Section 4 discusses risk budgeting based on AVaR and Section 5 is devoted to optimal portfolio problems. In Section 6, we remark on performance measures consistent with AVaR. Section 7 contains an empirical example with Russell 2000 universe.

2. Heavy-tailed and asymmetric models for assets returns

Specifying properly the distribution of assets returns is vital for risk management and optimal asset allocation. A failure may lead to significant underestimation of portfolio risk and, consequently, to wrong decisions.

The distributional modeling of financial variables has several dimensions. First, there should be a realistic model for the returns of each financial variable considered separately. That is, we should employ realistic one-dimensional models. Second, the model should capture properly the dependence between the one-dimensional variables. Therefore, we need a true multivariate model with the above two building blocks correctly specified.

2.1. One-dimensional models

The cornerstone theories in finance such as mean-variance model for portfolio selection and asset pricing models that have been developed rest upon the assumption that asset returns follow a normal distribution. Yet, there is little, if any, credible empirical evidence that supports this assumption for financial assets traded in most markets throughout the world. Moreover, the evidence is clear that financial return series are heavy-tailed and, possibly, skewed. Fortunately, several papers have analyzed the consequences of relaxing the normality assumption and developed generalizations of prevalent concepts in financial theory that can accommodate heavy-tailed returns (see Rachev and Mittnik (2000) and Rachev (2003) and references therein).

Mandelbrot (1963) strongly rejected normality as a distributional model for asset returns, conjecturing that financial return processes behave like non-Gaussian stable processes. To distinguish between Gaussian and non-Gaussian stable distributions, the latter are commonly referred to as "stable Paretian" distributions or "Levy stable" distributions.[1]

While there have been several studies in the 1960s that have extended Mandelbrot's investigation of financial return processes, probably, the most notable is Fama (1963) and Fama (1965). Fama's work and others led to a consolidation of the stable Paretian hypothesis. In the 1970s, however, closer empirical scrutiny of the "stability" of fitted stable Paretian distributions also produced evidence that was not consistent with the stable Paretian hypothesis. Specifically, it was often reported that fitted characteristic exponents (or tail-indices) did not remain constant under temporal aggregation.[2] Partly in response to these empirical "inconsistencies," various alternatives to the stable law were proposed in the literature, including fat-tailed distributions being only in the domain of attraction of a stable Paretian law, finite mixtures of normal distributions, the Student *t*-distribution, and the hyperbolic distribution, see Bibby and Sorensen (2003).

Recent attacks on Mandelbrot's stable Paretian hypothesis focus on the claim that empirical asset return distributions are not as heavy-tailed as the non-Gaussian stable law suggests. Studies that come to such conclusions are typically based on tail-index estimates obtained with the Hill estimator. Because sample sizes beyond 100,000 are required to obtain reasonably accurate estimates, the Hill estimator is highly unreliable for testing the stable hypothesis. More importantly, Mandelbrot's stable Paretian hypothesis is interpreted too narrowly, if one focuses solely on the *marginal* distribution of return processes. The hypothesis involves more than simply fitting marginal asset return distributions. Stable Paretian laws describe the fundamental "building blocks" (e.g., innovations) that drive asset return processes. In addition to describing these "building blocks," a complete model should be rich enough to encompass relevant stylized facts, such as

- non-Gaussian, heavy-tailed and skewed distributions
- volatility clustering (ARCH-effects)
- temporal dependence of the tail behavior
- short- and long-range dependence

An attractive feature of stable models – not shared by other distributional models – is that they allow us to generalize Gaussian-based financial theories and, thus, to build a coherent and more general framework for financial modeling. The generalizations are

[1] Stable Paretian is used to emphasize that the tails of the non-Gaussian stable density have Pareto power-type decay. "Levy stable" is used in recognition of the seminal work of Paul Levy's introduction and characterization of the class of non-Gaussian stable laws.
[2] For a more recent study, see Akgiray and Booth (1988) and Akgiray and Lamoureux (1989).

only possible because of specific probabilistic properties that are unique to (Gaussian and non-Gaussian) stable laws, namely, the stability property, the Central Limit Theorem, and the Invariance Principle for stable processes. Detailed accounts of properties of stable distributed random variables can be found in Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994).

Stable distributions are defined by the means of their characteristic functions, $\varphi_X(t) = Ee^{itX}$. The characteristic function has the following form,

$$\varphi_X(t) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left[1 - i\beta \frac{t}{|t|} \tan \frac{\pi\alpha}{2}\right] + i\mu t\right), \alpha \neq 1 \\ \exp\left(-\sigma |t| \left[1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \log|t|\right] + i\mu t\right), \alpha = 1 \end{cases} \tag{1}$$

In the general case, no closed-form expressions are known for the probability density and distribution functions of stable distributions. The formula in (1) implies that they are described by four parameters: α , called the index of stability, which determines the tail weight or density's kurtosis with $0 < \alpha \leq 2$, β , called the skewness parameter, which determines the density's skewness with $-1 \leq \beta \leq 1$, $\sigma > 0$ which is a scale parameter, and μ which is a location parameter. Stable distributions allow for skewed distributions when $\beta \neq 0$ and when β is zero, the distribution is symmetric around μ . Stable Paretian laws have fat tails, meaning that extreme events have high probability relative to the normal distribution when $\alpha < 2$. The Gaussian distribution is a stable distribution with $\alpha = 2$. (For more details on the properties of stable distributions, see Samorodnitsky, Taqqu (1994).) Of the four parameters, α and β are most important as they identify two fundamental properties that are atypical of the normal distribution – heavy tails and asymmetry.

Rachev *et al* (2006) consider the daily return distribution of 382 U.S. stocks in the framework of two probability models – the homoskedastic independent, identical distributed model and the conditional heteroskedastic ARMA-GARCH model. In both models, the Gaussian hypothesis is strongly rejected in favor of the stable Paretian hypothesis which better explains the tails and the central part of the return distribution. The companies in the study are the constituents of the S&P 500 with complete history in the 12-year time period from January 1, 1992 to December 12, 2003. Figure 1 illustrates the estimated (α, β) pairs from historical data. The estimated parameters suggest significant heavy-tail and asymmetry which are phenomena that cannot be accounted for by the normal distribution.

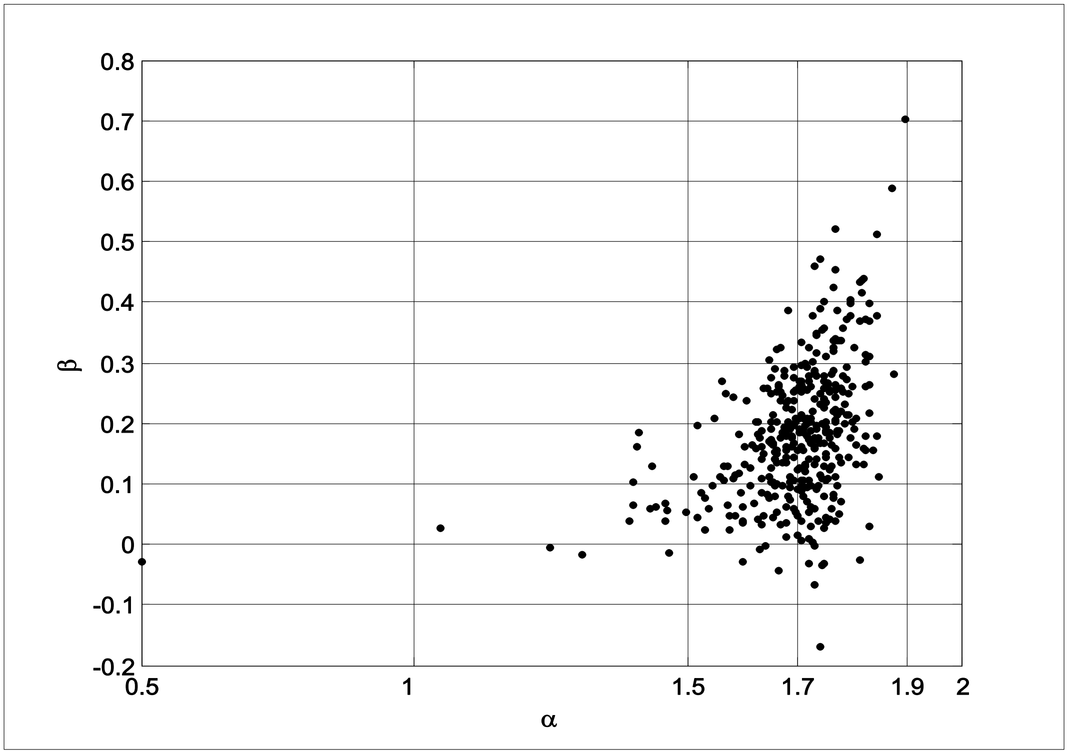

Figure 1. Scatter plot of the index of stability and the skewness parameter for the daily returns of 382 stocks. (Reproduced from Figure 1.1 in Rachev *et al* (2006).)

2.2.Multivariate models

For the purposes of portfolio risk estimation, constructing one-dimensional models for the instruments is incomplete. Failure to account for the dependencies between the instruments may be fatal for the analysis.

There are two ways to build a complete multivariate model. It is possible to hypothesize a multivariate distribution directly (i.e., the dependence between stock returns as well as their one-dimensional behavior). Assumptions of this type include the multivariate normal, the multivariate Student *t*, the more general elliptical family, the multivariate stable, etc. Sometimes, in analyzing dependence, an explicit assumption is not made, for instance, the covariance matrix is very often relied on. While an explicit multivariate assumption is not present, it should be kept in mind that this is consistent with the mutivariate normal hypothesis. More generally, the covariance matrix can describe only linear dependencies and this is a basic limitation.

In the last decade, a second approach has become popular. One can specify separately the one-dimensional hypotheses and the dependence structure through a function called copula. This is a more general and more appealing method because one is free to choose separately different parametric models for the stand-alone variables and a parametric copula function. For more information, see Embrechts *et al* (2003).

2.3. Generalized Stable Distribution modeling

Figure 1 indicates that the tail behavior of financial variables may vary. Generalized stable distribution modeling is based on fitting univariate stable distributions for each one dimensional set of returns or risk factors, each with its own parameter estimates α_i, β_i, μ_i, σ_i, i=1,2,…,K , where K is the number of risk factors, along with a dependency structure.

One way to produce the cross-sectional dependency structure is through a scale mixing process (called a "subordinated" process in the mathematical finance literature) as follows.

a) compute a robust mean vector and covariance matrix estimate of the risk factors to get rid of the outliers, and have a good covariance matrix estimate for the central bulk of the data.
b) multiply each of the random variable component of the scenarios by a strictly positive stable random variable with index $\alpha_i/2$, i=1,2,…,K. The vector of stable random variable scale multipliers is usually independent of the normal scenario vectors, but it can also be dependent. See, for example, Rachev and Mittnik (2000).

Another very promising approach to building the cross-sectional dependence model is through the use of copulas, an approach that is quite attractive because it allows for modeling higher correlations during extreme market movements, thereby accurately reflecting lower portfolio diversification at such times. The next section briefly discusses copulas.

2.4. Copula dependence models

Correlation is a widespread concept in modern finance and insurance and stands for a measure of dependence between random variables. However, this term is very often incorrectly used to mean any notion of dependence. Actually correlation is one particular measure of dependence among many. In the world of multivariate normal distribution and, more generally in the world of spherical and elliptical distributions, it is the accepted measure.

Financial theories and risk management analysis rely crucially on the dependence structure of assets. A major limitation of correlation as a measure of the dependence between two random variables is that zero correlation does not imply independence for non-Gaussian distributions. Furthermore, correlation is symmetric and, in order to be more realistic, we need a more general notion which can reflect the local variation in dependence that is related to the level of returns, in particular, those shapes that correspond to higher correlations with extreme co-movements in returns than with small to modest co-movements.

From a mathematical viewpoint, a copula function C is nothing more than a probability distribution function on the d-dimensional hypercube

$$C(u_1, u_2, \cdots, u_n), \quad u_i \in [0,1] \text{ for } i = 1, 2, \cdots, n$$

where $C(u_i) = u_i$, $i = 1, \ldots n$.

It is known that for any multivariate cumulative distribution function:

$$F(x_1, x_2, \cdots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \cdots X_n \le x_n)$$

there exists a copula C such that

$$F(x_1, x_2, \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n))$$

where the $F_i(x_i)$ are the marginal distributions of $F(x_1, x_2, \cdots, x_n)$, and conversely for any copula C the right-hand-side of the above equation defines a multivariate distribution function $F(x_1, x_2, \cdots, x_n)$. See, for example, Bradley and Taqqu (2001), Sklar (1996), and Embrechts *et al* (2003).

The main idea behind the use of copulas is that one can first specify the marginal distributions in whatever way makes sense (e.g. fitting marginal distribution models to risk factor data, and then specify a copula C to capture the multivariate dependency structure in the best suited manner).

A possible approach for choosing a flexible copula model is to adopt the copula of a parametric multivariate distribution. In this way, the copula itself will have a parametric form. There are many multivariate laws mentioned in the literature, which can be used for this purpose. One such example is the Gaussian copula, i.e. the copula of the multivariate normal distribution. It is easy to work with but it has one major drawback: It implies that extreme events are asymptotically independent. Thus, the probability of joint occurrence of large in absolute value negative returns of two stocks is significantly underestimated. An alternative to the Gaussian copula is the Student's *t* copula (i.e., the copula of the multivariate Student's *t* distribution). It models better the probability of joint extreme events but it has the disadvantage that it is symmetric. Thus, the probability of joint occurrence of very large returns is the same as the probability of joint occurrence of very small returns. This deficiency is not present in the skewed Student's *t* copula which we believe is a much more realistic model of dependency. This is the copula of the multivariate skewed Student's *t* distribution defined by means of the following stochastic representation,

$$X = \mu + \gamma W + Z\sqrt{W}$$

where $W \in IG(v/2, v/2)$, i.e. W is inverse gamma distributed, Z is multivariate normal random variable, $Z \in N_n(0,\Sigma)$, W and Z are independent, and the constants μ and γ are such that the sign of a given component of γ controls the asymmetry of the corresponding component of X and μ is a location parameter contributing to the mean of X. The skewed Student's t copula has the following parametric form,

$$C(u_1,\ldots,u_n) = \int_{-\infty}^{t_{v,\gamma}^{-1}(u_1)} \cdots \int_{-\infty}^{t_{v,\gamma}^{-1}(u_n)} f(x_1,\ldots,x_n)dx_1\ldots dx_n$$

where $t_{v,\gamma}^{-1}(u_1)$ is the inverse cdf of the one-dimensional skewed Student's t distribution, and $f(x_1,\ldots,x_n)$ is the density of the multivariate skewed Student's t distribution,

$$f(x_1,\ldots,x_n) = \frac{2^{1-(v+n)/2}}{\Gamma(v/2)(\pi v)^{d/2}|\Sigma|^{1/2}} * \frac{\exp\left((x-\mu)'\Sigma^{-1}\gamma\right)}{\left(1+\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{v}\right)^{\frac{v+d}{2}}} * \frac{K_{\frac{v+d}{2}}\left(\sqrt{\left(v+(x-\mu)'\Sigma^{-1}(x-\mu)\right)\gamma'\Sigma^{-1}\gamma}\right)}{\left(\sqrt{\left(v+(x-\mu)'\Sigma^{-1}(x-\mu)\right)\gamma'\Sigma^{-1}\gamma}\right)^{-\frac{v+d}{2}}}$$

in which $K_\lambda(x)$ stands for the modified Bessel function of the third kind. The skewed Student's t copula has the following features which make it a flexible and attractive model

- it has a parametric form which makes the copula an attractive model in higher dimensions
- the underlying stochastic representation facilitates scenario generation from the copula
- it can describes tail dependence, if present in the data
- it can describes asymmetric dependence, if present in the data

3. Average Value-at-Risk

A major activity in many financial institutions is to recognize the sources of risk, then manage and control them. This is possible only if risk is quantified. If we can measure the risk of a portfolio, then we can identify the financial assets which constitute the main risk contributors, reallocate the portfolio, and, in this way, minimize the potential loss by minimizing the portfolio risk.

From a historical perspective, Markowitz (1952) was the first to recognize the relationship between risk and reward and introduced the standard deviation as a proxy for risk. The standard deviation is not a good choice for a risk measure because it penalizes symmetrically both the negative and the positive deviations from the mean. It is an uncertainty measure and cannot account for the asymmetric nature of risk, i.e. risk concerns losses only. The deficiencies of the standard deviation as a risk measure were acknowledged by Markowitz who was the first to suggest the semi-standard deviation as a substitute, Markowitz (1959).

A risk measure which has been widely accepted since 1990s is the value-at-risk (VaR). In the late 1980s, it was integrated by JP Morgan on a firmwide level into its risk-management system. In this system, JP Morgan developed a service called RiskMetrics which was later spun off into a separate company called RiskMetrics Group. It is usually thought that JP Morgan first formulated the VaR measure. In fact, similar ideas had been used by large financial institutions. For more information about risk measures, the reader is referred to Rachev *et al* (2008) and the references therein.

Although VaR has been widely adopted as a standard risk measure in the financial industry, it has a number of deficiencies recognized by financial professionals. One important deficiency is that VaR cannot always account for the risk diversification effect. There are examples in which portfolio VaR is larger than the sum of the VaRs of the portfolio constituents. Another important deficiency is that VaR is not informative about the extreme losses beyond the VaR level. A risk measure which lacks these deficiencies is the average value-at-risk (AVaR). It is defined as the average VaR beyond a given VaR level. Not only does it have an intuitive definition, but there are also convenient ways of computing and estimating it. As a result, AVaR turns into a superior alternative to VaR suitable for management of portfolio risk and optimal portfolio problems. The average of VaRs is computed through the integral,

$$AVaR_{\varepsilon}(X) = \frac{1}{\varepsilon}\int_0^{\varepsilon} VaR_p(X)dp$$

where ε denotes the tail probability and $VaR_p(X) = -\inf\{X : P(X \le x) \ge p\}$ is the VaR of X at tail probability p. For additional information about AVaR, see Rachev, *et al* (2008). If the distribution of X is absolutely continuous, then the notion of AVaR coincides with the expected tail loss (ETL) defined through the conditional expectation,

$$ETL_{\varepsilon}(X) = -E(X \mid X < -VaR_{\varepsilon}(X)).$$

For this reason, in stable Paretian models for asset returns distributions, we can use both terms interchangeably. However, even though from a mathematical viewpoint both terms are equivalent for absolutely continuous distributions, we choose the notion of ETL when combining with stable distributions for asset returns modeling since ETL is intuitively linked to the tail behavior which is a central notion in stable Paretian distributions.

We summarize the attractive properties of AVaR below:

- AVaR gives an informed view of losses beyond VaR.

- AVaR is a convex, smooth function of portfolio weights, and is therefore attractive to optimize portfolios (see Uryasev and Rockafellar, (2000)).

- AVaR is sub-additive and satisfies a set of intuitively appealing coherent risk measure properties (see Artzner *et al*, (1999)).

- AVaR is a form of expected loss (i.e., a conditional expected loss) and is a very convenient form for use in scenario-based portfolio optimization. It is also quite a natural risk-adjustment to expected return (see STARR, or Stable Tail Adjusted Return Ratio).

Even though AVaR is not widely adopted, we expect it to become an accepted risk measure as portfolio and risk managers become more familiar with its attractive properties. For portfolio optimization, we recommend the use of Stable distribution ETL (SETL), and limiting the use of historical, normal or stable VaR to required regulatory reporting purposes only. Finally, organizations should consider the advantages of SETL for risk assessment purposes and non-regulatory reporting purposes.

4. Risk decomposition based on SETL

The concept of SETL allows for scenario-based risk decomposition which is a concept similar to the standard deviation based percentage contribution to risk (PCTR). The practical issue is to identify the contribution of each position to portfolio risk and since ETL is a tail risk measure, percentage contribution to ETL allows one to build a framework for tail risk budgeting. The approach largely depends on one of the properties of coherent risk measures given in Artzner *et al* (1999), which is the positive homogeneity property

$$ETL_{\varepsilon}(aX) = aETL_{\varepsilon}(X), \quad a > 0$$

There is a formula in calculus known as Euler's formula which is valid for such functions. According to it, the risk measure can be expressed in terms of a weighted average of the partial derivatives with respect to portfolio assets assuming that there exists a small cash account,

$$ETL_{\varepsilon}(w'r) = \sum_i w_i \frac{\partial ETL_{\varepsilon}(w'r)}{\partial w_i}$$

The cash account is used to finance the infinitesimal increase of portfolio holdings in order to compute the partial derivatives of the risk measure. The left hand-side of the equation equals total portfolio risk and if we divide both sides by it, we obtain the needed tail risk decomposition,

$$1 = \sum_i \frac{w_i}{ETL_{\varepsilon}(w'r)} \frac{\partial ETL_{\varepsilon}(w'r)}{\partial w_i}$$

The same idea can be applied if there is an underlying factor model in order to get the factor percentage contribution to tail risk or, on a more general level, the systematic and non-systematic percentage contribution. Furthermore, the partial derivatives of ETL can be computed from scenarios (see Zhang and Rachev (2006)).

5. Portfolio optimization with SETL

The solution of the optimal portfolio problem is a portfolio that minimizes a given risk measure provided that the expected return is constrained by some minimal value *R*. In our framework, we adopt the ETL as a risk measure:

$$\begin{aligned} &\min_{w} ETL_{\varepsilon}\left(w' r - r_b\right) \\ &s.t. \\ &w' Er - Er_b \geq R \\ &l \leq Aw \leq u \end{aligned} \qquad (2)$$

where the vector notation $w'r$ stands for the returns of a portfolio with composition $w = (w_1, w_2, \ldots, w_n)$, l is a vector of lower bounds, A is a matrix, u is a vector of upper bounds, and r_b is some benchmark (which could be set equal to zero). The set comprised by the double linear inequalities in matrix notation $l \leq Aw \leq u$ includes all feasible portfolios.

If the benchmark is zero, $r_b = 0$, and instead of ETL we use the standard deviation, which is an uncertainty measure, then the optimization problem transforms into the classical Markowitz problem. Optimal portfolio problems with a benchmark are called *active*. The benchmark could be non-stochastic or stochastic, for example the return of another portfolio or a market index. In case r_b is non-zero and we use the standard deviation instead of ETL, the problem transforms into the classical tracking error problem.

The set of all solutions of (2), when varying the value of the constraint, is called the efficient frontier. Along the efficient frontier, there is a portfolio that provides the maximum expected return per unit of risk; that is, this portfolio is a solution to the optimal ratio problem

$$\begin{aligned} &\max_{w} \frac{w^T Er - Er_b}{ETL_{\varepsilon}\left(w^T r - r_b\right)} \\ &s.t. \\ &l \leq Aw \leq u \end{aligned} \qquad (3)$$

An example of a reward-risk ratio is the celebrated Sharpe ratio or the information ratio depending on whether the benchmark is stochastic. In both cases, the standard deviation is used instead of the ETL. Beside the Sharpe ratio, or the information ratio, many more examples can be obtained by changing the risk and, possibly, the reward functional (see Biglova *et al* (2004) for an empirical study).

Problem (3) can be transformed into a simpler problem on the condition that the risk measure is strictly positive for all feasible portfolios

$$\begin{aligned}
&\min_{x,t} ETL_{\varepsilon}\left(x^T r - t r_b\right)\\
&s.t.\\
&x^T Er - tEr_b = 1\\
&tl \leq Ax \leq tu
\end{aligned} \tag{4}$$

where t is an additional variable. If (x_o, t_o) is a solution to (4), then $w_o = x_o/t_o$ is a solution to problem (3). There are other connections between problems (3) and (4), see Stoyanov *et al* (2007) for further details.

Following the approach in Uryasev and Rockafellar (2000), problem (4) can be solved by a linear programming problem,

$$\begin{aligned}
&\min_{w,\theta,d} \theta + \frac{1}{N\varepsilon}\sum_{k=1}^{N} d_k\\
&s.t.\\
&w^T Er - Er_b \geq R\\
&- w^T r^k + r_b^k - \theta \leq d_k, k = 1, N\\
&d_k \geq 0, k = 1, N\\
&l \leq Aw \leq u
\end{aligned} \tag{5}$$

where $\left(r^1,\ldots,r^N\right)$ and $\left(r_b^1,\ldots,r_b^N\right)$ are scenarios for the assets returns and the benchmark generated according to the generalized stable distribution framework. See also Rachev *et al* (2008) for geometric interpretations and further information on computational complexity.

6. Performance measures

The celebrated *Sharpe ratio* for a given portfolio p is defined as follows:

$$SR_p = \frac{ER_p - r_f}{\sigma_p}$$

where ER_p is the portfolio expected return, σ_p is the portfolio return standard deviation as a measure of portfolio risk, and r_f is the risk-free rate. While the Sharpe ratio is the single most widely used portfolio performance measure, it has several disadvantages due to its use of the standard deviation used as a proxy for risk measure:

- σ_p is a symmetric measure that does not focus on downside risk
- σ_p is not a coherent measure of risk (see Artzner *et al*, 1999)
- σ_p has an infinite value for non-Gaussian stable distributions.

Two alternative performance measures consistent with the SETL framework can be constructed, see Rachev *et al* (2007). One of them, the stable tail adjusted return ratio (STARR) defined as

$$STARR = \frac{w'Er - r_f}{ETL_\varepsilon(w'r)},$$

calculates the portfolio excess return per unit of downside risk measured by the ETL. The other performance measure is the Rachev ratio (R-ratio),

$$R - ratio = \frac{ETL_{\varepsilon_1}(w'(r_f - r))}{ETL_{\varepsilon_2}(w'(r - r_f))}$$

where ε_1 and ε_2 are two different tail probabilities and $r - r_f$ is the vector of asset excess returns. The R-ratio is a generalization of the STARR. Choosing appropriate levels ε_1 and ε_2 in optimizing the R-ratio the investor can seek the best risk/return profile of her portfolio. For example, an investor with portfolio allocation maximizing the R-ratio with $\varepsilon_1 = \varepsilon_2 = 0.01$ is seeking exceptionally high returns and protection against high losses.

7. An empirical example

Racheva-Iotova and Stoyanov (2006) provide a back-testing example of a long-only optimal portfolio strategy using the Russell 2000 universe. The back-testing time period is ten years (December 1993 to December 2004) with monthly frequency. In the optimization algorithm, they employ the proprietary stable model in Cognity Risk & Portfolio Optimization System in which the SETL methodology is implemented. In the strategies, the Russell 2000 index is used as the benchmark; that is r_b is the return of Russell 2000.

The optimization constraints are the following.

- 0% to 3% limit on single stock
- +/- 3% industry exposure with respect to the benchmark; the industries being defined by Ford Equity Research
- The active return is strictly positive
- The two-way turnover is below 15% per month. This constraint is used as a soft constraint (i.e., may be exceeded at times). Also, no limit is imposed in July because the benchmark is adjusted in July.

The back-testing is performed in the following way. They use 450 stocks as initial universe. One year of daily data is used to calibrate the model and monthly scenarios are produced by it. Then a version of the optimal portfolio problem (8) is solved in which a tail probability of 5% is selected for the ETL. At the end of the month, the portfolio present value is calculated. The process is repeated next month. Figure 2 shows the stable ETL portfolio present values compared to the Russell 2000 index.

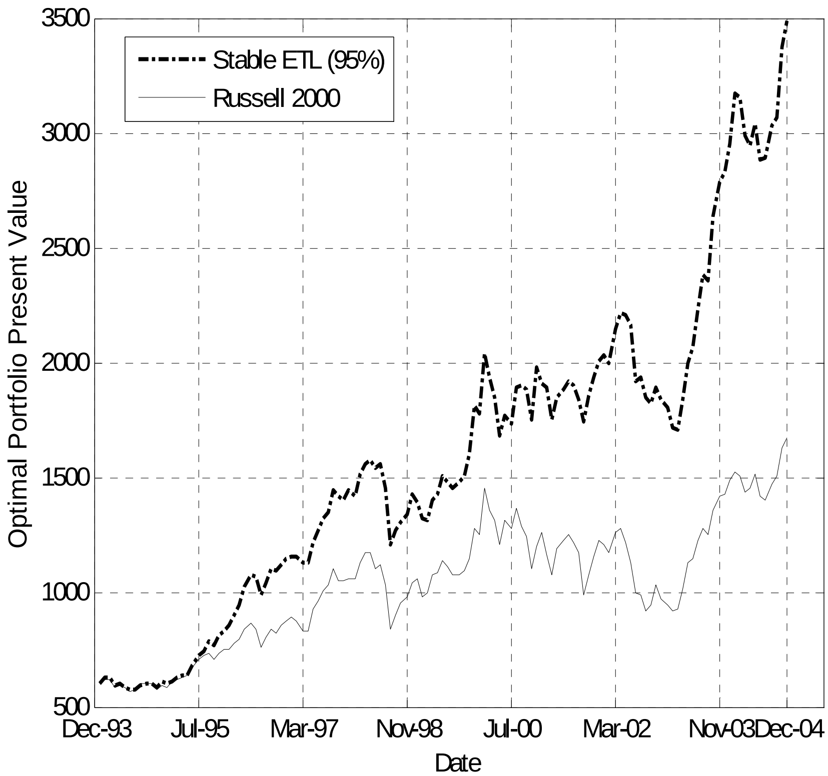

Figure 2. The time evolution of the present values of the stable ETL portfolio compared to the Russell 2000 index. (Reproduced from Figure 1 in Racheva-Iotova and Stoyanov (2006).)

In addition to the stable method, monthly back-testing is performed for a version of the Markowitz problem (6). Racheva-Iotova and Stoyanov use a factor model of eight factors and five years of monthly data to calibrate it. Each month the covariance matrix is estimated through the factor model and the optimization problem is solved. The portfolio present value is calculated at the end of month. Figure 3 shows the evolution of the portfolios present values. Note that the present value of the stable portfolio is scaled to start with the same capital as the Markowitz model.

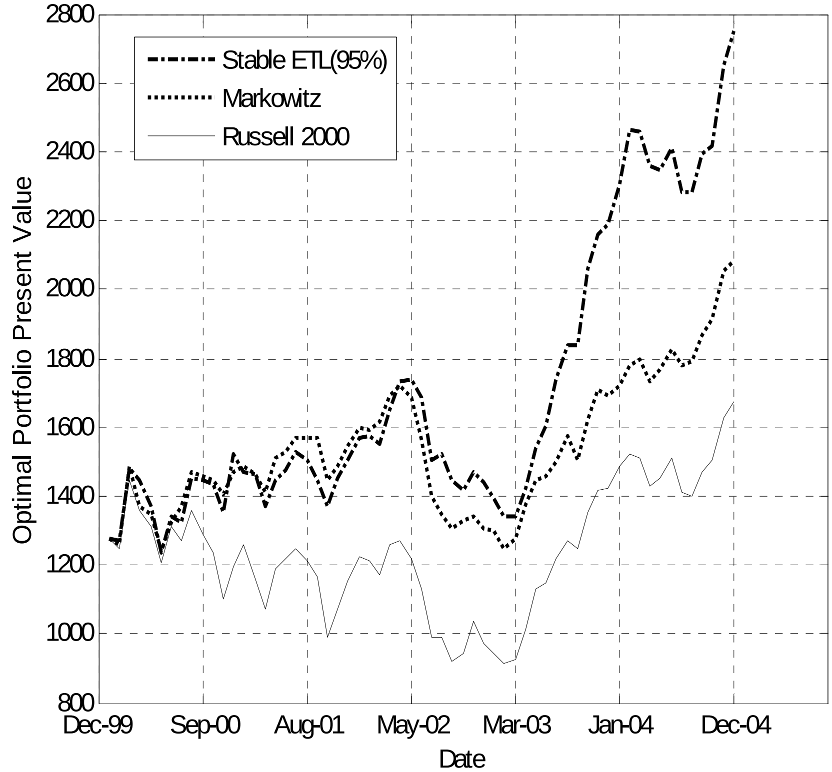

Figure 3. The time evolution of the present values of the Markowitz and the stable ETL (scaled) portfolios compared to the Russell 2000 index (Reproduced from Figure 2 in Racheva-Iotova and Stoyanov (2006).)

Additional information is given in Tables 1 and 2. The average monthly turnover is defined as the dollar-weighted purchases plus the dollar weighted sales. Tables 3 and 4 provide details on return-risk ratios. The information ratio is the active return per unit of tracking error.

	Stable ETL	Markowitz
10 year	112	
5 year	105	137
3 year	102	110
2 year	100	104
1 year	104	100

Table 1. Average number of holdings (Reproduced from Table 1 in Racheva-Iotova and Stoyanov (2006).)

	Stable ETL	Markowitz
11 months	16%	18%
July	163%	85%
All months	27%	24%

Table 2. Average monthly turnover (Reproduced from Table 2 in Racheva-Iotova and Stoyanov (2006).).

	Stable ETL	Markowitz
10 year	0.74	
5 year	0.71	0.29
3 year	0.93	-0.24
2 year	0.74	-0.57
1 year	1.22	1.03

Table 3.Annualized information ratio (Reproduced from Table 3 in Racheva-Iotova and Stoyanov (2006).).

	Stable ETL	Markowitz	Russell 2000
10 year	1.01		0.42
5 year	0.92	0.68	0.36
3 year	1.22	0.71	0.58
2 year	2.13	1.99	1.82
1 year	1.66	2.16	1.19

Table 4. Sharpe ratios (Reproduced from Table 4 in Racheva-Iotova and Stoyanov (2006).).

8. Conclusion

In this paper, we described and discussed the SETL framework which is implemented in Cognity Risk Management and Portfolio Optimization product. The SETL framework is appealing because it is based on realistic assumptions about asset return distributions, incorporates a downside risk measure, and can be used for risk budgeting and portfolio optimization. With the help of empirical examples, we demonstrated that the SETL framework is more realistic than the traditional models based on the normal distribution and may lead to better performance.

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